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IDENTITIES IN 3-PRIME NEAR-RINGS WITH LEFT MULTIPLIERS

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Abstract. Let $\mathcal{N}$ be a 3-prime near-ring with the center $Z(\mathcal{N})$ and $n \geq 1$ be a fixed positive integer. In the present paper it is shown that a 3-prime near-ring $\mathcal{N}$ is a commutative ring if and only if it admits a left multiplier $\mathcal{F}$ satisfying any one of the following properties: (i) $\mathcal{F}^n([x,y]) \in Z(\mathcal{N})$, (ii) $\mathcal{F}^n(x \circ y) \in Z(\mathcal{N})$, (iii) $\mathcal{F}^n([x,y]) \pm (x \circ y) \in Z(\mathcal{N})$ and (iv) $\mathcal{F}^n([x,y]) \pm x \circ y \in Z(\mathcal{N})$, for all $x, y \in \mathcal{N}$.

1. Introduction

Let $\mathcal{N}$ be a right near-ring with multiplicative center $Z(\mathcal{N})$. Define $\mathcal{N}$ to be 3-prime if for $a, b \in \mathcal{N}$, $a\mathcal{N}b = \{0\}$ implies that $a = 0$ or $b = 0$ and call $\mathcal{N}$ 2-torsion-free if $(\mathcal{N}, +)$ has no elements of order 2. A right near-ring $\mathcal{N}$ is called zero-symmetric if $x0 = 0$ for all $x \in \mathcal{N}$ (recalling that right distributivity yields $0x = 0$). For any pair of elements $x, y \in \mathcal{N}$, $[x, y]$ denotes the commutator $xy - yx$, while the symbol $x \circ y$ denotes the anticommutator $xy + yx$. A derivation on $\mathcal{N}$ is an additive endomorphism d of $\mathcal{N}$ such that $d(xy) = xd(y) + d(x)y$ for all $x, y \in \mathcal{N}$, or equivalently, as noted in [20], that $d(xy) = d(x)y + xd(y)$ for all $x, y \in \mathcal{N}$. The concept of derivation in rings has been generalized in several ways by various authors. Generalized derivation has been introduced already in rings by M. Brešar [10]. Also the notions of generalized derivation has been introduced in near-rings by Öznur Gölbasi [14]. An additive mapping $\mathcal{F} : \mathcal{N} \longrightarrow \mathcal{N}$ is called a right generalized derivation

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with associated derivation d if $\mathcal{F}(xy) = \mathcal{F}(x)y + xd(y)$, for all $x, y \in \mathcal{N}$ and $\mathcal{F}$ is called a left generalized derivation with associated derivation d if $\mathcal{F}(xy) = d(x)y + x\mathcal{F}(y)$, for all $x, y \in \mathcal{N}$. $\mathcal{F}$ is called a generalized derivation with associated derivation d if it is both a left as well as a right generalized derivation with associated derivation d. An additive mapping $\mathcal{F} : \mathcal{N} \to \mathcal{N}$ is said to be a left (resp. right) multiplier (or centralizer) if $\mathcal{F}(xy) = \mathcal{F}(x)y$ (resp. $\mathcal{F}(xy) = x\mathcal{F}(y)$) holds for all $x, y \in \mathcal{N}$. $\mathcal{F}$ is said to be a multiplier if it is both left as well as right multiplier. Notice that a right (resp. left) generalized derivation with associated derivation $d = 0$ is a left (resp. right) multiplier. Several authors investigated the commutativity in prime and semiprime rings admitting derivations and generalized derivations which satisfy appropriate algebraic conditions on suitable subset of the rings. For example, we refer the reader to [1], [3], [11], [12], [15], [18], [19], where further references can be found. In [11], Daif and Bell proved that if $\mathcal{R}$ is a prime ring admitting a derivation d and I a nonzero ideal of $\mathcal{R}$ such that $d([x, y]) - [x, y] = 0$ for all $x, y \in I$ or $d([x, y]) + [x, y] = 0$ for all $x, y \in I$, then $\mathcal{R}$ is commutative. Further, Hongan [15] generalized the above result and proved that if $\mathcal{R}$ is a semiprime ring with a nonzero ideal I and d is a derivation of $\mathcal{R}$ such that $d([x, y]) \pm [x, y] \in Z(\mathcal{R})$ for all $x, y \in I$, then I is a central ideal. In particular, if $I = \mathcal{R}$, then $\mathcal{R}$ is commutative. Recently, Dhara [12] generalized this result by replacing derivation d with a generalized derivation F in a prime ring $\mathcal{R}$. More precisely, he proved that if $\mathcal{R}$ is a prime ring and I a nonzero ideal of $\mathcal{R}$ which admits a generalized derivation $\mathcal{F}$ associated with a nonzero derivation d such that either $(i)\mathcal{F}([x, y]) - [x, y] \in Z(\mathcal{R})$ for all $x, y \in I$, or $(ii)\mathcal{F}(x \circ y) + x \circ y \in Z(\mathcal{R})$ for all $x, y \in I$, then $\mathcal{R}$ is commutative. There has been a great deal of work concerning left (or right) multiplier in prime or semiprime rings (see for reference [4], [16], [17], [21], where more references can be found). Recently the second author together with Ali [4] proved that if a prime ring $\mathcal{R}$ admits a left multiplier $\mathcal{F} : \mathcal{R} \to \mathcal{R}$ such that $\mathcal{F}([x, y]) = [x, y]$ with $\mathcal{F} \neq Id_{\mathcal{R}}$ for all $x, y \in I$, a nonzero ideal of $\mathcal{R}$, then $\mathcal{R}$ is commutative. In this line of investigation, it is more interesting to study the identities replacing ring with near-ring. In the present paper, we study all these cases in 3-prime near-ring. It is shown that a 3-prime near-ring $\mathcal{N}$ is a commutative ring if and only if $\mathcal{N}$ admits a left multiplier $\mathcal{F}$ such that any one of the identities (i) $\mathcal{F}^n([x, y]) \in Z(\mathcal{N})$, (ii) $\mathcal{F}^n(x \circ y) \in Z(\mathcal{N})$, (iii) $\mathcal{F}^n(x \circ y) \pm [x, y] \in Z(\mathcal{N})$ and (iv) $\mathcal{F}^n([x, y]) \pm x \circ y \in Z(\mathcal{N})$, holds for all $x, y \in \mathcal{N}$ and $n \geq 1$ a fixed positive integer.

2. Some preliminaries

In this section, we give some well known results of near-rings which will be used extensively in the remaining part of the paper. Proof of Lemma 2.1 can be seen in [6], while Lemmas 2.2 and 2.3 are essentially proved in [9].

Lemma 2.1. *Let $\mathcal{N}$ be a 3-prime near-ring.*

(i) *If $z \in Z(\mathcal{N}) \setminus \{0\}$ and $xz \in Z(\mathcal{N})$, then $x \in Z(\mathcal{N})$.*
(ii) *If x is an element of $\mathcal{N}$ such that $\mathcal{N}x = \{0\}$ (resp. $x\mathcal{N} = \{0\}$), then $x = 0$.*
(iii) *If $\mathcal{N} \subseteq Z(\mathcal{N})$, then $\mathcal{N}$ is a commutative ring.*

Lemma 2.2. *Let $\mathcal{N}$ be a 3-prime near-ring. If $\mathcal{N}$ admits a nonzero derivation d, then the following assertions are equivalent:*

(i) *$[x, y] \in Z(\mathcal{N})$ for all $x, y \in \mathcal{N}$.*
(ii) *$\mathcal{N}$ is a commutative ring.*

Lemma 2.3. *Let $\mathcal{N}$ be a 2-torsion 3-prime near-ring. If $\mathcal{N}$ admits a nonzero derivation d, then the following assertions are equivalent:*

(i) *$x \circ y \in Z(\mathcal{N})$ for all $x, y \in \mathcal{N}$.*
(ii) *$\mathcal{N}$ is a commutative ring.*

Lemma 2.4. *Let $\mathcal{R}$ be a ring and $\mathcal{F}$ a map of $\mathcal{R}$. Then the following assertions are equivalent:*

(i) *$\mathcal{F}$ is a left multiplier;*
(ii) *$\mathcal{F} - Id_{\mathcal{R}}$ is a left multiplier;*
(iii) *$\mathcal{F} + Id_{\mathcal{R}}$ is a left multiplier.*
(iv) *for each positive integer $n \geq 1$, $\mathcal{F}^n$ is a left multiplier.*

Proof. $(i) \Rightarrow (ii)$ Suppose that $\mathcal{F}$ is a left multiplier. If we set $G = \mathcal{F} - Id_{\mathcal{R}}$, then we find that

$$\begin{aligned} G(xy) &= \mathcal{F}(xy) - xy \\ &= (\mathcal{F}(x) - x)y \\ &= G(x)y \quad \text{for all} \quad x, y \in \mathcal{R}. \end{aligned}$$

$(i) \Rightarrow (iii)$ Now, we set $H = \mathcal{F} + Id_{\mathcal{R}}$, using the similar method as above, we obtain the required result.
$(ii) \Rightarrow (i)$ Assume that $G = \mathcal{F} - Id_{\mathcal{R}}$ is a left multiplier. It is obvious that $\mathcal{F} = G + Id_{\mathcal{R}}$ is a left multiplier by the proof of $(i) \Rightarrow (iii)$.
$(iii) \Rightarrow (i)$ Suppose that $H = \mathcal{F} + Id_{\mathcal{R}}$ is a left multiplier, it is very easy to see $\mathcal{F} = H - Id_{\mathcal{R}}$ is a left multiplier by proof of $(i) \Rightarrow (ii)$.
$(i) \Rightarrow (iv)$ Suppose $\mathcal{F}$ is a left multiplier. We proceed by induction,

when $n = 1$ our result remains true.
Now assume that $\mathcal{F}^n$ is a left multiplier. We have

$$\begin{aligned}\mathcal{F}^{n+1}(xy) &= \mathcal{F}(\mathcal{F}^n(xy)) \\ &= \mathcal{F}(\mathcal{F}^n(x)y) \\ &= \mathcal{F}^{n+1}(x)y \quad \text{for all } x, y \in \mathcal{R}.\end{aligned}$$

This proves that $\mathcal{F}^{n+1}$ is a left multiplier, then by induction hypothesis we get that the theorem is true and hence this completes the proof.
$(i) \Rightarrow (iv)$ Simply take $n = 1$.

∎

3. Main result

Theorem 3.1. *Let $\mathcal{N}$ be a 3-prime near-ring. If $\mathcal{N}$ admits a nonzero left multiplier $\mathcal{F}$, then the following assertions are equivalent:*

(i) $\mathcal{F}([x, y]) \in Z(\mathcal{N})$ for all $x, y \in \mathcal{N}$;
(ii) $\mathcal{N}$ is a commutative ring.

Proof. It is obvious that (ii) implies (i). So we need to prove that $(i) \Rightarrow (ii)$.
$(i) \Rightarrow (iii)$ Suppose that

$$\mathcal{F}([x, y]) \in Z(\mathcal{N}) \quad \text{for all } x, y \in \mathcal{N}. \tag{3.1}$$

Replacing y by yx in (3.1), we get

$$\mathcal{F}([x, y])x \in Z(\mathcal{N}) \quad \text{for all } x, y \in \mathcal{N}.$$

By Lemma 2.1 (i) and using (3.1), we obtain

$$\mathcal{F}([x, y]) = 0 \text{ or } x \in Z(\mathcal{N}) \quad \text{for all } x, y \in \mathcal{N}. \tag{3.2}$$

This reduces to

$$\mathcal{F}([x, y]) = 0 \quad \text{for all } x, y \in \mathcal{N}, \tag{3.3}$$

which means that

$$\mathcal{F}(x)y = \mathcal{F}(y)x \quad \text{for all } x, y \in \mathcal{N}. \tag{3.4}$$

Putting yt instead of y in (3.4), we obtain

$$\mathcal{F}(x)yt = \mathcal{F}(y)tx \quad \text{for all } x, y, t \in \mathcal{N}. \tag{3.5}$$

Replacing x by $[u, v]$ where $u, v \in \mathcal{N}$ in (3.6) and invoking (3.3), we get

$$\mathcal{F}(y)t[u, v] = 0 \quad \text{for all } u, v, y, t \in \mathcal{N}, \tag{3.6}$$

which implies that

$$\mathcal{F}(y)\mathcal{N}[u, v] = \{0\} \quad \text{for all } u, v, y \in \mathcal{N} \tag{3.7}$$

By 3-primeness of $\mathcal{N}$ together the fact that $\mathcal{F} \neq 0$, we obtain $[u, v] = 0$ for all $u, v \in \mathcal{N}$, and hence by Lemma 2.2, we conclude that $\mathcal{N}$ is a commutative ring. ∎

Using Lemma 2.4, we obtain the following results:

Corollary 3.2. *Let $\mathcal{N}$ be a 3-prime near-ring and $n \geq 1$ be a fixed positive integer. If $\mathcal{N}$ admits a left multiplier $\mathcal{F}$, then the following assertions are equivalent:*

(i) *$\mathcal{F}^n([x, y]) \in Z(\mathcal{N})$ for all $x, y \in \mathcal{N}$;*
(ii) *$\mathcal{N}$ is a commutative ring.*

Corollary 3.3. *Let $\mathcal{R}$ be a prime ring and $n \geq 1$ be a fixed positive integer. If $\mathcal{R}$ admits a left multiplier $\mathcal{F}$ such that $\mathcal{F} \neq Id_{\mathcal{R}}$, then the following assertions are equivalent:*

(i) *$\mathcal{F}^n([x, y]) - [x, y] \in Z(\mathcal{R})$ for all $x, y \in \mathcal{R}$;*
(ii) *$\mathcal{F}^n([x, y]) + [x, y] \in Z(\mathcal{R})$ for all $x, y \in \mathcal{R}$;*
(iii) *$\mathcal{R}$ is commutative.*

Theorem 3.4. *Let $\mathcal{N}$ be a 2-torsion free 3-prime near-ring. If $\mathcal{N}$ admits a nonzero left multiplier $\mathcal{F}$, then the following assertions are equivalent*

(i) *$\mathcal{F}(x \circ y) \in Z(\mathcal{N})$ for all $x, y \in \mathcal{N}$;*
(ii) *$\mathcal{N}$ is a commutative ring.*

Proof. It is obvious that (ii) implies (i). So we need to prove that $(i) \Rightarrow (ii)$.
$(i) \Rightarrow (ii)$ Assume that

$$\mathcal{F}(x \circ y) \in Z(\mathcal{N}) \quad \text{for all } \ x, y \in \mathcal{N}. \tag{3.8}$$

Using the same techniques that was used after (3.1), we arrive at

$$\mathcal{F}(x \circ y) = 0 \text{ or } \ x \in Z(\mathcal{N}) \quad \text{for all } \ x, y \in \mathcal{N}. \tag{3.9}$$

Suppose there exists $x_0 \in Z(\mathcal{N}) \setminus \{0\}$. From $\mathcal{F}(x_0 \circ t) \in Z(\mathcal{N})$ it follows that $(\mathcal{F}(t) + \mathcal{F}(t))x_0 \in Z(\mathcal{N})$ for all $t \in \mathcal{N}$ by Lemma 2.1 (i) and 2-torsion freeness of $\mathcal{N}$, we obtain

$$\mathcal{F}(t) \in Z(\mathcal{N}) \text{ or } \ x_0 = 0 \quad \text{for all } \ t \in \mathcal{N}. \tag{3.10}$$

Using (3.10), then (3.9) becomes

$$\mathcal{F}(x \circ y) = 0 \text{ or } \ \mathcal{F}(t) \in Z(\mathcal{N}) \quad \text{for all } \ x, y, t \in \mathcal{N}. \tag{3.11}$$

If $\mathcal{F}(t) \in Z(\mathcal{N})$ for all $t \in \mathcal{N}$, replacing t by ty and using Lemma 2.1 (i), we obtain $\mathcal{F}(t) = 0$ or $y \in Z(\mathcal{N})$ for all $y, t \in \mathcal{N}$. Since $\mathcal{F} \neq 0$, then $\mathcal{N} \subseteq Z(\mathcal{N})$ and by Lemma 2.1 (iii), we conclude that $\mathcal{N}$ is a commutative ring.

If $\mathcal{F}(x \circ y) = 0$ for all $x, y \in \mathcal{N}$, then $\mathcal{F}(x)y = -\mathcal{F}(y)x$ for all $x, y \in \mathcal{N}$. Substituting yt for y in the last relation, we obtain

$$\mathcal{F}(x)yt = \mathcal{F}(-y)tx \quad \text{for all } \; x, y, t \in \mathcal{N}. \tag{3.12}$$

Putting $u \circ v$ instead of x in (3.12) and invoking the fact that $\mathcal{F}(x \circ y) = 0$ for all $x, y \in \mathcal{N}$, we arrive at

$$\mathcal{F}(-y)\mathcal{N}(u \circ v) = \{0\} \quad \text{for all } \; u, v, y \in \mathcal{N}. \tag{3.13}$$

Since $\mathcal{F} \neq 0$, then the 3-primeness of $\mathcal{N}$ forces that $u \circ v = 0$ for all $u, v \in \mathcal{N}$. For $u = v$, the 2-torsion freeness of $\mathcal{N}$ implies that $u^2 = 0$ for all $u \in \mathcal{N}$. Therefore, $(u + v)^2 u = 0$ for all $u, v \in \mathcal{N}$ which means that $u\mathcal{N}u = \{0\}$ for all $u \in \mathcal{N}$ by 3-primeness of $\mathcal{N}$, we conclude that $u = 0$ for all $u \in \mathcal{N}$; a contradiction. ∎

By applying Lemma 2.4, we easily find the following results:

Corollary 3.5. *Let $\mathcal{N}$ be a 2-torsion free 3-prime near-ring and and $n \geq 1$ be a fixed positive integer. If $\mathcal{N}$ admits a nonzero left multiplier $\mathcal{F}$, then the following assertions are equivalent:*

(i) *$\mathcal{F}^n(x \circ y) \in Z(\mathcal{N})$ for all $x, y \in \mathcal{N}$;*

(ii) *$\mathcal{N}$ is a commutative ring.*

Corollary 3.6. *Let $\mathcal{R}$ be a 2-torsion free prime ring and and $n \geq 1$ be a fixed positive integer. If $\mathcal{R}$ admits a left multiplier $\mathcal{F}$ such that $\mathcal{F} \neq \pm Id_{\mathcal{R}}$, then the following assertions are equivalent:*

(i) *$\mathcal{F}^n(x \circ y) - x \circ y \in Z(\mathcal{R})$ for all $x \in \mathcal{R}$;*

(ii) *$\mathcal{F}^n(x \circ y) + x \circ y \in Z(\mathcal{R})$ for all $x \in \mathcal{R}$;*

(iii) *$\mathcal{R}$ is commutative.*

Theorem 3.7. *Let $\mathcal{N}$ be a 2-torsion free 3-prime near-ring. If $\mathcal{N}$ admits a left multiplier $\mathcal{F}$, then the following assertions are equivalent:*

(i) *$\mathcal{F}(x \circ y) \pm [x, y] \in Z(\mathcal{N})$ for all $x, y \in \mathcal{N}$;*

(ii) *$\mathcal{F}([x, y]) \pm x \circ y \in Z(\mathcal{N})$ for all $x, y \in \mathcal{N}$;*

(iii) *$\mathcal{N}$ is a commutative ring.*

Proof. It is obvious that (iii) implies (i) and (ii) . So we need to prove that $(i) \Rightarrow (iii)$ and $(ii) \Rightarrow (iii)$.

$(i) \Rightarrow (iii)$ If $\mathcal{F} = 0$, then $[x, y] \in Z(\mathcal{N})$ for all $x, y \in \mathcal{N}$, using Lemma 2.2, we obtain $\mathcal{N}$ is a commutative ring.

If $\mathcal{F} \neq 0$, then we have

$$\mathcal{F}(x \circ y) - [x, y] \in Z(\mathcal{N}) \quad \text{for all } \; x, y \in \mathcal{N}. \tag{3.14}$$

For $y = x$, (3.14) becomes $\mathcal{F}(x^2) \in Z(\mathcal{N})$ for all $x \in \mathcal{N}$ so $\mathcal{F}(x)x \in Z(\mathcal{N})$ for all $x \in \mathcal{N}$, replacing x by x^2 in the last expression and using it again with Lemma 2.1 (i), we conclude that

$$\mathcal{F}(x^2) = 0 \text{ or } \ x^2 \in Z(\mathcal{N}) \quad \text{for all } \ x \in \mathcal{N}. \tag{3.15}$$

If there exists $x_0 \in \mathcal{N}$ such that $x_0^2 \in Z(\mathcal{N})$, then (3.14) implies that $\mathcal{F}(y \circ x_0^2) \in Z(\mathcal{N})$ for all $y \in \mathcal{N}$. This implies that $2\mathcal{F}(y)x_0^2 \in Z(\mathcal{N})$ for all $y \in \mathcal{N}$ by 2-torsion freeness of $\mathcal{N}$ and Lemma 2.1(i), we get

$$\mathcal{F}(y) \in Z(\mathcal{N}) \text{ or } \ x_0^2 = 0 \quad \text{for all } \ y \in \mathcal{N}. \tag{3.16}$$

In view of (3.16), (3.15) becomes

$$\mathcal{F}(y) \in Z(\mathcal{N}) \text{ or } \ \mathcal{F}(x^2) = 0 \quad \text{for all } \ x, y \in \mathcal{N}. \tag{3.17}$$

If $\mathcal{F}(y) \in Z(\mathcal{N})$ for all $y \in \mathcal{N}$, using the same technique as used previously, we get $\mathcal{N}$ is a commutative ring.
If $\mathcal{F}(x^2) = 0$ for all $x \in \mathcal{N}$, then using (3.14), we obtain

$$\mathcal{F}(t)x^2 - [x^2, t] \in Z(\mathcal{N}) \quad \text{for all } \ x, t \in \mathcal{N}. \tag{3.18}$$

Replacing t by tx in (3.18), we obtain

$$(\mathcal{F}(t)x^2 - [x^2, t])x \in Z(\mathcal{N}) \quad \text{for all } \ x, t \in \mathcal{N}. \tag{3.19}$$

By Lemma 2.1 (i), (3.19) becomes

$$\mathcal{F}(t)x^2 = [x^2, t] \text{ or } \ x \in Z(\mathcal{N}) \quad \text{for all } \ x, t \in \mathcal{N}. \tag{3.20}$$

If there exists $x_0 \in Z(\mathcal{N})$, then $x_0^2 \in Z(\mathcal{N})$ and (3.19) implies that $\mathcal{F}(u)x_0^2 \in Z(\mathcal{N})$ for all $u \in \mathcal{N}$. By Lemma 2.1 (i), we obtain either $x_0^2 = 0$ or $\mathcal{F}(u) \in Z(\mathcal{N})$ for all $u \in \mathcal{N}$ in this case (3.20) becomes

$$\mathcal{F}(t)x^2 = [x^2, t] \text{ or } \ \mathcal{F}(u) \in Z(\mathcal{N}) \quad \text{for all } \ u, x, t \in \mathcal{N}. \tag{3.21}$$

If $\mathcal{F}(u) \in Z(\mathcal{N})$ for all $u \in \mathcal{N}$. Using similar arguments as above, we can easily prove $\mathcal{N}$ is a commutative ring.
If $\mathcal{F}(t)x^2 = [x^2, t]$ for all $x, t \in \mathcal{N}$, then replacing t by z^2 where $z \in Z(\mathcal{N})$, we obtain $\mathcal{F}(z)zx^2 = 0$ for all $x \in \mathcal{N}$. This reduces to $\mathcal{F}(z)\mathcal{N}z\mathcal{N}x^2 = \{0\}$ for all $x \in \mathcal{N}$, and hence by 3-primeness of $\mathcal{N}$ we obtain

$$\mathcal{F}(z) = 0 \text{ or } \ z = 0 \text{ or } \ x^2 = 0 \quad \text{for all } \ x \in \mathcal{N}, z \in Z(\mathcal{N}). \tag{3.22}$$

If there exists $z_0 \in Z(\mathcal{N})$, then by (3.14) and 2-torsion freeness, we obtain $\mathcal{F}(x)z_0 \in Z(\mathcal{N})$ for all $x \in \mathcal{N}$. By Lemma 2.1, we obtain $z_0 = 0$ or $\mathcal{F}(x) \in Z(\mathcal{N})$ for all $x \in \mathcal{N}$ which forces that $z_0 = 0$ or $\mathcal{N}$ is a commutative ring. In this case (3.22) implies that $x^2 = 0$ for all $x \in \mathcal{N}$, $Z(\mathcal{N}) = \{0\}$ or $\mathcal{N}$ is a commutative ring.
If $x^2 = 0$ for all $x \in \mathcal{N}$, using the same techniques as used previously,

we find a contradiction.
If $Z(\mathcal{N}) = \{0\}$, then by (3.14) we have

$$\mathcal{F}(x \circ y) = [x, y] \quad \text{for all } \; x, y \in \mathcal{N}. \tag{3.23}$$

For $x = y$, (3.23) gives $2\mathcal{F}(x^2) = 0$ for all $x \in \mathcal{N}$ and hence by 2-torsion freeness we obtain $\mathcal{F}(x^2) = 0$ for all $x \in \mathcal{N}$. Replacing x by x^2 in (3.23) and invoking the last expression, we obtain

$$\mathcal{F}(y)x^2 = [x^2, y] \quad \text{for all } \; x, y \in \mathcal{N}. \tag{3.24}$$

In the same way, putting y^2 instead of y in (3.23), we have

$$\mathcal{F}(x)y^2 = [x, y^2] \quad \text{for all } \; x, y \in \mathcal{N}. \tag{3.25}$$

For $x = y$, (3.25) implies that

$$\mathcal{F}(y)x^2 = [y, x^2] \quad \text{for all } \; x, y \in \mathcal{N}. \tag{3.26}$$

Comparing (3.24) and (3.26), we arrive at

$$[x^2, y] = [y, x^2] \quad \text{for all } \; x, y \in \mathcal{N}. \tag{3.27}$$

This means that

$$2[x^2, y] = 0 \quad \text{for all } \; x, y \in \mathcal{N}. \tag{3.28}$$

Using 2-torsion freeness of $\mathcal{N}$, we find that $x^2 \in Z(\mathcal{N})$ for all $x \in \mathcal{N}$. Since $Z(\mathcal{N}) = \{0\}$, then $x^2 = 0$ for all $x \in \mathcal{N}$ which gives a contradiction as shown above.
Now suppose the case that $\mathcal{F}(x \circ y) + [x, y] \in Z(\mathcal{N})$ for all $x, y \in \mathcal{N}$, using similar arguments as above, we can easily prove the required result.
$(ii) \Rightarrow (iii)$ If $\mathcal{F} = 0$, then $x \circ y \in Z(\mathcal{N})$ for all $x, y \in \mathcal{N}$, using Lemma 2.3, we obtain $\mathcal{N}$ is a commutative ring.
If $\mathcal{F} \neq 0$, then we have

$$\mathcal{F}([x, y]) - x \circ y \in Z(\mathcal{N}) \quad \text{for all } \; x, y \in \mathcal{N}. \tag{3.29}$$

For $x = y$, (3.29) implies that

$$-(x + x)x = -(x^2 + x^2) \in Z(\mathcal{N}) \quad \text{for all } \; x \in \mathcal{N}. \tag{3.30}$$

Replacing x by x^2 in (3.30) and hence using it again, we get $-(x^2+x^2) = 0$ or $x^2 \in Z(\mathcal{N})$ for all $x \in \mathcal{N}$ by 2-torsion freeness of $\mathcal{N}$ the last expression forces $x^2 \in Z(\mathcal{N})$ for all $x \in \mathcal{N}$. In this case, replacing x by x^2 in (3.29), we get $2x^2y \in Z(\mathcal{N})$ for all $x, y \in \mathcal{N}$ and using Lemma 2.1 (i) together with 2-torsion freeness of $\mathcal{N}$, we arrive at $x^2 = 0$ or $y \in \mathcal{N}$ for all $x, y \in \mathcal{N}$ by the same techniques as used previously the above expression forces that $\mathcal{N}$ is a commutative ring.
If $\mathcal{F}([x, y]) + x \circ y \in Z(\mathcal{N})$ for all $x, y \in \mathcal{N}$, using similar approach with the necessary variations, we can prove that the required result. ∎

Corollary 3.8. *Let $\mathcal{N}$ be a 2-torsion free 3-prime near-ring and $n \geq 1$ be a fixed positive integer. If $\mathcal{N}$ admits a left multiplier $\mathcal{F}$, then the following assertions are equivalent:*

(*i*) $\mathcal{F}^n(x \circ y) \pm [x, y] \in Z(\mathcal{N})$ *for all* $x, y \in \mathcal{N}$;
(*ii*) $\mathcal{F}^n([x, y]) \pm x \circ y \in Z(\mathcal{N})$ *for all* $x, y \in \mathcal{N}$;
(*iii*) $\mathcal{N}$ *is a commutative ring.*

The following example demonstrates that our results are not true for arbitrary near-rings.

Example 3.9. Suppose that S is any right near-ring, $n \geq 1$ be a fixed positive integer. Let $\mathcal{N} = \left\{ \begin{pmatrix} 0 & a & b \\ 0 & 0 & c \\ 0 & 0 & 0 \end{pmatrix} \mid a, b, c \in S \right\}$. Define a map $\mathcal{F} : \mathcal{N} \to \mathcal{N}$ such that $\mathcal{F}\begin{pmatrix} 0 & a & b \\ 0 & 0 & c \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & a \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$. Then, it is easy to see that $\mathcal{N}$ is a right near-ring and $\mathcal{F}$ is a left multiplier on $\mathcal{N}$ satisfying the following properties:

(*i*) $\mathcal{F}^n([x, y]) \in Z(\mathcal{N})$,
(*ii*) $\mathcal{F}^n(x \circ y) \in Z(\mathcal{N})$,
(*iii*) $\mathcal{F}^n([x, y]) \pm x \circ y \in Z(\mathcal{N})$,
(*iv*) $\mathcal{F}^n(x \circ y) \pm [x, y] \in Z(\mathcal{N})$

for all $x, y \in \mathcal{N}$. However, $\mathcal{N}$ is not commutative.

The following example shows that for $n \geq 1$ the conditions $\mathcal{F}^n([x, y]) \in Z(\mathcal{N}), \mathcal{F}^n(x \circ y) \in Z(\mathcal{N}), \mathcal{F}^n([x, y]) \pm x \circ y \in Z(\mathcal{N}), \mathcal{F}^n(x \circ y) \pm [x, y] \in Z(\mathcal{N})$ for all $x, y \in \mathcal{N}$ are crucial.

Example 3.10. Let $\mathcal{N} = M_2(\mathbb{Z})$ be the 2×2 matrix ring over $\mathbb{Z}$, $n \geq 1$ a fixed positive integer and $\mathcal{F} : \mathcal{N} \to \mathcal{N}$ such that

$$\mathcal{F}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix} \text{ for all } a, b, c, d \in \mathbb{Z}.$$

It is easy to verify that $\mathcal{N}$ is a non-commutative prime ring which is 2-torsion free and $\mathcal{F}$ is a left multiplier of $\mathcal{N}$. Moreover, for $A = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}$, $B = \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}$, we have

(*i*) $\mathcal{F}^n([A, B]) \notin Z(\mathcal{N})$,
(*ii*) $\mathcal{F}^n(A \circ B) \notin Z(\mathcal{N})$,

(iii) $\mathcal{F}^n([A,B]) \pm A \circ B \notin Z(\mathcal{N})$,
(iv) $\mathcal{F}^n(A \circ B) + [A,B] \notin Z(\mathcal{N})$.

The following example demonstrate that the existence of 2-torsion free in the hypotheses of Theorem 3.7 is essential.

Example 3.11. Let $\mathcal{N} = M_2(\mathbb{Z}_2)$ be the 2×2 matrix ring over the field $\mathbb{Z}_2, n \geq 1$ a fixed positive integer and $\mathcal{F} = Id_{\mathcal{N}}$. It is easy to see that $\mathcal{N}$ is a non-commutative prime ring which is not 2-torsion free. Moreover, $\mathcal{N}$ satisfies the condition (i) $\mathcal{F}(x \circ y) \pm [x,y] \in Z(\mathcal{N})$ and (ii) $\mathcal{F}([x,y]) \pm x \circ y \in Z(\mathcal{N})$ for all $x, y \in \mathcal{N}$.

Remark. It can be easily seen that the above results which are obtained for left multipliers are also true in the case of right multipliers.

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