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Bases for polynomial-based spaces

Maryam Mohammadi[†*] and Maryam Bahrkazemi[‡]

[†] *Faculty of Mathematical Sciences and Computer, Kharazmi University, Tehran, Iran.*

[‡] *School of Mathematics, Iran University of Science and Technology, Tehran, Iran.*

Emails: m.mohammadi@khu.ac.ir, maryam_bahrkazemi@alumni.iust.ac.ir

Abstract. Since it is well-known that the Vandermonde matrix is ill-conditioned, this paper surveys the choices of other bases. These bases are data-dependent and are categorized into discretely ℓ^2-orthonormal and continuously L^2-orthonormal bases. The first one is defined via a decomposition of the Vandermonde matrix while the latter is given by a decomposition of the Gramian matrix corresponding to monomial bases. A discussion of various matrix decomposition (e.g. Cholesky, QR and SVD) provides a variety of different bases with different properties. Special attention is given to duality. Numerical results show that the matrices of values of the new bases have smaller condition numbers than the common monomial bases. It can also be pointed out that the new introduced bases are good candidates for interpolation.

Keywords: Polynomial interpolation, interpolation bases, monomial bases, duality, Vandermonde matrix, Gramian Matrix, matrix decomposition.
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1 Introduction

Interpolation by polynomials or other functions is a rather old method in applied mathematics. This is already indicated by the fact that, apparently, the word "interpolation" itself has been introduced by Wallis as early as

[*]Corresponding author.

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1655 as it is claimed in [6]. The concept of interpolation is the selection of a function $p(x)$ from a given class of functions in such a way that the graph of $y = p(x)$ passes through a finite set of given data points. Polynomial interpolation theory has a number of important uses. Its primary use is to furnish some mathematical tools that are used in developing methods in the areas of approximation theory, numerical integration, and the numerical solution of differential equations. A second use is in developing means for working with functions that are stored in tabular form [2]. If the underlying interpolating basis is the usual family of monomials then the polynomial interpolation leads to solving a linear system involving the Vandermonde matrix. Vandermonde matrix is an $n \times n$ matrix where the first row is the first point x_0 evaluated at each of the n monomials, the second row is the second point x_1 evaluated at each of the n monomials, and so on. In [7] it is proved that the Vandermonde matrix of a large size is badly ill-conditioned unless its knots are more or less equally spaced on or about the circle $C(0,1) = \{x : |x| = 1\}$. Several representations for the interpolating polynomial exist: Lagrange, Newton, orthogonal polynomials etc. Each representation is characterized by some basis functions. Transformations between the basis functions which map a specific representation to another is investigated in [4] . In this paper, the choices of other bases are surveyed. These bases are data-dependent and are categorized into discretely ℓ^2-orthonormal and continuously L^2-orthonormal bases. The first one is defined via a decomposition of Vandermonde matrix while the latter is given by a decomposition of the Gramian matrix corresponding to monomial bases. A discussion of various matrix decomposition (e.g. Cholesky, QR and SVD) provides a variety of different bases with different properties. The rest of the paper is organized as follows. Some basic definitions and theorems are given in Section 2. We describe general data-dependent bases in Section 3. There is a discussion about dual of the new bases and the relation between value matrices of new bases and their dual in Section 4. Section 5 dedicates discretely ℓ^2-orthonormal bases. These bases are defined via SVD and QR decomposition of the Vandermonde matrix. In Section 6, continuously L^2-orthonormal bases are presented. These bases are given by SVD and Cholesky decomposition of the Gramian matrix corresponding to monomial bases. In Section 7, we provides some numerical examples to verify the effectiveness of the new bases. The paper ends with a brief conclusion.

2 Preliminaries

Given function values with $x_i \neq x_j$ for $i \neq j$ and (x_i, f_i), $i = 1, \ldots, n$ where $f_i = f(x_i)$. There exists a unique polynomial $P_n \in \mathbb{P}_n$ of degree less than or equal to n which interpolates these values, i.e.

$$P_n(x_i) = f_i, \quad i = 0, 1, \ldots, n. \tag{1}$$

Several representations of P_n are known. In this paper we consider the monomials among all of them. Consider the monomials

$$m(x) = [m_0(x), m_1(x), \ldots, m_n(x)] = [1, x, x^2, \ldots, x^n],$$

and the representation

$$P_n(x) = a_0 + a_1 x + \cdots + a_n x^n.$$

The coefficients $a = (a_0, a_1, \ldots, a_n)^T$ are determined by the interpolation condition (1), as the solution of the linear system $Aa = f$ with the Vandermonde matrix

$$A = \begin{pmatrix} 1 & x_0 & \cdots & {x_0}^n \\ 1 & x_1 & \cdots & {x_1}^n \\ \vdots & \vdots & \ddots & \vdots \\ 1 & x_n & \cdots & {x_n}^n \end{pmatrix},$$

and the right hand side $f = (f_0, f_1, \ldots, f_n)^T$. With this notation, the interpolating polynomial becomes $P_n(x) = m(x) \cdot a$. Since the Vandermonde matrix is badly ill-conditioned and dense, this tends to instability of solutions in the interpolation system. So we are looking for another bases in order to decrease condition number of coefficient matrices of the interpolation system. For simplicity we work with $[x_0, x_n] = [-1, 1]$, because every interval $[a, b]$ can be easily transferred to $[-1, 1]$. The following theorems and definitions are used in the next sections [1,3,5].

Definition 1. Let (E, μ) be a measure space and $0 < p < \infty$. The vector space $L^p(E)$ consists of measurable functions $f : E \to \mathbb{R}$ such that

$$\|f\|_P = \int_E |f(x)|^p d\mu(x) < \infty.$$

Example 1. Some important examples of L^p-spaces are provided by considering the counting measure on $\mathbb{N}$. In this case, the functions on $\mathbb{N}$ are denoted as sequences and integration is replaced by summation. These L^p-spaces are called the **little L^p-spaces**, and they are denoted by ℓ^p. In other words, if $0 < p < \infty$, then ℓ^p consists of all sequences $X = (x_1, x_1, \ldots)$ such that $\sum_{n=1}^{\infty} |x_n|^p < \infty$, and in this case $\|X\|_p = (\sum_{n=1}^{\infty} |x_n|^p)^{\frac{1}{p}}$.

Remark 1. *$L^2(E)$ is an inner product space under the inner product*

$$\langle f, g\rangle = \int_E f(x)g(x)d\mu(x), \quad \forall\ f, g \in L^2(E).$$

Remark 2. *ℓ^2 is an inner product space under the inner product*

$$\langle X, Y\rangle = \sum_{k=1}^{\infty} x_k y_k,$$

for all $X = (x_1, x_2, \ldots)$ and $Y = (y_1, y_2, \ldots)$ in ℓ^2.

Definition 2. The matrix A is said to be positive definite if for every nonzero vector $x \in \mathbb{R}^n$, $x^T Ax > 0$.

Definition 3. The **Gramian matrix** of a set of vectors $\{u_0, \ldots, u_n\}$ is a symmetric matrix of inner products, whose entries are given by $(G_{ij} = \langle u_i, u_j\rangle)_{0\le i,j\le n}$.

Example 2. Gramian matrix corresponding to monomial bases in the space $L^2[-1, 1]$ is given by

$$\begin{aligned} G_m &= \begin{pmatrix} \langle m_0, m_0\rangle & \langle m_0, m_1\rangle & \cdots & \langle m_0, m_n\rangle \\ \langle m_1, m_0\rangle & \langle m_1, m_1\rangle & \cdots & \langle m_1, m_n\rangle \\ \vdots & \vdots & \ddots & \vdots \\ \langle m_n, m_0\rangle & \langle m_n, m_1\rangle & \cdots & \langle m_n, m_n\rangle \end{pmatrix} \\ &= \begin{pmatrix} \int_{-1}^{1} 1dx & \cdots & \int_{-1}^{1} x^n dx \\ \vdots & \ddots & \vdots \\ \int_{-1}^{1} x^n dx & \cdots & \int_{-1}^{1} x^{2n} dx \end{pmatrix}. \end{aligned}$$

Theorem 1 (**QR Decomposition**)**.** *Let $A \in \mathbb{R}^{m\times n}$, $m \ge n$. There exists an orthogonal matrix Q and an upper triangular matrix R such that $A = QR$.*

Theorem 2 (**Cholesky Decomposition**)**.** *For a symmetric positive definite matrix A, there exists a unique decomposition $A = LL^T$, where L is a lower triangular matrix with positive diagonal entries.*

Theorem 3 (**Singular Value Decomposition (SVD)**)**.** *Let $A \in \mathbb{R}^{m\times n}$ with $rank(A) = r$. Then the eigenvalues of $n \times n$ symmetric matrix $A^T A$ are real and nonnegative. Let these eigenvalues be denoted by σ_i^2, where $\sigma_1^2 \ge \sigma_2^2 \ge \cdots \ge \sigma_n^2$ then $\sigma_1, \ldots, \sigma_n$ are called the singular values of A.*

Every $m \times n$ matrix A can be decomposed into $A = U\Sigma V^T$, where $U_{m\times m}$ and $V_{n\times n}$ are orthogonal and Σ is an $m \times n$ rectangular diagonal matrix with r nonzero elements, which are the nonzero singular values of A. This decomposition is called Singular Value Decomposition or SVD.

Definition 4. [**Dual basis in inner product spaces**] Let $(E, \langle ., .\rangle)$ be an inner product space of dimension n with basis $\Phi = [\phi_1, \ldots, \phi_n]$. Any $e \in E$ can be represented as a linear combination

$$e = \Phi.\alpha = \sum_{i=1}^{n} \alpha_i.\phi_i,$$

for some coefficient vector $\alpha \in \mathbb{R}^n$. The dual space E^* consists of all linear functionals on E, and among these one has the dual functionals λ_j such that $\lambda_j.\phi_i = \delta_{ij}$. With data map $\Lambda = [\lambda_1, \ldots, \lambda_n]$, the duality statement is simply

$$\Lambda^T \Phi = I,$$

where I the $n \times n$ identity matrix. Since Φ is a basis (hence linearly independent), then Λ is also linearly independent, and therefore a basis for the n-dimensional space E^*. By the *Riesz Representation Theorem*, every linear functional on E has a representer in E. That is, for each λ_j, there exists $D_j \in E$ such that $\lambda_j = \langle ., D_j\rangle$. Therefore, we associate $\Lambda = [\lambda_1, \ldots, \lambda_n]$ with the representers $D = [D_1, \ldots, D_n]$. Since Λ is linearly independent in E^* and dual to Φ, then D is linearly independent in E and also dual to Φ. That is, $\langle D_j, \Phi_i\rangle = \delta_{ij}$. Hence, D is a so-called **dual basis**.

3 General data-dependent Bases

Let $X = \{x_0, x_1, \ldots, x_n\}$, any basis $u_0, \ldots, u_n$ of $\mathbb{P}_n$ can be arranged into a row vector

$$U(x) = (u_0(x), \ldots, u_n(x)) \in \mathbb{R}^{n+1}.$$

and it can be expressed by monomial bases $m(x)$ and a coefficient or construction matrix C_U via

$$\begin{aligned} U(x) &= m(x).C_U, \qquad (2) \\ u_k(x) &= \sum_{j=0}^{n} m_j(x) c_{jk}, \qquad 0 \le k \le n. \end{aligned}$$

Here we just note that the full set of possible bases U can be parametrized by arbitrary matrices C_U. Thus we shall express formulas for features of bases U mainly in term of C_U, but there are other parameterizations as well as we shall see.

The *evaluation* operator E based on the set X will map functions f into columns

$$E(f) = f_i = (f(x_0), \ldots, f(x_n))^T \in \mathbb{R}^{n+1},$$

of values on X, and rows of functions into matrices, such that

$$E(m) = E(m_j(x_i))_{0 \le i,j \le n} = A, \tag{3}$$

is the Vandermonde matrix. Similarly, for a general basis U we can form the value matrix

$$V_U = E(U) = (u_j(x_i))_{0 \le i,j \le n} = \begin{pmatrix} u_0(x_0) & u_1(x_0) & \cdots & u_n(x_0) \\ \vdots & \vdots & \ddots & \vdots \\ u_0(x_n) & u_n(x_n) & \cdots & u_n(x_n) \end{pmatrix}. \tag{4}$$

From the identity

$$V_U = E(U) = E(m).C_U = A.C_U, \tag{5}$$

for the monomial bases m, the decomposition (5) is $A = A.I$. Note that (5) also shows that we could as well parametrized the set of bases U via the value matrices V_U, using $C_U = A^{-1}.V_U$ to come back to the parametrization via C_U.

4 Functionals and duality

In this section we discuss dual basis U^* of U. We find value matrix V_{U^*} and also find the relationship between V_{U^*} and V_U. At first the functionals λ_j are defined. Interpolants $P_n(x)$ to values $E(f)$ of some function f can be written as

$$P_n(x) = m(x).a,$$

with a coefficient vector $a \in \mathbb{R}^{n+1}$ satisfying the linear system

$$A.a = E(f).$$

This is well-known, but also follows immediately from,

$$E(P_n(x)) = E(m).a = A.a = E(f),$$

using our notation. For general bases, the interpolant takes the form

$$P_n(x) = m(x).A^{-1}.E(f) = U(x).\underbrace{C_U^{-1}.A^{-1}.E(f)}_{=\Lambda_U(f)} = \sum_{i=0}^{n} u_i(x)\lambda_j(f),$$

with a column vector

$$\Lambda_U(f) = (\lambda_0(f), \ldots, \lambda_n(f))^T = C_U^{-1}.A^{-1}.E(f), \tag{6}$$

of values of linear $functionals$. By using the *Reisz Representation Theorem*, for each λ_j, there exists u_j^* such that

$$\Lambda_U(.) = (\lambda_0, \lambda_1, \ldots, \lambda_n)^T = (\langle ., u_0^*\rangle, \ldots, \langle ., u_n^*\rangle)^T. \tag{7}$$

Therefore, we associate $\Lambda_U = (\lambda_0, \ldots, \lambda_n)^T$ with representers

$$U^* = (u_0^*, \ldots, u_n^*).$$

Hence, U^* is a dual basis of U. Now, we find the value matrix V_{U^*} of the dual basis U^*. For this purpose, we use the following theorem in [5].

Theorem 4. *Let (E, Φ, D) be an inner product space of dimension n over the field $\mathbb{R}$ with basis Φ and dual basis D. Then, $D = \Phi\mathcal{C}$ with*

$$\mathcal{C} = (\Phi^T\Phi)^{-1},$$

a symmetric positive-definite $n \times n$ matrix.

Theorem 5. *The dual basis U^* to a data-dependent basis U satisfies*

$$\begin{aligned} V_{U^*} &= (V_U^T)^{-1}, \\ \langle u_i, u_j^*\rangle &= \delta_{ij}, \qquad 0 \le i, j \le n. \end{aligned}$$

Proof. According to Theorem 4 and considering $\mathcal{C} \in \mathbb{R}^{(n+1)\times(n+1)}$, we have

$$U^* = U\mathcal{C} = U(U^TU)^{-1} = UU^{-1}(U^T)^{-1} = (U^T)^{-1}.$$

By using *evaluation* operator, we get

$$\begin{aligned} E(U^*) &= (E(U))^{-T}, \\ V_{U^*} &= (V_U^T)^{-1}. \end{aligned}$$

Now, by considering (6) and (7), we have

$$\begin{aligned}\langle u_i, u_j^*\rangle &= \lambda_j(u_i)\\ &= e_j^T \Lambda_U(u_i)\\ &= e_j^T \Lambda_U(Ue_i)\\ &= e_j^T \Lambda_U(U)e_i\\ &= e_j^T \underbrace{C_U^{-1}A^{-1}}_{V_U^{-1}} \underbrace{E(U)}_{V_U} e_i\\ &= e_j^T e_i = \delta_{ij}, \qquad 0 \le i,j \le n.\end{aligned}$$

This proves the last assertion and shows that the functionals of Λ_U always are a biorthogonal basis with respect to U. □

5 Discretely ℓ^2-orthonormal bases

Discrete $\ell^2(X)$ inner products form a Gramian Γ_u via

$$\begin{aligned}\Gamma_U &= \big(\langle u_i, u_j\rangle_{\ell^2(X)}\big)_{0\le i,j\le n} = (\sum_{k=0}^{n} u_i(x_k)u_j(x_k))_{0\le i,j\le n}\\ &= V_U^T V_U = C_U^T A^T A C_U. \end{aligned} \tag{8}$$

Theorem 6. *Each data-dependent discretely ℓ^2-orthonormal basis arises from a decomposition*

$$A = Q.B,$$

with $Q = V_U$ orthogonal matrix and $B = C_U^{-1} = Q^T.A$.

Proof. By the formula (8), put $\Gamma_U = C_U^T A^T A C_U = I$, and set $Q = A.C_U$. □

We conclude that every discretely ℓ^2-orthonormal basis is obtained by decomposition of the Vandermonde matrix. Thus, we have two special cases:

- QR Decomposition
  A standard QR decomposition $A = QR$ into an orthogonal matrix Q and an upper triangular matrix R will lead to a basis with $C_U = R^{-1}$, $V_U = Q$. This is nothing but the Gram-Schmidt orthonormalization of the monomial bases.

- SVD
  The second case comes from rescaling an SVD basis. In fact any $SVD(A) = Q\Sigma\widetilde{Q}^T$, with an orthogonal matrix Q and a diagonal matrix Σ having the singular values of A on its diagonal, can be splitted into $A = Q.B$ with $B = \Sigma\widetilde{Q}^T$.

Remark 3. *For discretely ℓ^2-orthonormal bases, the value matrix becomes orthonormal, while the ill-conditioning is completely shifted into the construction matrix.*

6 Continuously L^2-orthonormal bases

The Gramian matrix of the continuously L^2-orthonormal bases in the space $L^2[-1,1]$ is given by

$$\begin{aligned}
G_U &= \big(\langle u_i, u_j\rangle_{L^2[-1,1]}\big)_{0\le i,j\le n} = (\langle \sum_{k=0}^{n} m_k(x)c_{ki}, \sum_{L=0}^{n} m_L(x)c_{Lj}\rangle)_{0\le i,j\le n} \\
&= \begin{pmatrix} \langle \sum_{k=0}^{n} m_k(x)c_{k0}, \sum_{L=0}^{n} m_L(x)c_{L0}\rangle & \cdots & \langle \sum_{k=0}^{n} m_k(x)c_{k0}, \sum_{L=0}^{n} m_L(x)c_{Ln}\rangle \\ \vdots & \ddots & \vdots \\ \langle \sum_{k=0}^{n} m_k(x)c_{kn}, \sum_{L=0}^{n} m_L(x)c_{L0}\rangle & \cdots & \langle \sum_{k=0}^{n} m_k(x)c_{kn}, \sum_{L=0}^{n} m_L(x)c_{Ln}\rangle \end{pmatrix} \\
&= \begin{pmatrix} \sum_{k=0}^{n}\sum_{L=0}^{n} c_{k0}c_{L0}\langle m_k(x), m_L(x)\rangle & \cdots & \sum_{k=0}^{n}\sum_{L=0}^{n} c_{k0}c_{Ln}\langle m_k(x), m_L(x)\rangle \\ \vdots & \ddots & \vdots \\ \sum_{k=0}^{n}\sum_{L=0}^{n} c_{kn}c_{L0}\langle m_k(x), m_L(x)\rangle & \cdots & \sum_{k=0}^{n}\sum_{L=0}^{n} c_{kn}c_{Ln}\langle m_k(x), m_L(x)\rangle \end{pmatrix} \\
&= C_U^T G_m C_U, \qquad (9)
\end{aligned}$$

where

$$G_m = \begin{pmatrix} \langle m_0, m_0\rangle & \cdots & \langle m_0, m_n\rangle \\ \vdots & \cdots & \vdots \\ \langle m_n, m_0\rangle & \cdots & \langle m_n, m_n\rangle \end{pmatrix},$$

is the Gramian matrix corresponding to monomials in the space $L^2[-1,1]$ and

$$C_U = \begin{pmatrix} c_{00} & c_{01} & \cdots & c_{0n} \\ c_{10} & c_{11} & \cdots & c_{1n} \\ \vdots & \vdots & \ddots & \vdots \\ c_{n0} & c_{n1} & \cdots & c_{nn} \end{pmatrix}.$$

Theorem 7. *The Gramian matrix corresponding to monomials in the inner product space $L^2[-1,1]$ is a symmetric positive definite matrix.*

Proof. It is evident that G_m is symmetric. We need to prove that G_m is positive definite. For every $x \in \mathbb{R}^{n+1}$ we have,

$$\begin{aligned} x^T G_m x &= \sum_{i,j} x_i G_{ij} x_j = \sum_{i,j} x_i \langle m_i, m_j \rangle x_j = \sum_j \langle \sum_i x_i m_i, m_j \rangle x_j \\ &= \langle \sum_i x_i m_i, \sum_j m_j x_j \rangle = \langle y, y \rangle = \|y\|_2^2 \geq 0. \end{aligned}$$

Now assume that $y = 0$. Since $m(x)$ is a basis, we have $x_i = 0$, and so $x = 0$. Then G_m is symmetric and positive definite. □

Theorem 8. *Each data-dependent continuously L^2-orthonormal basis arises from a decomposition*

$$G_m = B^T.B, \ \text{with}\ B = C_U^{-1}. \tag{10}$$

Proof. By using (9), we have $G_U = C_U^T.G_m.C_U = I$. Therefore $G_m = (C_U^{-1})^T C_U^{-1}$. □

We conclude that every continuously L^2-orthonormal basis is obtained by decomposition of the Gramian matrix G_m. Thus, we have two special cases:

- Cholesky Decomposition
  By using Cholesky decomposition, we have $G_m = L.L^T$ with a non-singular lower triangular matrix L. Considering (10), we get $V_U = A.C_U = A.(L^{-1})^T$.

- SVD
  The other case is induced by SVD in the form $G_m = Q\Sigma Q^T$ with an orthogonal matrix Q and a diagonal matrix Σ having the eigenvalues of G_m on its diagonal. This SVD basis satisfies

$$B = \sqrt{\Sigma} Q^T, \qquad C_U = Q(\sqrt{\Sigma})^{-1}, \qquad V_U = AQ(\sqrt{\Sigma})^{-1}.$$

Remark 4. *According to (5), the following relation holds between the condition number of value matrices corresponding to new bases and monomials*

$$\operatorname{cond}(V_U) \leq \operatorname{cond}(A).\operatorname{cond}(C_U). \tag{11}$$

Remark 5. *Let*

$$s(x) = \alpha_0 u_0(x) + \alpha_1 u_1(x) + \cdots + \alpha_n u_n(x) = U(x).\alpha,$$

be the interpolant of the new bases, where $\alpha = (\alpha_0, \dots, \alpha_n)^T$ *can be determined by solving the system of linear equations* $V_U.\alpha = f$*. If the values of the new bases are needed at other points like* $\tilde{X} = \{\tilde{x}_0, \dots, \tilde{x}_m\} \in [-1, 1]$*, we use*

$$[s(\tilde{x}_0), \dots, s(\tilde{x}_m)]^T = \tilde{V}.\alpha,$$

where

$$\tilde{V} = (u_j(\tilde{x}_i))_{0\le j\le n,\ 0\le i\le m} = \begin{pmatrix} u_0(\tilde{x}_0) & \cdots & u_n(\tilde{x}_0) \\ \vdots & \ddots & \vdots \\ u_0(\tilde{x}_m) & \cdots & u_n(\tilde{x}_m) \end{pmatrix}.$$

Then equation (2) *leads to*

$$\tilde{V} = \tilde{A}.\ C_U,$$

where

$$\tilde{A} = \begin{pmatrix} 1 & \tilde{x}_0 & \cdots & \tilde{x}_0^n \\ \vdots & \vdots & \ddots & \vdots \\ 1 & \tilde{x}_m & \cdots & \tilde{x}_m^n \end{pmatrix}.$$

7 Numerical results

In this section we present the numerical results of condition numbers of Vandermonde matrix and value matrices of the new bases as well as plots of interpolants and relative errors distribution of the new bases. Codes are written in MATLAB R2018a. We interpolate the function $f(x) = \mathrm{e}^x$ in $[-1, 1]$ by using n uniform set of points

$$x_i = (-1) + \frac{2}{n-1} i, \qquad i = 0, 1, \dots, n-1.$$

Condition numbers are calculated in 2-norm. Table 1 compares condition numbers of value and structure matrices of discretely ℓ^2-orthonormal bases and Vandermonde matrix for different values of n. As we stated in the Remark 3, Table 1 verifies that the condition number of value matrices of discretely ℓ^2-orthonormal bases are equal to one and the ill-conditioning is completely shifted into construction matrix. Table 2 compares condition numbers of value and structure matrices of continuously L^2-orthonormal

Table 1: Comparing condition numbers of value and structures matrices corresponding to discretely ℓ^2-orthonormal bases and Vandermonde matrix.

n	$\kappa(A)$	$\kappa(V_{QR})$	$\kappa(C_{QR})$	$\kappa(V_{SVD})$	$\kappa(C_{SVD})$
5	63.8273	1	63.8273	1	63.8273
10	1.3959×10^4	1	1.3959×10^4	1	1.3959×10^4
15	3.2800×10^6	1	3.2800×10^6	1	3.2800×10^6
20	8.3138×10^8	1	8.3138×10^8	1	8.3138×10^8
25	2.1314×10^{11}	1	2.1314×10^{11}	1	2.1314×10^{11}

Table 2: Comparing condition numbers of value and structures matrices corresponding to continuously L^2-orthonormal bases and Vandermonde matrix.

n	$\kappa(A)$	$\kappa(V_{Chol})$	$\kappa(C_{Chol})$	$\kappa(V_{SVD})$	$\kappa(C_{SVD})$
5	63.8273	4.8630	43.2480	4.8630	43.2480
10	1.3959×10^4	40.2449	3.0730×10^3	40.2449	3.0730×10^3
15	3.2800×10^6	727.4846	2.2989×10^5	727.4846	2.2989×10^5
20	8.3138×10^8	1.6183×10^4	1.7697×10^7	1.6183×10^4	1.7697×10^7
24	7.0535×10^{10}	2.0493×10^5	4.0956×10^8	2.0668×10^5	5.1674×10^8

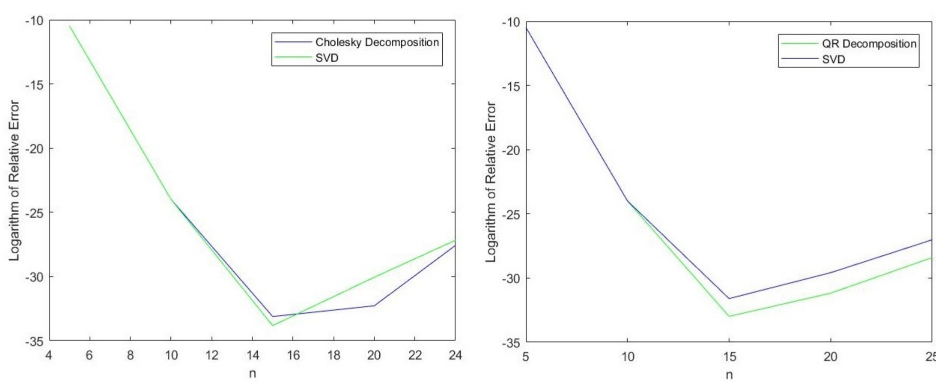

Figure 1: Logarithmic plots of relative errors of interpolation. Discretely ℓ^2-orthonormal bases (right); Continuously L^2-orthonormal bases (left).

bases and Vandermonde matrix for different values of n. It can be noted from Table 2 that the condition number of value matrices by both SVD and Cholesky decompositions increases as long as n increases, but they are still much smaller than the condition numbers of the Vandermonde matrix. The logarithmic plots of relative errors of interpolation are given in Figure 1 by using both discretely ℓ^2-orthonormal and continuously L^2-orthonormal

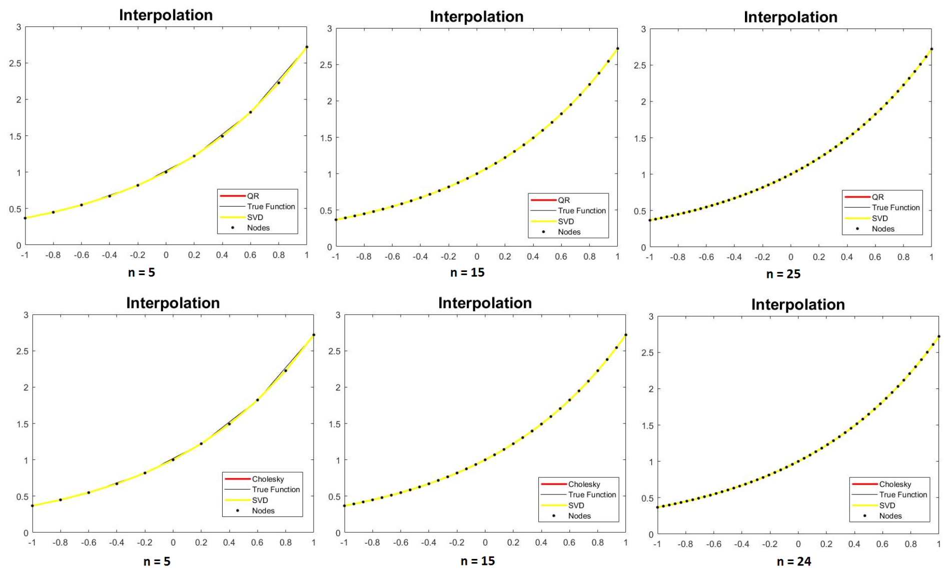

Figure 2: Plots of interpolants. Discretely ℓ^2-orthonormal bases (top); Continuously L^2-orthonormal bases (bottom).

bases. It can be seen that the errors are very close to zero. In Figure 2, we plotted the interpolants of function $f(x) = \mathrm{e}^x$ by using new bases for different values of n. The numerical results are in good agreement with the exact solution.

8 Conclusions

Since it is well-known that the Vandermonde matrix is ill-conditioned, we had an account of the possibilities to construct bases of data-dependent polynomial-based spaces. QR and SVD decomposition of the Vandermonde matrix lead to discretely ℓ^2-orthonormal bases while Cholesky and SVD decomposition of the Gramian matrix corresponding to monomial bases lead to continuously L^2-orthonormal bases. Dual of the new bases and the relation between value matrices of new bases and their dual are also investigated. Numerical results show that the matrices of values of the new bases have smaller condition numbers than the common monomial bases. It can also be pointed out that the new introduced bases are good candidates for interpolation.

References

[1] C.D. Aliprantis and O. Burkinshaw, *Principles of real analysis,* Academic Press, New York, 3nd Edition, 1998.

[2] K.E. Atkinson, *An introduction to numerical analysis,* John Wiley & Sons, New York, 2nd Edition, 1989.

[3] B.N. Datta, *Numerical linear algebra and applications,* SIAM, Philadelphia, 2nd Edition, 2010.

[4] W. Gander, *Change of basis in polynomial interpolation,* Numer. Linear Algebra Appl. **12** (2005) 769–778.

[5] S. Keresy, *Dual basis functions in subspaces of inner product spaces,* Appl. Math. Comput. **219** (2013) 10012–10024.

[6] F. Klein and W.F. Meyer, *Encyklopädie der mathematischen Wissenschaften,* Teubner, Leipzig, 6pp. 1900–1904.

[7] V.Y. Pan, *How bad are Vandermonde matrices?,* SIAM J. Matrix Anal. Appl. **37** (2016) 676–694.