**Abstract.** In this paper, $\mathcal{N}$-fuzzy UP-subalgebras (resp., $\mathcal{N}$-fuzzy UP-filters, $\mathcal{N}$-fuzzy UP-ideals and $\mathcal{N}$-fuzzy strongly UP-ideals) of UP-algebras are introduced and considered their generalizations and characteristic $\mathcal{N}$-fuzzy sets of UP-subalgebras (resp., UP-filters, UP-ideals and strongly UP-ideals). Further, we discuss the relations between $\mathcal{N}$-fuzzy UP-subalgebras (resp., $\mathcal{N}$-fuzzy UP-filters, $\mathcal{N}$-fuzzy UP-ideals and $\mathcal{N}$-fuzzy strongly UP-ideals) and its level subsets.

1. Introduction

A fuzzy subset $f$ of a set $S$ is a function from $S$ to a closed interval $[0, 1]$. The concept of a fuzzy subset of a set was first considered by Zadeh [19] in 1965. The fuzzy set theories developed by Zadeh and others have found many applications in the domain of mathematics and elsewhere.


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MSC(2010): Primary: 03G25; Secondary: 03E72

Keywords: UP-algebra, $\mathcal{N}$-fuzzy UP-subalgebra, $\mathcal{N}$-fuzzy UP-filter, $\mathcal{N}$-fuzzy UP-ideal, $\mathcal{N}$-fuzzy strongly UP-ideal.

This work was financially supported by the University of Phayao.

Received: 30 April 2018, Accepted: 21 June 2018.

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Iampan [4] introduced a new branch of the logical algebra, called a UP-algebra. Later Guntasow et al. [3] studied fuzzy translations of a fuzzy set in UP-algebras. Senapati et al. [13, 14] applied cubic set and interval-valued intuitionistic fuzzy structure in UP-algebras. In this paper, we introduce the notion of $\mathcal{N}$-fuzzy UP-subalgebras (resp., $\mathcal{N}$-fuzzy UP-filters, $\mathcal{N}$-fuzzy UP-ideals and $\mathcal{N}$-fuzzy strongly UP-ideals) of UP-algebras and prove its generalizations and characteristic $\mathcal{N}$-fuzzy sets of UP-subalgebras (resp., UP-filters, UP-ideals and strongly UP-ideals). Further, we discuss the relations between $\mathcal{N}$-fuzzy UP-subalgebras (resp., $\mathcal{N}$-fuzzy UP-filters, $\mathcal{N}$-fuzzy UP-ideals and $\mathcal{N}$-fuzzy strongly UP-ideals) and its level subsets.

2. Basic Results on UP-Algebras

Before we begin our study, we will introduce the definition of a UP-algebra.

**Definition 2.1.** [4] An algebra $A = (A, \cdot, 0)$ of type $(2, 0)$ is called a UP-algebra where $A$ is a nonempty set, $\cdot$ is a binary operation on $A$, and $0$ is a fixed element of $A$ (i.e., a nullary operation) if it satisfies the following axioms: for any $x, y, z \in A$,

- **(UP-1):** $(y \cdot z) \cdot ((x \cdot y) \cdot (x \cdot z)) = 0$,
- **(UP-2):** $0 \cdot x = x$,
- **(UP-3):** $x \cdot 0 = 0$, and
- **(UP-4):** $x \cdot y = 0$ and $y \cdot x = 0$ imply $x = y$.

From [4], we know that the notion of UP-algebras is a generalization of KU-algebras.

**Example 2.2.** [4] Let $X$ be a universal set. Define two binary operations $\cdot$ and $*$ on the power set of $X$ by putting $A \cdot B = B \cap A'$ and $A * B = B \cup A'$ for all $A, B \in \mathcal{P}(X)$. Then $(\mathcal{P}(X), \cdot, 0)$ and $(\mathcal{P}(X), *, X)$ are UP-algebras and we shall call it the power UP-algebra of type 1 and the power UP-algebra of type 2, respectively.
Example 2.3. Let $A = \{0, 1, 2, 3, 4, 5, 6\}$ be a set with a binary operation $\cdot$ defined by the following Cayley table:

\[
\begin{array}{cccccccc}
    & 0 & 1 & 2 & 3 & 4 & 5 & 6 \\
\hline
0 & 0 & 1 & 2 & 3 & 4 & 5 & 6 \\
1 & 0 & 0 & 2 & 3 & 2 & 3 & 6 \\
2 & 0 & 1 & 0 & 3 & 1 & 5 & 3 \\
3 & 0 & 1 & 2 & 0 & 4 & 1 & 2 \\
4 & 0 & 0 & 0 & 3 & 0 & 3 & 3 \\
5 & 0 & 0 & 2 & 0 & 2 & 0 & 2 \\
6 & 0 & 1 & 0 & 0 & 1 & 1 & 0 \\
\end{array}
\]

Then $(A, \cdot, 0)$ is a UP-algebra.

In what follows, let $A$ be a UP-algebra unless otherwise specified. The following proposition is very important for the study of UP-algebras.

Proposition 2.4. [4] In a UP-algebra $A$, the following properties hold:

for any $x, y, z \in A$,

1. $x \cdot x = 0$,
2. $x \cdot y = 0$ and $y \cdot z = 0$ imply $x \cdot z = 0$,
3. $x \cdot y = 0$ implies $(z \cdot x) \cdot (z \cdot y) = 0$,
4. $x \cdot y = 0$ implies $(y \cdot z) \cdot (x \cdot z) = 0$,
5. $x \cdot (y \cdot x) = 0$,
6. $(y \cdot x) \cdot x = 0$ if and only if $x = y \cdot x$, and
7. $x \cdot (y \cdot y) = 0$.

Definition 2.5. [4] A subset $S$ of $A$ is called a UP-subalgebra of $A$ if the constant 0 of $A$ is in $S$, and $(S, \cdot, 0)$ itself forms a UP-algebra.

Jampani [4] proved the useful criteria that a nonempty subset $S$ of a UP-algebra $A = (A, \cdot, 0)$ is a UP-subalgebra of $A$ if and only if $S$ is closed under the $\cdot$ multiplication on $A$.

Definition 2.6. [4] A subset $B$ of $A$ is called a UP-ideal of $A$ if it satisfies the following properties:

1. the constant 0 of $A$ is in $B$, and
2. for any $x, y, z \in A, x \cdot (y \cdot z) \in B$ and $y \in B$ imply $x \cdot z \in B$.

Definition 2.7. [15] A subset $F$ of $A$ is called a UP-filter of $A$ if it satisfies the following properties:

1. the constant 0 of $A$ is in $F$, and
2. for any $x, y \in A, x \cdot y \in F$ and $x \in F$ imply $y \in F$.

Definition 2.8. [3] A subset $C$ of $A$ is called a strongly UP-ideal of $A$ if it satisfies the following properties:
(1) the constant 0 of $A$ is in $C$, and
(2) for any $x, y, z \in A, (z \cdot y) \cdot (z \cdot x) \in C$ and $y \in C$ imply $x \in C$.

Guntasow et al. [3] proved the generalization that the notion of UP-subalgebras is a generalization of UP-filters, the notion of UP-filters is a generalization of UP-ideals, and the notion of UP-ideals is a generalization of strongly UP-ideals. Moreover, they also proved that a UP-algebra $A$ is the only one strongly UP-ideal of itself.

3. $\mathcal{N}$-Fuzzy Sets

In this section, we introduce the notion of $\mathcal{N}$-fuzzy UP-subalgebras (resp., $\mathcal{N}$-fuzzy UP-filters, $\mathcal{N}$-fuzzy UP-ideals and $\mathcal{N}$-fuzzy strongly UP-ideals) of UP-algebras, provide the necessary examples and prove its generalizations.

**Definition 3.1.** [8] A negative fuzzy set (briefly $\mathcal{N}$-fuzzy set) in a nonempty set $X$ (or a negative fuzzy subset (briefly $\mathcal{N}$-fuzzy subset) of $X$) is an arbitrary function from the set $X$ into $[-1, 0]$ where $[-1, 0]$ is the unit segment of the real line. If $A \subseteq X$, the characteristic $\mathcal{N}$-fuzzy set $\chi_A$ of $X$ is a function of $X$ into $\{-1, 0\}$ defined as follows:

$$
\chi_A(x) = \begin{cases} 
-1 & \text{if } x \in A, \\
0 & \text{if } x \notin A.
\end{cases}
$$

Hence, $\chi_A$ is an $\mathcal{N}$-fuzzy set in $X$.

**Lemma 3.2.** Let $A$ be a subset of a nonempty set $X$. Then $\chi_A$ is constant if and only if $A = X$ or $A = \emptyset$.

**Proof.** Assume that $\chi_A$ is constant and $A \neq \emptyset$. Then there exists $a \in A$, that is, $\chi_A(a) = -1$. Thus $\chi_A(x) = -1$ for all $x \in X$, so $x \in A$. Hence, $X = A$.

Conversely, assume that $A = X$ or $A = \emptyset$. Then $\chi_A(x) = \chi_X(x) = -1$ for all $x \in X$ or $\chi_A(x) = \chi_\emptyset(x) = 0$ for all $x \in X$. Hence, $\chi_A$ is constant. \hfill \Box

**Definition 3.3.** Let $f$ be an $\mathcal{N}$-fuzzy set in a nonempty set $X$. The $\mathcal{N}$-fuzzy set $\overline{f}$ defined by $\overline{f}(x) = -1 - f(x)$ for all $x \in X$ is said to be the complement of $f$ in $X$.

**Definition 3.4.** An $\mathcal{N}$-fuzzy set $f$ in $A$ is called an $\mathcal{N}$-fuzzy UP-subalgebra of $A$ if for any $x, y \in A$,

$$f(x \cdot y) \leq \max\{f(x), f(y)\}.$$
Example 3.5. Let $A = \{0, 1, 2, 3, 4\}$ be a set with a binary operation $\cdot$ defined by the following Cayley table:

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Then $(A, \cdot, 0)$ is a UP-algebra. We define a mapping $f : A \to [-1, 0]$ as follows:

$f(0) = -1, f(1) = -0.7, f(2) = -0.5, f(3) = -0.3,$ and $f(4) = -0.1.$

Then $f$ is an $\mathcal{N}$-fuzzy UP-subalgebra of $A$.

Definition 3.6. An $\mathcal{N}$-fuzzy set $f$ in $A$ is called an $\mathcal{N}$-fuzzy UP-filter of $A$ if it satisfies the following properties: for any $x, y \in A$,

1. $f(0) \leq f(x)$, and
2. $f(y) \leq \max\{f(x \cdot y), f(x)\}$.

Example 3.7. Let $A = \{0, 1, 2, 3, 4\}$ be a set with a binary operation $\cdot$ defined by the following Cayley table:

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Then $(A, \cdot, 0)$ is a UP-algebra. We define a mapping $f : A \to [-1, 0]$ as follows:

$f(0) = -1, f(1) = -0.8, f(2) = -0.7, f(3) = -0.9,$ and $f(4) = -0.1.$

Then $f$ is an $\mathcal{N}$-fuzzy UP-filter of $A$.

Definition 3.8. An $\mathcal{N}$-fuzzy set $f$ in $A$ is called an $\mathcal{N}$-fuzzy UP-ideal of $A$ if it satisfies the following properties: for any $x, y, z \in A$,

1. $f(0) \leq f(x)$, and
2. $f(x \cdot z) \leq \max\{f(x \cdot (y \cdot z)), f(y)\}$.


Example 3.9. Let $A = \{0, 1, 2, 3\}$ be a set with a binary operation $\cdot$ defined by the following Cayley table:

\[
\begin{array}{cccc}
\cdot & 0 & 1 & 2 & 3 \\
0 & 0 & 1 & 2 & 3 \\
1 & 0 & 0 & 2 & 3 \\
2 & 0 & 1 & 0 & 3 \\
3 & 0 & 0 & 0 & 0 \\
\end{array}
\]

Then $(A, \cdot, 0)$ is a UP-algebra. We define a mapping $f : A \to [-1, 0]$ as follows:

\[
f(0) = -1, f(1) = -0.7, f(2) = -0.5, \text{ and } f(3) = -0.3.
\]

Then $f$ is an $\mathcal{N}$-fuzzy UP-ideal of $A$.

Definition 3.10. An $\mathcal{N}$-fuzzy set $f$ in $A$ is called an $\mathcal{N}$-fuzzy strongly UP-ideal of $A$ if it satisfies the following properties: for any $x, y, z \in A$,

1. $f(0) \leq f(x)$, and
2. $f(x) \leq \max\{f((z \cdot y) \cdot (z \cdot x)), f(y)\}$.

Example 3.11. Let $A = \{0, 1, 2, 3\}$ be a set with a binary operation $\cdot$ defined by the following Cayley table:

\[
\begin{array}{cccc}
\cdot & 0 & 1 & 2 & 3 \\
0 & 0 & 1 & 2 & 3 \\
1 & 0 & 0 & 1 & 3 \\
2 & 0 & 0 & 0 & 3 \\
3 & 0 & 1 & 1 & 0 \\
\end{array}
\]

Then $(A, \cdot, 0)$ is a UP-algebra. We define a mapping $f : A \to [-1, 0]$ as follows:

\[
f(x) = -0.7 \text{ for all } x \in A.
\]

Then $f$ is an $\mathcal{N}$-fuzzy strongly UP-ideal of $A$.

Theorem 3.12. An $\mathcal{N}$-fuzzy set in $A$ is constant if and only if it is an $\mathcal{N}$-fuzzy strongly UP-ideal of $A$.

Proof. Assume that $f$ is a constant $\mathcal{N}$-fuzzy set in $A$. Then for all $x \in A$, $f(0) = f(x)$ and so $f(0) \leq f(x)$. For all $x, y, z \in A$, $f(x) = f((z \cdot y) \cdot (z \cdot x)) = f(y)$, so $f(x) = \max\{f((z \cdot y) \cdot (z \cdot x)), f(y)\}$. Thus $f(x) \leq \max\{f((z \cdot y) \cdot (z \cdot x)), f(y)\}$. Hence, $f$ is an $\mathcal{N}$-fuzzy strongly UP-ideal of $A$.

Conversely, assume that $f$ is an $\mathcal{N}$-fuzzy strongly UP-ideal of $A$. Then $f(0) \leq f(x)$ and $f(x) \leq \max\{f((z \cdot y) \cdot (z \cdot x)), f(y)\}$ for all...
For any $x \in A$, we choose $z = x$ and $y = 0$. Then
\[ f(x) \leq \max\{f((x \cdot 0) \cdot (x \cdot x)), f(0)\} \]
\[ = \max\{f(0 \cdot (x \cdot x)), f(0)\} \quad \text{((UP-3))} \]
\[ = \max\{f(x \cdot x), f(0)\} \quad \text{((UP-2))} \]
\[ = \max\{f(0), f(0)\} \quad \text{(Proposition 2.4 (1))} \]
\[ = f(0). \]
Thus $f(0) = f(x)$ for all $x \in A$. Hence, $f$ is a constant $N$-fuzzy set in $A$. □

**Theorem 3.13.** Every $N$-fuzzy strongly UP-ideal of $A$ is an $N$-fuzzy UP-ideal.

**Proof.** Assume that $f$ is an $N$-fuzzy strongly UP-ideal of $A$. By Theorem 3.12, we have $f(x) = f(0)$ for all $x \in A$. For any $x, y, z \in A$, we have $f(0) \leq f(x)$ and
\[ f(x \cdot z) \leq f(0) = \max\{f(0), f(0)\} = \max\{f(x \cdot (y \cdot z)), f(y)\}. \]
Hence, $f$ is an $N$-fuzzy UP-ideal of $A$. □

**Example 3.14.** Let $A = \{0, 1, 2, 3, 4\}$ be a set with a binary operation $\cdot$ defined by the following Cayley table:

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Then $(A, \cdot, 0)$ is a UP-algebra. We define a mapping $f : A \to [-1, 0]$ as follows:
\[ f(0) = -1, f(1) = -0.8, f(2) = -0.5, f(3) = -0.2, \text{ and } f(4) = -0.1. \]
Then $f$ is an $N$-fuzzy UP-ideal of $A$. Since $f(2) = -0.5 \geq -0.8 = \max\{f((2 \cdot 1) \cdot (2 \cdot 2)), f(1)\}$, we have $f$ is not an $N$-fuzzy strongly UP-ideal of $A$.

**Theorem 3.15.** Every $N$-fuzzy UP-ideal of $A$ is an $N$-fuzzy UP-filter.

**Proof.** Assume that $f$ is an $N$-fuzzy UP-ideal of $A$. Then for all $x, y \in A$, $f(0) \leq f(x)$ and
\[ f(y) = f(0 \cdot y) \leq f(0 \cdot (x \cdot y)), f(x) \]
\[ = \max\{f(x \cdot y), f(x)\}. \quad \text{((UP-2))} \]
Hence, \( f \) is an \( \mathcal{N} \)-fuzzy UP-filter of \( A \). \( \square \)

**Example 3.16.** Let \( A = \{0, 1, 2, 3, 4\} \) be a set with a binary operation \( \cdot \) defined by the following Cayley table:

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</table>

Then \( (A, \cdot, 0) \) is a UP-algebra. We define a mapping \( f : A \to [-1, 0] \) as follows:

\[
f(0) = -1, \quad f(1) = -0.9, \quad f(2) = -0.7, \quad f(3) = -0.5, \quad \text{and} \quad f(4) = -0.5.
\]

Then \( f \) is an \( \mathcal{N} \)-fuzzy UP-filter of \( A \). Since \( f(3 \cdot 4) = -0.5 > -0.7 = \max\{f(3 \cdot 2 \cdot 4), f(2)\} \), we have \( f \) is not an \( \mathcal{N} \)-fuzzy UP-ideal of \( A \).

**Theorem 3.17.** Every \( \mathcal{N} \)-fuzzy UP-filter of \( A \) is an \( \mathcal{N} \)-fuzzy UP-subalgebra.

**Proof.** Assume that \( f \) is an \( \mathcal{N} \)-fuzzy UP-filter of \( A \). Then for all \( x, y \in A \),

\[
f(x \cdot y) \leq \max\{f(y \cdot (x \cdot y)), f(y)\}
\]

\[
= \max\{f(0), f(y)\} \quad \text{(Proposition 2.4 (5))}
\]

\[
= f(y)
\]

\[
\leq \max\{f(x), f(y)\}.
\]

Hence, \( f \) is an \( \mathcal{N} \)-fuzzy UP-subalgebra of \( A \). \( \square \)

**Example 3.18.** Let \( A = \{0, 1, 2, 3, 4\} \) be a set with a binary operation \( \cdot \) defined by the following Cayley table:

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<th>( \cdot )</th>
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Then \( (A, \cdot, 0) \) is a UP-algebra. We define a mapping \( f : A \to [-1, 0] \) as follows:

\[
f(0) = -1, \quad f(1) = -0.7, \quad f(2) = -0.4, \quad f(3) = -0.2, \quad \text{and} \quad f(4) = -0.1.
\]
Then $f$ is a $N$-fuzzy UP-subalgebra of $A$. Since $f(3) = -0.2 > -0.4 = \max\{f(2 \cdot 3), f(2)\}$, we have $f$ is not an $N$-fuzzy UP-filter of $A$.

By Theorem 3.13, 3.15, and 3.17 and Example 3.14, 3.16, and 3.18, we have that the notion of $N$-fuzzy UP-subalgebras is a generalization of $N$-fuzzy UP-filters, the notion of $N$-fuzzy UP-filters is a generalization of $N$-fuzzy UP-ideals, and the notion of $N$-fuzzy UP-ideals is a generalization of $N$-fuzzy strongly UP-ideals.

4. Characteristic $N$-Fuzzy Sets

In this section, we prove several theorems for characteristic $N$-fuzzy sets of UP-subalgebras (resp., UP-filters, UP-ideals and strongly UP-ideals).

**Theorem 4.1.** A nonempty subset $S$ of $A$ is a UP-subalgebra of $A$ if and only if the characteristic $N$-fuzzy set $\chi_S$ is an $N$-fuzzy UP-subalgebra of $A$.

**Proof.** Assume that $S$ is a UP-subalgebra of $A$. Let $x,y \in A$.

Case 1: $x,y \in S$. Then $\chi_S(x) = -1$ and $\chi_S(y) = -1$. Thus $\max\{\chi_S(x), \chi_S(y)\} = \max\{-1,-1\} = -1$. Since $S$ is a UP-subalgebra of $A$, we have $x \cdot y \in S$ and so $\chi_S(x \cdot y) = -1$. Therefore, $\chi_S(x \cdot y) = -1 \leq -1 = \max\{\chi_S(x), \chi_S(y)\}$.

Case 2: $x \notin S$ or $y \notin S$. Then $\chi_S(x) = 0$ or $\chi_S(y) = 0$. Thus $\max\{\chi_S(x), \chi_S(y)\} = 0$. Therefore, $\chi_S(x \cdot y) \leq 0 = \max\{\chi_S(x), \chi_S(y)\}$.

Hence, $\chi_S$ is an $N$-fuzzy UP-subalgebra of $A$.

Conversely, assume that $\chi_S$ is an $N$-fuzzy UP-subalgebra of $A$. Let $x,y \in S$. Then $\chi_S(x) = -1$ and $\chi_S(y) = -1$. Thus $\chi_S(x \cdot y) \leq \max\{\chi_S(x), \chi_S(y)\} = -1$ and so $\chi_S(x \cdot y) = -1$. Hence, $x \cdot y \in S$, that is, $S$ is a UP-subalgebra of $A$. \qed

**Lemma 4.2.** The constant 0 of $A$ is in a nonempty subset $B$ of $A$ if and only if $\chi_B(0) \leq \chi_B(x)$ for all $x \in A$.

**Proof.** If $0 \in B$, then $\chi_B(0) = -1$. Thus $\chi_B(0) = -1 \leq \chi_B(x)$ for all $x \in A$.

Conversely, assume that $\chi_B(0) \leq \chi_B(x)$ for all $x \in A$. Since $B$ is a nonempty subset of $A$, we have $a \in B$ for some $a \in A$. Then $\chi_B(0) \leq \chi_B(a) = -1$, so $\chi_B(0) = -1$. Hence, $0 \in B$. \qed

**Theorem 4.3.** A nonempty subset $F$ of $A$ is a UP-filter of $A$ if and only if the characteristic $N$-fuzzy set $\chi_F$ is an $N$-fuzzy UP-filter of $A$. 

Lemma 4.2 that \( \chi \) Assume that 

**Proof.** Assume that \( F \) is a UP-filter of \( A \). Since \( 0 \in F \), it follows from Lemma 4.2 that \( \chi_F(0) \leq \chi_F(x) \) for all \( x \in A \). Next, let \( x, y \in A \).

Case 1: \( x, y \in F \). Then \( \chi_F(x) = -1 \) and \( \chi_F(y) = -1 \). Thus \( \chi_F(y) = -1 \leq \chi_F(x \cdot y) = \max\{\chi_F(x \cdot y), -1\} = \max\{\chi_F(x \cdot y), \chi_F(x)\} \).

Case 2: \( x \notin F \) or \( y \notin F \). Then \( \chi_F(x) = 0 \) or \( \chi_F(y) = 0 \).

Case 2.1: If \( x \notin F \), then \( \chi_F(x) = 0 \). Thus \( \chi_F(y) \leq 0 = \max\{\chi_F(x \cdot y), 0\} = \max\{\chi_F(x \cdot y), \chi_F(x)\} \).

Case 2.2: If \( y \notin F \), then \( \chi_F(y) = 0 \). Since \( F \) is a UP-filter of \( A \), we have \( x \cdot y \notin F \) or \( x \notin F \). Then \( \chi_F(x \cdot y) = 0 \) or \( \chi_F(x) = 0 \). Thus \( \chi_F(y) \leq 0 = \max\{\chi_F(x \cdot y), \chi_F(x)\} \).

Hence, \( \chi_F \) is an \( N \)-fuzzy UP-filter of \( A \).

Conversely, assume that \( \chi_F \) is an \( N \)-fuzzy UP-filter of \( A \). Since \( \chi_F(0) \leq \chi_F(x) \) for all \( x \in A \), it follows from Lemma 4.2 that \( 0 \in F \). Next, let \( x, y, z \in A \) be such that \( x \cdot y \in F \) and \( x \in F \). Then \( \chi_F(x \cdot y) = -1 \) and \( \chi_F(x) = -1 \). Thus \( \chi_F(y) \leq \max\{\chi_F(x \cdot y), \chi_F(x)\} = -1 \), so \( \chi_F(y) = -1 \). Therefore, \( y \in F \) and so \( F \) is a UP-filter of \( A \). \( \square \)

**Theorem 4.4.** A nonempty subset \( B \) of \( A \) is a UP-ideal of \( A \) if and only if the characteristic \( N \)-fuzzy set \( \chi_B \) is an \( N \)-fuzzy UP-ideal of \( A \).

**Proof.** Assume that \( B \) is a UP-ideal of \( A \). Since \( 0 \in F \), it follows from Lemma 4.2 that \( \chi_B(0) \leq \chi_B(x) \) for all \( x \in A \). Next, let \( x, y, z \in A \).

Case 1: \( x \cdot (y \cdot z) \in B \) and \( y \in B \). Then \( \chi_B(x \cdot (y \cdot z)) = -1 \) and \( \chi_B(y) = -1 \). Thus \( \max\{\chi_B(x \cdot (y \cdot z)), \chi_B(y)\} = -1 \). Since \( x \cdot (y \cdot z) \in B \) and \( y \in B \), we have \( x \cdot z \in B \) and so \( \chi_B(x \cdot z) = -1 \). Therefore, \( \chi_B(x \cdot z) = -1 \leq -1 = \max\{\chi_B(x \cdot (y \cdot z)), \chi_B(y)\} \).

Case 2: \( x \cdot (y \cdot z) \notin B \) or \( y \notin B \). Then \( \chi_B(x \cdot (y \cdot z)) = 0 \) or \( \chi_B(y) = 0 \). Thus \( \max\{\chi_B(x \cdot (y \cdot z)), \chi_B(y)\} = 0 \). Therefore, \( \chi_B(x \cdot z) \leq 0 = \max\{\chi_B(x \cdot (y \cdot z)), \chi_B(y)\} \).

Hence, \( \chi_B \) is an \( N \)-fuzzy UP-ideal of \( A \).

Conversely, assume that \( \chi_B \) is an \( N \)-fuzzy UP-ideal of \( A \). Since \( \chi_B(0) \leq \chi_B(x) \) for all \( x \in A \), it follows from Lemma 4.2 that \( 0 \in B \). Next, let \( x, y, z \in A \) be such that \( x \cdot (y \cdot z) \in B \) and \( y \in B \). Then \( \chi_B(x \cdot (y \cdot z)) = -1 \) and \( \chi_B(y) = -1 \). Thus \( \chi_B(x \cdot z) = \max\{\chi_B(x \cdot (y \cdot z)), \chi_B(y)\} = -1 \), so \( \chi_B(x \cdot z) = -1 \). Therefore, \( x \cdot z \in B \) and so \( B \) is a UP-ideal of \( A \). \( \square \)

**Theorem 4.5.** A nonempty subset \( C \) of \( A \) is a strongly UP-ideal of \( A \) if and only if the characteristic \( N \)-fuzzy set \( \chi_C \) is an \( N \)-fuzzy strongly UP-ideal of \( A \).

**Proof.** Assume that \( C \) is a strongly UP-ideal of \( A \). Then \( C = A \). Thus \( \chi_C = \chi_A \), so \( \chi_C \) is constant. It follows from Theorem 3.12 that \( \chi_C \) is an \( N \)-fuzzy strongly UP-ideal of \( A \).
Conversely, assume that \( \chi_C \) is an \( \mathcal{N} \)-fuzzy strongly UP-ideal of \( A \). By Theorem 3.12, we have \( \chi_C \) is constant. By Lemma 3.2, we have \( C = A \). Hence, \( C \) is a strongly UP-ideal of \( A \). \( \square \)

5. Level Subsets of an \( \mathcal{N} \)-Fuzzy Set

In this section, we discuss the relationships among \( \mathcal{N} \)-fuzzy UP-subalgebras (resp., \( \mathcal{N} \)-fuzzy UP-filters, \( \mathcal{N} \)-fuzzy UP-ideals and \( \mathcal{N} \)-fuzzy strongly UP-ideals) and and its level subsets.

**Definition 5.1.** Let \( f \) be an \( \mathcal{N} \)-fuzzy set in \( A \). For any \( t \in [-1,0] \), the sets

\[
U(f; t) = \{ x \in A \mid f(x) \geq t \}
\]

and

\[
U^+(f; t) = \{ x \in A \mid f(x) > t \}
\]

are called an upper \( t \)-level subset and an upper \( t \)-strong level subset of \( f \), respectively. The sets

\[
L(f; t) = \{ x \in A \mid f(x) \leq t \}
\]

and

\[
L^-(f; t) = \{ x \in A \mid f(x) < t \}
\]

are called a lower \( t \)-level subset and a lower \( t \)-strong level subset of \( f \), respectively. The set

\[
E(f; t) = \{ x \in A \mid f(x) = t \}
\]

is called an equal \( t \)-level subset of \( f \). Then

\[
U(f; t) = U^+(f; t) \cup E(f; t)
\]

and

\[
L(f; t) = L^-(f; t) \cup E(f; t).
\]

The following lemma is easily verified, therefore the proof is omitted.

**Lemma 5.2.** Let \( f \) be an \( \mathcal{N} \)-fuzzy set in \( A \) and \( t \in [-1,0] \). Then the following statements hold:

1. \( U(f; t) = L(\overline{f}; -1 - t) \),
2. \( L(f; t) = U(\overline{f}; -1 - t) \),
3. \( U^+(f; t) = L^-(\overline{f}; -1 - t) \), and
4. \( L^-(f; t) = U^+(\overline{f}; -1 - t) \).

5.1. Lower \( t \)-Level Subsets of an \( \mathcal{N} \)-Fuzzy Set.

**Theorem 5.3.** An \( \mathcal{N} \)-fuzzy set \( f \) in \( A \) is an \( \mathcal{N} \)-fuzzy UP-subalgebra of \( A \) if and only if for all \( t \in [-1,0] \), \( L(f; t) \) is a UP-subalgebra of \( A \) if \( L(f; t) \) is nonempty.

**Proof.** Assume that \( f \) is an \( \mathcal{N} \)-fuzzy UP-subalgebra of \( A \). Let \( t \in [-1,0] \) be such that \( L(f; t) \neq \emptyset \) and let \( x, y \in L(f; t) \). Then \( f(x) \leq t \) and \( f(y) \leq t \), so \( t \) is an upper bound of \( \{f(x), f(y)\} \). Since \( f \) is an \( \mathcal{N} \)-fuzzy UP-subalgebra of \( A \), we have \( f(x \cdot y) \leq \max\{f(x), f(y)\} \leq t \) and thus \( x \cdot y \in L(f; t) \). Hence, \( L(f; t) \) is a UP-subalgebra of \( A \).

Conversely, assume that for all \( t \in [-1,0] \), \( L(f; t) \) is a UP-subalgebra of \( A \) if \( L(f; t) \) is nonempty. Let \( x, y \in A \). Then \( f(x), f(y) \in [-1,0] \).
Choose \( t = \max\{f(x), f(y)\} \). Then \( f(x) \leq t \) and \( f(y) \leq t \). Thus \( x, y \in L(f; t) \neq \emptyset \). By assumption, we have \( L(f; t) \) is a UP-subalgebra of \( A \) and thus \( x \cdot y \in L(f; t) \). So \( f(x \cdot y) \leq t = \max\{f(x), f(y)\} \). Hence, \( f \) is an \( \mathcal{N} \)-fuzzy UP-subalgebra of \( A \). \( \square \)

**Theorem 5.4.** An \( \mathcal{N} \)-fuzzy set \( f \) in \( A \) is an \( \mathcal{N} \)-fuzzy UP-filter of \( A \) if and only if for all \( t \in [-1, 0] \), \( L(f; t) \) is a UP-filter of \( A \) if \( L(f; t) \) is nonempty.

**Proof.** Assume that \( f \) is an \( \mathcal{N} \)-fuzzy UP-filter of \( A \). Let \( t \in [-1, 0] \) be such that \( L(f; t) \neq \emptyset \) and let \( a \in L(f; t) \). Then \( f(a) \leq t \). Since \( f \) is an \( \mathcal{N} \)-fuzzy UP-filter of \( A \), we have \( f(0) \leq f(a) \leq t \). Thus \( 0 \in L(f; t) \).

Next, let \( x, y \in A \) be such that \( x \cdot y \in L(f; t) \) and \( x \in L(f; t) \). Then \( f(x \cdot y) \leq t \) and \( f(x) \leq t \), so \( t \) is an upper bound of \( \{f(x \cdot y), f(x)\} \). If \( f \) is an \( \mathcal{N} \)-fuzzy UP-filter of \( A \), we have \( f(y) \leq \max\{f(x \cdot y), f(x)\} \). Thus \( y \in L(f; t) \). Hence, \( L(f; t) \) is a UP-filter of \( A \).

Conversely, assume that for all \( t \in [-1, 0] \), \( L(f; t) \) is a UP-filter of \( A \) if \( L(f; t) \) is nonempty. Let \( x \in A \). Then \( f(x) \in [-1, 0] \). Choose \( t = f(x) \). Then \( f(x) \leq t \). Thus \( x \in L(f; t) \neq \emptyset \). By assumption, we have \( L(f; t) \) is a UP-filter of \( A \). Then \( 0 \in L(f; t) \). Thus \( f(0) \leq t = f(x) \). Next, let \( x, y \in A \). Then \( f(x \cdot y), f(x) \in [-1, 0] \). Choose \( t = \max\{f(x \cdot y), f(x)\} \). Then \( f(x \cdot y) \leq t \) and \( f(x) \leq t \). Thus \( x \cdot y, x \in L(f; t) \neq \emptyset \). By assumption, we have \( L(f; t) \) is a UP-filter of \( A \). So \( y \in L(f; t) \).

Hence, \( f(y) \leq t = \max\{f(x \cdot y), f(x)\} \). Therefore, \( f \) is an \( \mathcal{N} \)-fuzzy UP-filter of \( A \). \( \square \)

**Theorem 5.5.** An \( \mathcal{N} \)-fuzzy set \( f \) in \( A \) is an \( \mathcal{N} \)-fuzzy UP-ideal of \( A \) if and only if for all \( t \in [-1, 0] \), \( L(f; t) \) is a UP-ideal of \( A \) if \( L(f; t) \) is nonempty.

**Proof.** Assume that \( f \) is an \( \mathcal{N} \)-fuzzy UP-ideal of \( A \). Let \( t \in [-1, 0] \) be such that \( L(f; t) \neq \emptyset \) and let \( a \in L(f; t) \). Then \( f(a) \leq t \). Since \( f \) is an \( \mathcal{N} \)-fuzzy UP-ideal of \( A \), we have \( f(0) \leq f(a) \leq t \). Thus \( 0 \in L(f; t) \). Next, let \( x, y, z \in A \) be such that \( x \cdot (y \cdot z) \in L(f; t) \) and \( y \in L(f; t) \). Then \( f(x \cdot (y \cdot z)) \leq t \) and \( f(y) \leq t \), so \( t \) is an upper bound of \( \{f(x \cdot (y \cdot z)), f(y)\} \). Since \( f \) is an \( \mathcal{N} \)-fuzzy UP-ideal of \( A \), we have \( f(x \cdot z) \leq \max\{f(x \cdot (y \cdot z)), f(y)\} \). Thus \( x \cdot z \in L(f; t) \). Hence \( L(f; t) \) is a UP-ideal of \( A \).

Conversely, assume that for all \( t \in [-1, 0] \), \( L(f; t) \) is a UP-ideal of \( A \) if \( L(f; t) \) is nonempty. Let \( x \in A \). Then \( f(x) \in [-1, 0] \). Choose \( t = f(x) \). Then \( f(x) \leq t \). Thus \( x \in L(f; t) \neq \emptyset \). By assumption, we have \( L(f; t) \) is a UP-ideal of \( A \). Then \( 0 \in L(f; t) \). Thus \( f(0) \leq t = f(x) \). Next, let \( x, y, z \in A \) be such that \( x \cdot (y \cdot z) \in L(f; t) \) and \( y \in L(f; t) \). Then \( f(x \cdot (y \cdot z)) \leq t \) and \( f(y) \leq t \), so \( t \) is an upper bound of \( \{f(x \cdot (y \cdot z)), f(y)\} \). Since \( f \) is an \( \mathcal{N} \)-fuzzy UP-ideal of \( A \), we have \( f(x \cdot z) \leq \max\{f(x \cdot (y \cdot z)), f(y)\} \). Thus \( x \cdot z \in L(f; t) \). Hence \( L(f; t) \) is a UP-ideal of \( A \).
\( x \cdot (y \cdot z), y \in L(f; t) \neq \emptyset \). By assumption, we have \( L(f; t) \) is a UP-ideal of \( A \). So \( x \cdot z \in L(f; t) \). Hence, \( f(x \cdot z) \leq t = \max\{f(x \cdot (y \cdot z)), f(y)\} \).

Therefore, \( f \) is an \( \mathcal{N} \)-fuzzy UP-ideal of \( A \).

Theorem 5.6. An \( \mathcal{N} \)-fuzzy set \( f \) in \( A \) is an \( \mathcal{N} \)-fuzzy strongly UP-ideal of \( A \) if and only if for all \( t \in [-1, 0] \), \( L(f; t) \) is a strongly UP-ideal of \( A \) if \( L(f; t) \) is nonempty.

Proof. Assume that \( f \) is an \( \mathcal{N} \)-fuzzy strongly UP-ideal of \( A \). Let \( t \in [-1, 0] \) be such that \( L(f; t) \neq \emptyset \) and let \( a \in L(f; t) \). Then \( f(a) \leq t \).

Since \( f \) is an \( \mathcal{N} \)-fuzzy strongly UP-ideal of \( A \), we have \( f(0) \leq f(a) \leq t \). Thus \( 0 \in L(f; t) \). Next, let \( x, y, z \in A \) be such that \((z \cdot y) \cdot (z \cdot x) \in L(f; t) \) and \( y \in L(f; t) \). Then \( f((z \cdot y) \cdot (z \cdot x)) \leq t \) and \( f(y) \leq t \), so \( t \) is an upper bound of \( \{f((z \cdot y) \cdot (z \cdot x)), f(y)\} \). Since \( f \) is an \( \mathcal{N} \)-fuzzy strongly UP-ideal of \( A \), we have \( f(x) \leq \max\{f((z \cdot y) \cdot (z \cdot x)), f(y)\} \leq t \). Thus \( x \in L(f; t) \). Hence, \( L(f; t) \) is a strongly UP-ideal of \( A \).

Conversely, assume that for all \( t \in [-1, 0] \), \( L(f; t) \) is a strongly UP-ideal of \( A \) if \( L(f; t) \) is nonempty. Let \( x \in A \). Then \( f(x) \in [-1, 0] \). Choose \( t = f(x) \). Then \( f(x) \leq t \). Thus \( x \in L(f; t) \neq \emptyset \). By assumption, we have \( L(f; t) \) is a strongly UP-ideal of \( A \). Then \( 0 \in L(f; t) \). Thus \( f(0) \leq t = f(x) \). Next, let \( x, y, z \in A \). Then \( f((z \cdot y) \cdot (z \cdot x)), f(y) \in [-1, 0] \). Choose \( t = \max\{f((z \cdot y) \cdot (z \cdot x)), f(y)\} \). Then \( f((z \cdot y) \cdot (z \cdot x)) \leq t \) and \( f(y) \leq t \). Thus \((z \cdot y) \cdot (z \cdot x), y \in L(f; t) \neq \emptyset \). By assumption, we have \( L(f; t) \) is a strongly UP-ideal of \( A \). So \( x \in L(f; t) \). Hence, \( f(x) \leq t = \max\{f((z \cdot y) \cdot (z \cdot x)), f(y)\} \). Therefore, \( f \) is an \( \mathcal{N} \)-fuzzy strongly UP-ideal of \( A \).

\[ \square \]

5.2. Lower \( t \)-Strong Level Subsets of an \( \mathcal{N} \)-Fuzzy Set.

Theorem 5.7. An \( \mathcal{N} \)-fuzzy set \( f \) in \( A \) is an \( \mathcal{N} \)-fuzzy UP-subalgebra of \( A \) if and only if for all \( t \in [-1, 0] \), \( L^- (f; t) \) is a UP-subalgebra of \( A \) if \( L^- (f; t) \) is nonempty.

Proof. Assume that \( f \) is an \( \mathcal{N} \)-fuzzy UP-subalgebra of \( A \). Let \( t \in [-1, 0] \) be such that \( L^- (f; t) \neq \emptyset \) and let \( x, y \in L^- (f; t) \). Then \( f(x) < t \) and \( f(y) < t \), so \( t \) is an upper bound of \( \{f(x), f(y)\} \). Since \( f \) is an \( \mathcal{N} \)-fuzzy UP-subalgebra of \( A \), we have \( f(x \cdot y) \leq \max\{f(x), f(y)\} < t \), so \( x \cdot y \in L^- (f; t) \). Hence, \( L^- (f; t) \) is a UP-subalgebra of \( A \).

Conversely, assume that for all \( t \in [-1, 0] \), \( L^- (f; t) \) is a UP-subalgebra of \( A \) if \( L^- (f; t) \) is nonempty. Assume that there exist \( x, y \in A \) such that \( f(x \cdot y) > \max\{f(x), f(y)\} \). Then \( f(x \cdot y) \in [-1, 0] \). Choose \( t = f(x \cdot y) \). Then \( f(x) < t \) and \( f(y) < t \). Thus \( x, y \in L^- (f; t) \neq \emptyset \). By assumption, we have \( L^- (f; t) \) is a UP-subalgebra of \( A \) and thus \( x \cdot y \in L^- (f; t) \). So \( f(x \cdot y) < t = f(x \cdot y) \), a contradiction. Hence,
\[ f(x \cdot y) \leq \max\{f(x), f(y)\}, \text{ for all } x, y \in A. \] Therefore, \( f \) is an \( \mathcal{N} \)-fuzzy UP-subalgebra of \( A \).

**Theorem 5.8.** An \( \mathcal{N} \)-fuzzy set \( f \) in \( A \) is an \( \mathcal{N} \)-fuzzy UP-filter of \( A \) if and only if for all \( t \in [-1, 0] \), \( L^-(f; t) \) is a UP-filter of \( A \) if \( L^-(f; t) \) is nonempty.

**Proof.** Assume that \( f \) is an \( \mathcal{N} \)-fuzzy UP-filter of \( A \). Let \( t \in [-1, 0] \) be such that \( L^-(f; t) \neq \emptyset \) and let \( a \in L^-(f; t) \). Then \( f(a) < t \). Since \( f \) is an \( \mathcal{N} \)-fuzzy UP-filter of \( A \), we have \( f(0) \leq f(a) < t \). Thus \( 0 \in L^-(f; t) \). Next, let \( x, y \in A \) be such that \( x \in L^-(f; t) \) and \( x \cdot y \in L^-(f; t) \) for all \( f \). Then \( f(x) < t \) and \( f(x \cdot y) < t \), so \( t \) is an upper bound of \( \{f(x), f(x \cdot y)\} \). Since \( f \) is an \( \mathcal{N} \)-fuzzy UP-filter of \( A \), we have \( f(y) \leq \max\{f(x), f(x \cdot y)\} < t \). Thus \( y \in L^-(f; t) \). Hence, \( L^-(f; t) \) is a UP-filter of \( A \).

Conversely, assume that for all \( t \in [-1, 0] \), \( L^-(f; t) \) is a UP-filter of \( A \) if \( L^-(f; t) \) is nonempty. Assume that there exist \( x \in A \) such that \( f(0) > f(x) \). Then \( f(0) \in [-1, 0] \). Choose \( t = f(0) \). Then \( f(x) < t \). Thus \( x \in L^-(f; t) \neq \emptyset \). By assumption, we have \( L^-(f; t) \) is a UP-filter of \( A \) and thus \( 0 \in L^-(f; t) \). So \( f(0) < t = f(0) \), a contradiction. Hence, \( f(0) \leq \max\{f(x), f(x \cdot y)\} \). Choose \( t = f(0) \). Then \( f(x) < t \) and \( f(x \cdot y) < t \). Thus \( x, x \cdot y \in L^-(f; t) \neq \emptyset \). By assumption, we have \( L^-(f; t) \) is a UP-filter of \( A \) and thus \( y \in L^-(f; t) \). So \( f(y) < t = f(y) \), a contradiction. Hence, \( f(y) \leq \max\{f(x), f(x \cdot y)\} \), for all \( x, y \in A \). Therefore, \( f \) is an \( \mathcal{N} \)-fuzzy UP-filter of \( A \).

**Theorem 5.9.** An \( \mathcal{N} \)-fuzzy set \( f \) in \( A \) is an \( \mathcal{N} \)-fuzzy UP-ideal of \( A \) if and only if for all \( t \in [-1, 0] \), \( L^-(f; t) \) is a UP-ideal of \( A \) if \( L^-(f; t) \) is nonempty.

**Proof.** Assume that \( f \) is an \( \mathcal{N} \)-fuzzy UP-ideal of \( A \). Let \( t \in [-1, 0] \) be such that \( L^-(f; t) \neq \emptyset \) and let \( a \in L^-(f; t) \). Then \( f(a) < t \). Since \( f \) is an \( \mathcal{N} \)-fuzzy UP-ideal of \( A \), we have \( f(0) \leq f(a) < t \). Thus \( 0 \in L^-(f; t) \). Next, let \( x, y, z \in A \) be such that \( x \cdot (y \cdot z) \in L^-(f; t) \) and \( y \in L^-(f; t) \) for all \( f \). Then \( f(x \cdot (y \cdot z)) < t \) and \( f(y) < t \), so \( t \) is an upper bound of \( \{f(x \cdot (y \cdot z)), f(y)\} \). Since \( f \) is an \( \mathcal{N} \)-fuzzy UP-ideal of \( A \), we have \( f(x \cdot z) \leq \max\{f(x \cdot (y \cdot z)), f(y)\} < t \). Thus \( x \cdot z \in L^-(f; t) \). Hence, \( L^-(f; t) \) is a UP-ideal of \( A \).

Conversely, assume that for all \( t \in [-1, 0] \), \( L^-(f; t) \) is a UP-ideal of \( A \) if \( L^-(f; t) \) is nonempty. Assume that there exist \( x \in A \) such that \( f(0) > f(x) \). Then \( f(0) \in [-1, 0] \). Choose \( t = f(0) \). Then \( f(x) < t \). Thus \( x \in L^-(f; t) \neq \emptyset \). By assumption, we have \( L^-(f; t) \) is a UP-ideal of \( A \) and thus \( 0 \in L^-(f; t) \). So \( f(0) < t = f(0) \), a contradiction.
Hence, \( f(0) \leq f(x) \), for all \( x \in A \). Assume that there exist \( x, y, z \in A \) such that \( f(x \cdot z) > \max\{f(x \cdot (y \cdot z)), f(y)\} \). Then \( f(x \cdot z) \in [-1, 0] \). Choose \( t = f(x \cdot z) \). Then \( f(x \cdot (y \cdot z)) < t \) and \( f(y) < t \). Thus \( x \cdot (y \cdot z), y \in L^{-}(f; t) \neq \emptyset \). By assumption, we have \( L^{-}(f; t) \) is a UP-ideal of \( A \) and thus \( x \cdot z \in L^{-}(f; t) \). So \( f(x \cdot z) < t = f(x \cdot z) \), a contradiction. Hence, \( f(x \cdot z) \leq \max\{f(x \cdot (y \cdot z)), f(y)\} \), for all \( x, y \in A \). Therefore, \( f \) is an \( \mathcal{N} \)-fuzzy UP-ideal of \( A \).

**Theorem 5.10.** An \( \mathcal{N} \)-fuzzy set \( f \) in \( A \) is an \( \mathcal{N} \)-fuzzy strongly UP-ideal of \( A \) if and only if for all \( t \in [-1, 0] \), \( L^{-}(f; t) \) is a strongly UP-ideal of \( A \) if \( L^{-}(f; t) \) is nonempty.

**Proof.** Assume that \( f \) is an \( \mathcal{N} \)-fuzzy strongly UP-ideal of \( A \). Let \( t \in [-1, 0] \) be such that \( L^{-}(f; t) \neq \emptyset \) and let \( a \in L^{-}(f; t) \). Then \( f(a) < t \).

Since \( f \) is an \( \mathcal{N} \)-fuzzy strongly UP-ideal of \( A \), we have \( f(0) \leq f(a) < t \). Thus \( 0 \in L^{-}(f; t) \). Next, let \( x, y, z \in A \) be such that \( (z \cdot y) \cdot (z \cdot x) \in L^{-}(f; t) \) and \( y \in L^{-}(f; t) \). Then \( f((z \cdot y) \cdot (z \cdot x)) < t \) and \( f(y) < t \), so \( t \) is an upper bound of \( \{f((z \cdot y) \cdot (z \cdot x)), f(y)\} \). Since \( f \) is an \( \mathcal{N} \)-fuzzy strongly UP-ideal of \( A \), we have \( f(x) \leq \max\{f((z \cdot y) \cdot (z \cdot x)), f(y)\} < t \).

Thus \( x \in L^{-}(f; t) \). Hence, \( L^{-}(f; t) \) is a strongly UP-ideal of \( A \).

Conversely, assume that for all \( t \in [-1, 0] \), \( L^{-}(f; t) \) is a strongly UP-ideal of \( A \), if \( L^{-}(f; t) \) is nonempty. Assume that there exist \( x \in A \) such that \( f(0) > f(x) \). Then \( f(0) \in [-1, 0] \). Choose \( t = f(0) \). Then \( f(x) < t \). Thus \( x \in L^{-}(f; t) \neq \emptyset \). By assumption, we have \( L^{-}(f; t) \) is a strongly UP-ideal of \( A \) and thus \( 0 \in L^{-}(f; t) \). So \( f(0) < t = f(0) \), a contradiction. Hence, \( f(0) \leq f(x) \), for all \( x \in A \). Assume that there exist \( x, y, z \in A \) such that \( f(x) > \max\{f((z \cdot y) \cdot (z \cdot x)), f(y)\} \). Then \( f(x) \in [-1, 0] \). Choose \( t = f(x) \). Then \( f((z \cdot y) \cdot (z \cdot x)) < t \) and \( f(y) < t \).

Thus \( (z \cdot y) \cdot (z \cdot x), y \in L^{-}(f; t) \neq \emptyset \). By assumption, we have \( L^{-}(f; t) \) is a strongly UP-ideal of \( A \) and thus \( x \in L^{-}(f; t) \). So \( f(x) < t = f(x) \), a contradiction. Hence, \( f(x) \leq \max\{f((z \cdot y) \cdot (z \cdot x)), f(y)\} \), for all \( x, y \in A \). Therefore, \( f \) is an \( \mathcal{N} \)-fuzzy strongly UP-ideal of \( A \).

**Theorem 5.12.** The complement \( \overline{f} \) in \( A \) is an \( \mathcal{N} \)-fuzzy UP-subalgebra of \( A \) if and only if for all \( t \in [-1, 0] \), \( U(f; t) \) is a UP-subalgebra of \( A \) if \( U(f; t) \) is nonempty.

5.3. **Upper \( t \)-Level Subsets of an \( \mathcal{N} \)-Fuzzy Set.**

The following lemma is easily proved.

**Lemma 5.11.** Let \( f \) be an \( \mathcal{N} \)-fuzzy set in \( A \). Then the following statements hold: for any \( x, y \in A \),

1. \(-1 - \max\{f(x), f(y)\} = \min\{-1 - f(x), -1 - f(y)\} \), and
2. \(-1 - \min\{f(x), f(y)\} = \max\{-1 - f(x), -1 - f(y)\} \).

**Theorem 5.12.** The complement \( \overline{f} \) in \( A \) is an \( \mathcal{N} \)-fuzzy UP-subalgebra of \( A \) if and only if for all \( t \in [-1, 0] \), \( U(f; t) \) is a UP-subalgebra of \( A \) if \( U(f; t) \) is nonempty.
Proof. Assume that $\mathcal{F}$ is an $\mathcal{N}$-fuzzy UP-subalgebra of $A$. Let $t \in [-1,0]$ be such that $U(f; t) \neq \emptyset$ and let $x, y \in U(f; t)$. Then $f(x) \geq t$ and $f(y) \geq t$, so $t$ is a lower bound of $\{f(x), f(y)\}$. Since $\mathcal{F}$ is an $\mathcal{N}$-fuzzy UP-subalgebra of $A$, we have $\mathcal{F}(x \cdot y) \leq \max\{\mathcal{F}(x), \mathcal{F}(y)\}$.

By Lemma 5.11 (2), we have $-1 - f(x \cdot y) = \max\{-1 - f(x), -1 - f(y)\} = -1 - \min\{f(x), f(y)\}$. Thus $f(x \cdot y) = \min\{f(x), f(y)\} \geq t.$

So $x \cdot y \in U(f; t)$. Hence, $U(f; t)$ is a UP-filter of $A$.

Conversely, assume that for all $t \in [-1,0]$, $U(f; t)$ is a UP-subalgebra of $A$ if $U(f; t)$ is nonempty. Let $x, y \in A$. Then $f(x), f(y) \in [-1,0]$. Choose $t = \min\{f(x), f(y)\}$. Then $f(x) \geq t$ and $f(y) \geq t$. Thus $x, y \in U(f; t) \neq \emptyset$. By assumption, we have $U(f; t)$ is a UP-subalgebra of $A$ and thus $x \cdot y \in U(f; t)$. So $f(x \cdot y) \geq t = \min\{f(x), f(y)\}$. By Lemma 5.11 (2), we have

$$
\mathcal{F}(x \cdot y) = -1 - f(x \cdot y) \\
\leq -1 - \min\{f(x), f(y)\} \\
= \max\{-1 - f(x), -1 - f(y)\} \\
= \max\{\mathcal{F}(x), \mathcal{F}(y)\}.
$$

Therefore, $\mathcal{F}$ is an $\mathcal{N}$-fuzzy UP-subalgebra of $A$. $\square$

Theorem 5.13. The complement $\overline{\mathcal{F}}$ in $A$ is an $\mathcal{N}$-fuzzy UP-filter of $A$ if and only if for all $t \in [-1,0], U(f; t)$ is a UP-filter of $A$ if $U(f; t)$ is nonempty.

Proof. Assume that $\mathcal{F}$ is an $\mathcal{N}$-fuzzy UP-filter of $A$. Let $t \in [-1,0]$ be such that $U(f; t) \neq \emptyset$ and let $a \in U(f; t).$ Then $f(a) \geq t$. Since $\mathcal{F}$ is an $\mathcal{N}$-fuzzy UP-filter of $A$, we have $\mathcal{F}(0) \leq \mathcal{F}(a)$. Thus $-1 - f(0) \leq -1 - f(a)$, so $f(0) \geq f(a) \geq t$. Hence, $0 \in U(f; t)$. Next, let $x, y \in A$ be such that $x \in U(f; t)$ and $x \cdot y \in U(f; t)$. Then $f(x) \geq t$ and $f(x \cdot y) \geq t$, so $t$ is a lower bound of $\{f(x), f(x \cdot y)\}$. Since $\mathcal{F}$ is an $\mathcal{N}$-fuzzy UP-filter of $A$, we have $\mathcal{F}(y) \leq \max\{\mathcal{F}(x), \mathcal{F}(x \cdot y)\}$. By Lemma 5.11 (2), we have $-1 - f(y) \leq \max\{-1 - f(x), -1 - f(x \cdot y)\} = -1 - \min\{f(x), f(x \cdot y)\}$. Thus $f(y) \geq \min\{f(x), f(x \cdot y)\} \geq t$. So $y \in U(f; t)$. Hence, $U(f; t)$ is a UP-filter of $A$.

Conversely, assume that for all $t \in [-1,0], U(f; t)$ is a UP-filter of $A$ if $U(f; t)$ is nonempty. Let $x \in A$. Then $f(x) \in [-1,0]$. Choose $t = f(x)$. Then $f(x) \geq t$. Thus $x \in U(f; t) \neq \emptyset$. By assumption, we have $U(f; t)$ is a UP-filter of $A$ and thus $0 \in U(f; t)$. So $f(0) \geq f(x)$. Hence, $\mathcal{F}(0) = -1 - f(0) \leq -1 - f(x) = \mathcal{F}(a)$. Next, let $x, y \in A$. Then $f(x), f(x \cdot y) \in [-1,0]$. Choose $t = \min\{f(x), f(x \cdot y)\}$. Then $f(x) \geq t$ and $f(x \cdot y) \geq t$. Thus $x, x \cdot y \in U(f; t) \neq \emptyset$. By assumption,
we have $U(f; t)$ is a UP-filter of $A$ and thus $y \in U(f; t)$. Thus $f(y) \geq t = \min\{f(x), f(x \cdot y)\}$. By Lemma 5.11 (2), we have

$$
\overline{f}(y) = -1 - f(y) \\
\leq -1 - \min\{f(x), f(x \cdot y)\} \\
= \max\{-1 - f(x), -1 - f(x \cdot y)\} \\
= \max\{\overline{f}(x), \overline{f}(x \cdot y)\}.
$$

Therefore, $\overline{f}$ is an $\mathcal{N}$-fuzzy UP-filter of $A$.

\[\square\]

**Theorem 5.14.** The complement $\overline{f}$ in $A$ is an $\mathcal{N}$-fuzzy UP-ideal of $A$ if and only if for all $t \in [-1, 0]$, $U(f; t)$ is a UP-ideal of $A$ if $U(f; t)$ is nonempty.

**Proof.** Assume that $\overline{f}$ is an $\mathcal{N}$-fuzzy UP-ideal of $A$. Let $t \in [-1, 0]$ be such that $U(f; t) \neq \emptyset$ and let $a \in U(f; t)$. Then $f(a) \geq t$. Since $\overline{f}$ is an $\mathcal{N}$-fuzzy UP-ideal of $A$, we have $\overline{f}(0) \leq \overline{f}(a)$. Thus $-1 - f(0) \leq -1 - f(a)$, so $f(0) \geq f(a) \geq t$. Hence, $0 \in U(f; t)$. Next, let $x, y, z \in A$ be such that $x \cdot (y \cdot z) \in U(f; t)$ and $y \in U(f; t)$. Then $f(x \cdot (y \cdot z)) \geq t$ and $f(y) \geq t$, so $t$ is a lower bound of $\{f(x \cdot (y \cdot z)), f(y)\}$. Since $\overline{f}$ is an $\mathcal{N}$-fuzzy UP-ideal of $A$, we have $\overline{f}(x \cdot z) \leq \max\{\overline{f}(x \cdot (y \cdot z)), \overline{f}(y)\}$. By Lemma 5.11 (2), we have $-1 - f(x \cdot z) \leq \max\{-1 - f(x \cdot (y \cdot z)), -1 - f(y)\} = -1 - \min\{f(x \cdot (y \cdot z)), f(y)\}$. Thus $f(x \cdot z) \geq \min\{f(x \cdot (y \cdot z)), f(y)\} \geq t$. So $x \cdot z \in U(f; t)$. Hence, $U(f; t)$ is a UP-ideal of $A$.

Conversely, assume that for all $t \in [-1, 0]$, $U(f; t)$ is a UP-ideal of $A$ if $U(f; t)$ is nonempty. Let $x \in A$. Then $f(x) \in [-1, 0]$. Choose $t = f(x)$. Then $f(x) \geq t$ and thus $x \in U(f; t) \neq \emptyset$. By assumption, we have $U(f; t)$ is a UP-ideal of $A$ and thus $0 \in U(f; t)$. So $f(0) \geq f(x)$. Hence, $\overline{f}(0) = -1 - f(0) \leq -1 - f(x) = \overline{f}(a)$. Next, let $x, y, z \in A$. Then $f(x \cdot (y \cdot z)), f(y) \in [-1, 0]$. Choose $t = \min\{f(x \cdot (y \cdot z)), f(y)\}$. Then $f(x \cdot (y \cdot z)) \geq t$ and $f(y) \geq t$. Thus $x \cdot (y \cdot z), y \in U(f; t) \neq \emptyset$. By assumption, we have $U(f; t)$ is a UP-ideal of $A$ and thus $x \cdot z \in U(f; t)$. Thus $f(x \cdot z) \geq t = \min\{f(x \cdot (y \cdot z)), f(y)\}$. By Lemma 5.11 (2), we have

$$
\overline{f}(x \cdot z) = -1 - f(x \cdot z) \\
\leq -1 - \min\{f(x \cdot (y \cdot z)), f(y)\} \\
= \max\{-1 - f(x \cdot (y \cdot z)), -1 - f(y)\} \\
= \max\{\overline{f}(x \cdot (y \cdot z)), \overline{f}(y)\}.
$$

Therefore, $\overline{f}$ is an $\mathcal{N}$-fuzzy UP-ideal of $A$. \[\square\]
**Theorem 5.15.** The complement $\overline{f}$ in $A$ is an $N$-fuzzy strongly UP-ideal of $A$ if and only if for all $t \in [-1, 0]$, $U(f; t)$ is a strongly UP-ideal of $A$ if $U(f; t)$ is nonempty.

**Proof.** Assume that $\overline{f}$ is an $N$-fuzzy strongly UP-ideal of $A$. Let $t \in [-1, 0]$ be such that $U(f; t) \neq \emptyset$ and let $a \in U(f; t)$. Then $f(a) \geq t$. Since $\overline{f}$ is an $N$-fuzzy strongly UP-ideal of $A$, we have $\overline{f}(0) \leq \overline{f}(a)$. Thus $-1 - f(0) \leq -1 - f(a)$, so $f(0) \geq f(a) \geq t$. Hence, $0 \in U(f; t)$. Next, let $x, y, z, A$ be such that $(z \cdot y) \cdot (z \cdot x) \in U(f; t)$ and $y \in U(f; t)$. Then $f((z \cdot y) \cdot (z \cdot x)) \geq t$ and $f(y) \geq t$, so $t$ is a lower bound of $\{f((z \cdot y) \cdot (z \cdot x)), f(y)\}$. Since $\overline{f}$ is an $N$-fuzzy strongly UP-ideal of $A$, we have $\overline{f}(x) \leq \max\{\overline{f}((z \cdot y) \cdot (z \cdot x)), \overline{f}(y)\}$. By Lemma 5.11 (2), we have $-1 - f(x) \leq \max\{-1 - f((z \cdot y) \cdot (z \cdot x)), -1 - f(y)\} = -1 - \min\{f((z \cdot y) \cdot (z \cdot x)), f(y)\}$. Thus $f(x) \geq \min\{f((z \cdot y) \cdot (z \cdot x)), f(y)\} \geq t$. So $x \in U(f; t)$. Hence, $U(f; t)$ is a strongly UP-ideal of $A$.

Conversely, assume that for all $t \in [-1, 0]$, $U(f; t)$ is a strongly UP-ideal of $A$ if $U(f; t)$ is nonempty. Let $x \in A$. Then $f(x) \in [-1, 0]$. Choose $t = f(x)$. Then $f(x) \geq t$. Thus $x \in U(f; t) \neq \emptyset$. By assumption, we have $U(f; t)$ is a strongly UP-ideal of $A$ and thus $0 \in U(f; t)$. So $f(0) \geq f(x)$. Hence, $\overline{f}(0) = -1 - f(0) \leq -1 - f(x) = \overline{f}(a)$. Next, Let $x, y, z \in A$. Then $f((z \cdot y) \cdot (z \cdot x)), f(y) \in [-1, 0]$. Choose $t = \min\{f((z \cdot y) \cdot (z \cdot x)), f(y)\}$. Then $f((z \cdot y) \cdot (z \cdot x)) \geq t$ and $f(y) \geq t$. Thus $(z \cdot y) \cdot (z \cdot x), y \in U(f; t) \neq \emptyset$. By assumption, we have $U(f; t)$ is a strongly UP-ideal of $A$ and thus $y \in U(f; t)$. Thus $f(y) \geq t = \min\{f((z \cdot y) \cdot (z \cdot x)), f(y)\}$. By Lemma 5.11 (2), we have

\[
\overline{f}(x) = -1 - f(x)
\leq -1 - \min\{f((z \cdot y) \cdot (z \cdot x)), f(y)\}
= \max\{-1 - f((z \cdot y) \cdot (z \cdot x)), -1 - f(y)\}
= \max\{\overline{f}((z \cdot y) \cdot (z \cdot x)), \overline{f}(y)\}.
\]

Therefore, $\overline{f}$ is an $N$-fuzzy strongly UP-ideal of $A$. 

\[\square\]

5.4. **Upper t-Strong Level Subsets of an N-Fuzzy Set.**

**Theorem 5.16.** The complement $\overline{f}$ in $A$ is an $N$-fuzzy UP-subalgebra of $A$ if and only if for all $t \in [-1, 0]$, $U^+(f; t)$ is a UP-subalgebra of $A$ if $U^+(f; t)$ is nonempty.

**Proof.** Assume that $\overline{f}$ is an $N$-fuzzy UP-subalgebra of $A$. Let $t \in [-1, 0]$ be such that $U^+(f; t) \neq \emptyset$ and let $x, y \in U^+(f; t)$. Then $f(x) > t$ and $f(y) > t$, so $t$ is a lower bound of $\{f(x), f(y)\}$. Since $\overline{f}$ is an $N$-fuzzy UP-subalgebra of $A$, we have $\overline{f}(x \cdot y) \leq \max\{\overline{f}(x), \overline{f}(y)\}$. By
Lemma 5.11 (2), we have $-1 - f(x \cdot y) \leq \max\{-1 - f(x), -1 - f(y)\} = -1 - \min\{f(x), f(y)\}$. Thus $f(x \cdot y) \geq \min\{f(x), f(y)\} > t$. So $x \cdot y \in U^+(f; t)$. Hence, $U^+(f; t)$ is a UP-subalgebra of $A$.

Conversely, assume that for all $t \in [-1, 0]$, $U^+(f; t)$ is a UP-subalgebra of $A$ if $U^+(f; t)$ is nonempty. Assume that there exist $x, y \in A$ such that $\overline{f}(x \cdot y) > \max\{\overline{f}(x), \overline{f}(y)\}$. By Lemma 5.11 (2), we have

$$-1 - f(x \cdot y) > \max\{-1 - f(x), -1 - f(y)\} = -1 - \min\{f(x), f(y)\}.$$ 

Thus $f(x \cdot y) < \min\{f(x), f(y)\}$. Now $f(x \cdot y) \in [-1, 0]$, we choose $t = f(x \cdot y)$. Then $f(x) > t$ and $f(y) > t$. Thus $x, y \in U^+(f; t) \neq \emptyset$. By assumption, we have $U^+(f; t)$ is a UP-subalgebra of $A$ and thus $x \cdot y \in U^+(f; t)$. So $f(x \cdot y) > t = f(x \cdot y)$, a contradiction. Hence, $\overline{f}(x \cdot y) \leq \max\{\overline{f}(x), \overline{f}(y)\}$, for all $x, y \in A$. Therefore, $\overline{f}$ is an $\mathcal{N}$-fuzzy UP-subalgebra of $A$. 

**Theorem 5.17.** The complement $\overline{f}$ in $A$ is an $\mathcal{N}$-fuzzy UP-filter of $A$ if and only if for all $t \in [-1, 0]$, $U^+(f; t)$ is a UP-filter of $A$ if $U^+(f; t)$ is nonempty.

**Proof.** Assume that $\overline{f}$ is an $\mathcal{N}$-fuzzy UP-filter of $A$. Let $t \in [-1, 0]$ be such that $U^+(f; t) \neq \emptyset$ and let $a \in U^+(f; t)$. Then $f(a) > t$. Since $\overline{f}$ is an $\mathcal{N}$-fuzzy UP-filter of $A$, we have $\overline{f}(0) \leq \overline{f}(a)$. Thus $-1 - f(0) \leq -1 - f(a)$. So $f(0) \geq f(a) > t$. Hence, $0 \in U^+(f; t)$.

Next, let $x, y \in A$ be such that $x \in U^+(f; t)$ and $x \cdot y \in U^+(f; t)$. Then $f(x) > t$ and $f(x \cdot y) > t$, so $t$ is a lower bound of $\{f(x), f(x \cdot y)\}$. Since $\overline{f}$ is an $\mathcal{N}$-fuzzy UP-filter of $A$, we have $\overline{f}(y) \leq \max\{\overline{f}(x), \overline{f}(x \cdot y)\}$. By Lemma 5.11 (2), we have $-1 - f(y) \leq \max\{-1 - f(x), -1 - f(x \cdot y)\} = -1 - \min\{f(x), f(x \cdot y)\}$. So $f(y) \geq \min\{f(x), f(x \cdot y)\} > t$ and thus $y \in U^+(f; t)$. Hence, $U^+(f; t)$ is a UP-filter of $A$.

Conversely, assume that for all $t \in [-1, 0]$, $U^+(f; t)$ is a UP-filter of $A$ if $U^+(f; t)$ is nonempty. Assume that there exist $x \in A$ such that $\overline{f}(0) > \overline{f}(x)$. Then $-1 - f(0) > -1 - f(x)$. Thus $f(0) < f(x)$. Now $f(0) \in [-1, 0]$, we choose $t = f(0)$. Then $f(x) > t$. Thus $x \in U^+(f; t) \neq \emptyset$. By assumption, we have $U^+(f; t)$ is a UP-filter of $A$ and thus $0 \in U^+(f; t)$. So $f(0) > t = f(0)$, a contradiction. Hence, $\overline{f}(0) \leq \overline{f}(x)$, for all $x \in A$. Assume that there exist $x, y \in A$ such that $\overline{f}(y) > \max\{\overline{f}(x), \overline{f}(x \cdot y)\}$. By Lemma 5.11 (2), we have

$$-1 - f(y) > \max\{-1 - f(x), -1 - f(x \cdot y)\} = -1 - \min\{f(x), f(x \cdot y)\}.$$ 

Thus $f(y) < \min\{f(x), f(x \cdot y)\}$. Now $f(y) \in [-1, 0]$, we choose $t = f(y)$. Then $f(x) > t$ and $f(x \cdot y) > t$. Then $x, x \cdot y \in U^+(f; t) \neq \emptyset$. By assumption, we have $U^+(f; t)$ is a UP-filter of $A$ and thus $y \in U^+(f; t)$.
So \( f(y) > t = f(y) \), a contradiction. Hence, \( \overline{f}(y) \leq \max\{\overline{f}(x), \overline{f}(x \cdot y)\} \), for all \( x, y \in A \). Therefore, \( \overline{f} \) is an \( N \)-fuzzy UP-filter of \( A \).

\[ \square \]

**Theorem 5.18.** The complement \( \overline{f} \) in \( A \) is an \( N \)-fuzzy UP-ideal of \( A \) if and only if for all \( t \in [-1, 0] \), \( U^+(f; t) \) is a UP-ideal of \( A \) if \( U^+(f; t) \) is nonempty.

**Proof.** Assume that \( \overline{f} \) is an \( N \)-fuzzy UP-ideal of \( A \). Let \( t \in [-1, 0] \) be such that \( U^+(f; t) \neq \emptyset \) and let \( a \in U^+(f; t) \). Then \( f(a) > t \). Since \( \overline{f} \) is an \( N \)-fuzzy UP-ideal of \( A \), we have \( \overline{f}(0) \leq \overline{f}(a) \). Thus \(-1 - f(0) \leq -1 - f(a) \). So \( f(0) \geq f(a) > t \). Hence, \( 0 \in U^+(f; t) \).

Next, let \( x, y, z \in A \) be such that \( x \cdot (y \cdot z) \in U^+(f; t) \) and \( y \in U^+(f; t) \).

Then \( f(x \cdot (y \cdot z)) > t \) and \( f(y) > t \), so \( t \) is a lower bound of \( \{f(x \cdot (y \cdot z)), f(y)\} \). Since \( \overline{f} \) is an \( N \)-fuzzy UP-ideal of \( A \), we have \( \overline{f}(x \cdot z) \leq \max\{\overline{f}(x \cdot (y \cdot z)), \overline{f}(y)\} \). By Lemma 5.11 (2), we have \(-1 - f(x \cdot z) \leq \max\{-1 - f(x \cdot (y \cdot z)), -1 - f(y)\} = -1 - \min\{f(x \cdot (y \cdot z)), f(y)\} \).

Thus \( f(x \cdot z) \geq \min\{f(x \cdot (y \cdot z)), f(y)\} > t \) and thus \( x \cdot z \in U^+(f; t) \).

Hence, \( U(f; t) \) is a UP-ideal of \( A \).

Conversely, assume that for all \( t \in [-1, 0] \), \( U^+(f; t) \) is a UP-filter of \( A \) if \( U^+(f; t) \) is nonempty. Assume that there exist \( x \in A \) such that \( \overline{f}(0) > \overline{f}(x) \). Then \(-1 - f(0) > -1 - f(x) \). Thus \( f(0) < f(x) \).

Now \( f(0) \in [-1, 0] \), we choose \( t = f(0) \). Then \( f(x) > t \). Thus \( x \in U^+(f; t) \neq \emptyset \).

By assumption, we have \( U^+(f; t) \) is a UP-filter of \( A \) and thus \( 0 \in U^+(f; t) \). So \( f(0) > t = f(0) \), a contradiction. Hence, \( \overline{f}(0) \leq \overline{f}(x) \), for all \( x \in A \). Assume that there exist \( x, y, z \in A \) such that \( \overline{f}(x \cdot z) > \max\{\overline{f}(x \cdot (y \cdot z)), \overline{f}(y)\} \).

By Lemma 5.11 (2), we have \(-1 - f(x \cdot z) > \max\{-1 - f(x \cdot (y \cdot z)), -1 - f(y)\} = -1 - \min\{f(x \cdot (y \cdot z)), f(y)\} \).

Thus \( f(x \cdot z) \leq \min\{f(x \cdot (y \cdot z)), f(y)\} \).

Thus \( x \cdot (y \cdot z) \in U^+(f; t) \). So \( f(x \cdot z) > t = f(x \cdot z) \), a contradiction. Hence, \( \overline{f}(x \cdot z) \leq \max\{\overline{f}(x \cdot (y \cdot z)), \overline{f}(y)\} \), for all \( x, y, z \in A \). Therefore, \( \overline{f} \) is an \( N \)-fuzzy UP-ideal of \( A \).

\[ \square \]

**Theorem 5.19.** The complement \( \overline{f} \) in \( A \) is an \( N \)-fuzzy strongly UP-ideal of \( A \) if and only if for all \( t \in [-1, 0] \), \( U^+(f; t) \) is a strongly UP-ideal of \( A \) if \( U^+(f; t) \) is nonempty.

**Proof.** Assume that \( \overline{f} \) is an \( N \)-fuzzy strongly UP-ideal of \( A \). Let \( t \in [-1, 0] \) be such that \( U^+(f; t) \neq \emptyset \) and let \( a \in U^+(f; t) \). Then \( f(a) > t \).

Since \( \overline{f} \) is an \( N \)-fuzzy strongly UP-ideal of \( A \), we have \( \overline{f}(0) \leq \overline{f}(a) \).

Thus \(-1 - f(0) \leq -1 - f(a) \). So \( f(0) \geq f(a) > t \). Hence, \( 0 \in U^+(f; t) \).

Next, let \( x, y, z \in A \) be such that \((z \cdot y) \cdot (z \cdot x) \in U^+(f; t) \) and \( y \in
$U^+(f; t)$. Then $f((z \cdot y) \cdot (z \cdot x)) > t$ and $f(y) > t$, so $t$ is a lower bound of $\{f((z \cdot y) \cdot (z \cdot x)), f(y)\}$. Since $\overline{f}$ is an $\mathcal{N}$-fuzzy strongly UP-ideal of $A$, we have $\overline{f}(x) \leq \max\{\overline{f}(z \cdot y) \cdot (z \cdot x), \overline{f}(y)\}$. By Lemma 5.11 (2), we have $-1 - f(x) \leq \max\{-1 - f((z \cdot y) \cdot (z \cdot x)), -1 - f(y)\} = -1 - \min\{f((z \cdot y) \cdot (z \cdot x)), f(y)\}$. So $f(x) \geq \min\{f((z \cdot y) \cdot (z \cdot x)), f(y)\} > t$ and thus $x \in U^+(f; t)$. Hence, $U(f; t)$ is a strongly UP-ideal of $A$.

Conversely, assume that for all $t \in [-1, 0]$, $U^+(f; t)$ is a strongly UP-ideal of $A$ if $U^+(f; t)$ is nonempty. Assume that there exist $x, y, z \in A$ such that $\overline{f}(0) > \overline{f}(x)$. Then $-1 - f(0) > -1 - f(x)$. Thus $f(0) < f(x)$. Now $f(0) \in [-1, 0]$, we choose $t = f(0)$. Then $f(x) > t$. Thus $x \in U^+(f; t) \neq \emptyset$. By assumption, we have $U^+(f; t)$ is a strongly UP-ideal of $A$ and thus $0 \in U^+(f; t)$. So $f(0) > t = f(0)$, a contradiction. Hence, $\overline{f}(0) \leq \overline{f}(x)$, for all $x \in A$. Assume that there exist $x, y, z \in A$ such that $\overline{f}(x) > \max\{\overline{f}(z \cdot y) \cdot (z \cdot x), \overline{f}(y)\}$. By Lemma 5.11 (2), we have $-1 - f(x) > \max\{-1 - f((z \cdot y) \cdot (z \cdot x)), -1 - f(y)\} = -1 - \min\{f((z \cdot y) \cdot (z \cdot x)), f(y)\}$.

Now $f(x) \in [-1, 0]$, we choose $t = f(x)$. Then $f((z \cdot y) \cdot (z \cdot x)) > t$ and $f(y) > t$. Then $(z \cdot y) \cdot (z \cdot x), y \in U^+(f; t) \neq \emptyset$. By assumption, we have $U^+(f; t)$ is a strongly UP-ideal of $A$ and thus $x \in U^+(f; t)$. So $f(x) > t = f(x)$, a contradiction. Hence, $\overline{f}(x) \leq \max\{\overline{f}(z \cdot y) \cdot (z \cdot x), \overline{f}(y)\}$, for all $x, y, z \in A$. Therefore, $\overline{f}$ is an $\mathcal{N}$-fuzzy strongly UP-ideal of $A$.

\[ \square \]

5.5. Equal $t$-Level Subsets of an $\mathcal{N}$-Fuzzy Set.

**Corollary 5.20.** If $f$ is an $\mathcal{N}$-fuzzy UP-subalgebra of $A$, then for all $t \in [-1, 0]$, $E(f; t)$ is a UP-subalgebra of $A$ where $E(f; t)$ is nonempty and $L^-(f; t)$ is empty.

**Proof.** Assume that $f$ is an $\mathcal{N}$-fuzzy UP-subalgebra of $A$. Let $t \in [-1, 0]$ be such that $E(f; t) \neq \emptyset$ and $L^-(f; t) = \emptyset$. Since $E(f; t) \subseteq L(f; t)$, we have $L(f; t) \neq \emptyset$. By Theorem 5.3, we have $E(f; t) = \emptyset \cup E(f; t) = L^-(f; t) \cup E(f; t) = L(f; t)$ is a UP-subalgebra of $A$. \[ \square \]

**Example 5.21.** Let $A = \{0, 1, 2, 3\}$ be a set with a binary operation $\cdot$ defined by the following Cayley table:

\[
\begin{array}{c|cccc}
  & 0 & 1 & 2 & 3 \\
\hline
  0 & 0 & 1 & 2 & 3 \\
  1 & 0 & 0 & 1 & 3 \\
  2 & 0 & 0 & 0 & 3 \\
  3 & 0 & 1 & 1 & 0 \\
\end{array}
\]

Then $(A, \cdot, 0)$ is a UP-algebra. We define a mapping $f : A \rightarrow [-1, 0]$ as follows:
that Proof. Let $L^1$ have as follows:

If $t \neq -1$, then $L^1(f; t) \neq \emptyset$. If $t = -1$, then $L^1(f; t) = \emptyset$ and $E(f; t) = \{0\}$. Clearly, $E(f; t)$ is a UP-subalgebra of $A$. Since $f(3 \cdot 2) = -0.1 > -0.2 = \max\{f(3), f(2)\}$, we have $f$ is not an $\mathcal{N}$-fuzzy UP-subalgebra of $A$.

**Corollary 5.22.** If $f$ is an $\mathcal{N}$-fuzzy UP-filter of $A$, then for all $t \in [-1, 0]$, $E(f; t)$ is a UP-filter of $A$ where $E(f; t)$ is nonempty and $L^1(f; t)$ is empty.

**Proof.** Assume that $f$ is an $\mathcal{N}$-fuzzy UP-filter of $A$. Let $t \in [-1, 0]$ be such that $E(f; t) \neq \emptyset$ and $L^1(f; t) = \emptyset$. Since $E(f; t) \subseteq L(f; t)$, we have $L(f; t) \neq \emptyset$. By Theorem 5.4, we have $E(f; t) = \emptyset \cup E(f; t) = L^1(f; t) \cup E(f; t) = L(f; t)$ is a UP-filter of $A$. \qed

**Example 5.23.** Let $A = \{0, 1, 2, 3\}$ be a set with a binary operation $\cdot$ defined by the following Cayley table:

<table>
<thead>
<tr>
<th></th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>0</td>
<td>2</td>
<td>3</td>
</tr>
<tr>
<td>1</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>3</td>
</tr>
<tr>
<td>2</td>
<td>0</td>
<td>0</td>
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<td>3</td>
</tr>
<tr>
<td>3</td>
<td>0</td>
<td>1</td>
<td>2</td>
<td>0</td>
</tr>
</tbody>
</table>

Then $(A, \cdot, 0)$ is a UP-algebra. We define a mapping $f : A \to [-1, 0]$ as follows:

$f(0) = -1, f(1) = -0.9, f(2) = -0.7$, and $f(3) = -0.5$.

If $t \neq -1$, then $L^1(f; t) \neq \emptyset$. If $t = -1$, then $L^1(f; t) = \emptyset$ and $E(f; t) = \{0\}$. Clearly, $E(f; t)$ is a UP-filter of $A$. Since $f(2) = -0.7 > -0.9 = \max\{f(1 \cdot 2), f(1)\}$, we have $f$ is not an $\mathcal{N}$-fuzzy UP-filter of $A$.

**Corollary 5.24.** If $f$ is an $\mathcal{N}$-fuzzy UP-ideal of $A$, then for all $t \in [-1, 0]$, $E(f; t)$ is a UP-ideal of $A$ where $E(f; t)$ is nonempty and $L^1(f; t)$ is empty.

**Proof.** Let $f$ is an $\mathcal{N}$-fuzzy UP-ideal of $A$. Let $t \in [-1, 0]$ be such that $E(f; t) \neq \emptyset$ and $L^1(f; t) = \emptyset$. Since $E(f; t) \subseteq L(f; t)$, we have $L(f; t) \neq \emptyset$. By Theorem 5.5, we have $E(f; t) = \emptyset \cup E(f; t) = L^1(f; t) \cup E(f; t) = L(f; t)$ is a UP-ideal of $A$. \qed
Example 5.25. Let $A = \{0, 1, 2, 3\}$ be a set with a binary operation $\cdot$ defined by the following Cayley table:

<table>
<thead>
<tr>
<th></th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>1</td>
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<tr>
<td>1</td>
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<td>2</td>
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<tr>
<td>3</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>

Then $(A, \cdot, 0)$ is a UP-algebra. We define a mapping $f : A \to [-1, 0]$ as follows:

$f(0) = -1, f(1) = -0.8, f(2) = -0.5, \text{ and } f(3) = -0.2$.

If $t \neq -1$, then $L^-(f; t) \neq \emptyset$. If $t = -1$, then $L^-(f; t) = \emptyset$ and $E(f; t) = \{0\}$. Clearly, $E(f; t)$ is a UP-ideal of $A$. Since $f(1 \cdot 3) = -0.2 > -0.5 = \max\{f(1 \cdot (2 \cdot 3)), f(2)\}$, we have $f$ is not an $\mathcal{N}$-fuzzy UP-ideal of $A$.

Corollary 5.26. An $\mathcal{N}$-fuzzy set $f$ in $A$ is an $\mathcal{N}$-fuzzy strongly UP-ideal of $A$ if and only if for all $t \in [-1, 0]$, $E(f; t)$ is a strongly UP-ideal of $A$ where $E(f; t)$ is nonempty.

Proof. It is straightforward by Theorem 3.12 and 5.6, and $A$ is the only one strongly UP-ideal of itself. \hfill \Box

6. Conclusions

In the present paper, we have introduced the notion of $\mathcal{N}$-fuzzy UP-subalgebras (resp., $\mathcal{N}$-fuzzy UP-filters, $\mathcal{N}$-fuzzy UP-ideals and $\mathcal{N}$-fuzzy strongly UP-ideals) of UP-algebras and proved its generalizations and characteristic $\mathcal{N}$-fuzzy sets of UP-subalgebras (resp., UP-filters, UP-ideals and strongly UP-ideals). We think this work would enhance the scope for further study in UP-algebras and related algebraic systems. It is our hope that this work would serve as a foundation for the further study in a new concept of UP-algebras.

Acknowledgments

The author wish to express their sincere thanks to the referees for the valuable suggestions which lead to an improvement of this paper.

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