Journal of Mathematical Modeling
Vol. 6, No. 1, 2018, pp. 77-90

Solution of Troesche's problem by double exponential Sinc collocation method

Mohammad Nabati[†*], Soudabeh Nikmanesh[†] and Mehdi Jalalvand[‡]

[†] *Department of Basic Sciences, Abadan Faculty of Petroleum Engineering, Petroleum University of Technology, Abadabn, Iran*

[‡] *Department of Mathematics, Faculty of Mathematical Sciences and Computer, Shahid Chamran University of Ahvaz, Ahvaz, Iran*

emails: nabati@put.ac.ir, soudabeh.nikmanesh@put.ac.ir, m.jalalvand@scu.ac.ir

Abstract. In this investigation, the Sinc collocation method based on double exponential transformation is developed to solve the Troesche's problem. Properties of this method are utilized to reduce the system of strongly nonlinear two point boundary value problem to same nonlinear algebraic equations. Combining double exponential transformation through Sinc collocation method causes the remarkable results. To illustrate the high accuracy of the method, the obtained solutions are compared with results of other methods in open literature. The demonstrated results show the simplicity and considerably accuracy of this method in comparison with other methods.

Keywords: Sinc function, collocation method, double exponential transformation, nonlinear Troesche's problem.
AMS Subject Classification: 65L10, 65L60, 65H10, 41A30.

1 Introduction

The Troesche's problem, defined by

$$\begin{cases} y''(x) = \eta \sinh(\eta\, y(x)), & 0 < x < 1, \\ y(0) = 0, \qquad y(1) = 1, \end{cases} \tag{1}$$

[*]Corresponding author.

Received: 17 February 2018 / Revised: 20 March 2018 / Accepted: 11 April 2018.
DOI 10.22124/jmm.2018.2808

with η being a positive constant, is a nonlinear two-point boundary value problem. This problem arises in an investigation of the confinement of a plasma column by radiation pressure [23] and also in the theory of gas porous electrodes [6, 12]. The closed form solution of this problem in terms of the Jacobian elliptic function has been given [5] by

$$y(x) = \frac{2}{\eta}\sinh^{-1}\left\{\frac{y'(0)}{2}Sc\Big(\eta\; x|1-\frac{1}{4}y'(0)^2\Big)\right\}, \tag{2}$$

where $y'(0)$, the derivation of y at $x = 0$, is given by expression $y'(0) = 2\sqrt{1-m}$, with m being the solution of the transcendental equation

$$\frac{\sinh(\frac{\eta}{2})}{\sqrt{1-m}} = Sc(\eta|m), \tag{3}$$

where the Jacobian elliptic function $Sc(\eta|m)$ is defined by $Sc(\eta|m) = \tan\phi$, where ϕ, η are related by the integral

$$\eta = \int_0^{\phi} \frac{1}{\sqrt{1-m-\sin^2\theta}}d\theta. \tag{4}$$

It has been shown that $y'(x)$ has a singularity located approximately at [16, 22]

$$\chi_s = \frac{1}{\eta}\ln\Big(\frac{8}{y'(0)}\Big), \tag{5}$$

which implies that the singularity lies in the integration range of $y'(0) > 8e^{-\eta}$.

This problem is strongly nonlinear and it is suitable criterion for testing ability and reliability for each methods. This is a motivation to many researchers for considering Troesch's problem. Chang [2] applied shooting method, Feng et al. [5] used a modified homotopy perturbation method, Zarebnia et al. [24] used Sinc-Galerkin method based on single exponential transformation (SE). They found that the rate of convergence by using Sinc-Galerkin method based on SE transformation is $\mathcal{O}\Big(\exp(-k\sqrt{n})\Big)$. EL-Gamel [4] applied Sinc collocation method based on SE transformation. Deeba et al. [3] used decomposition method approximation. Recently, Nabati et al. [13] developed Sinc Galerkin method based on double exponential transformation and showed the rate of convergency is $\mathcal{O}(\exp(-k'n/\log n))$ with some positive k'.

Sinc method based on single exponential transformation was developed by Stenger and other authors in several fields of applied mathematics [4, 11, 15, 18, 24]. They have shown that, theoretically and numerically, the rate of convergency is $\mathcal{O}(\exp(-k\sqrt{n}))$. Later Mori and Sugihara, and some of researchers [13, 14, 19–21] were developed Sinc approximation method based on double exponential to solve problems. They were achieved to rate of convergency as $\mathcal{O}(\exp(-k'n/\log n))$.

In this study, Sinc collocation method based on double exponential transformation (DE-Sinc collocation), with novel strategy, has been developed to solve Troesch's problem. The rest of the paper is organized into five sections. Some preliminary definitions, theorems and notations that are employed to derive the formulations of the Sinc-collocation method is presented in section 2. In section 3, Sinc-collocation method based on suitable transformation to solve Troesch's problem is developed. The properties of this method are utilized to reduce the solution of strongly nonlinear BVP to solution of nonlinear algebraic equations. To use programming language, the matrix-vector form of this nonlinear system and jacobian form have been achieved. Solving obtained system and comparison between calculated solution with numerical results of the other existing methods have been tabulated in section 4. These tables demonstrated efficiency, rapidly convergency and simplicity of the method. Finally, in section 5, conclusions of the study have been brought.

2 Preliminaries

In this section, some preliminary definitions, theorems and notations of the Sinc function from references of [11, 18, 21], which are required, have been reviewed.

The Sinc function is defined on the whole real line, $-\infty < x < \infty$, by

$$\mathrm{Sinc}(x) = \begin{cases} \frac{\sin(\pi x)}{\pi x}, & x \neq 0, \\ 1, & x = 0. \end{cases}$$

For any $h > 0$, the translated Sinc function with evenly spaced nodes are given as

$$\mathrm{S}(j,h)(x) = \mathrm{Sinc}(\frac{x - jh}{h}), \quad j = 0, \pm 1, \pm 2, \ldots. \tag{6}$$

The $S(j,h)$ is called the $\mathbf{j}^{th}$ Sinc function with step size h at x.

Lemma 1. *([11]) Let $S(k,h)(x)$ be the $\boldsymbol{k}^{th}$ Sinc function with step h. Then*

$$\delta^{(0)}_{jk} = S(j,h)(kh) = \begin{cases} 1, & j = k, \\ 0, & j \neq k, \end{cases}$$

$$\delta_{jk}^{(1)} = h\frac{d}{dz}[S(j,h)(z)]_{z=kh} = \begin{cases} 0, & j = k, \\ \frac{(-1)^{k-j}}{k-j}, & j \neq k, \end{cases}$$

$$\delta_{jk}^{(2)} = h^2\frac{d^2}{dz^2}[S(j,h)(z)]_{z=kh} = \begin{cases} \frac{-\pi^2}{3}, & j = k, \\ \frac{-2(-1)^{k-j}}{(k-j)^2}, & j \neq k. \end{cases}$$

For the assembly of the discrete system, it is convenient to define the following matrices $I^{(l)} = [\delta_{jk}^{(l)}]$, $l = 0, 1, 2$, where $\delta_{jk}^{(l)}$ denotes the $(j,k)^{th}$ element of the matrix $I^{(l)}$. The matrix $I^{(0)}$ is the $m \times m$ identity matrix. The matrix $I^{(1)}$ is the skew symmetric Toeplitz matrix and $I^{(2)}$ is the symmetric Toeplitz matrix.

The following notation will be necessary for writing down the system. Let $D(g)$ be the $m \times m$ diagonal matrix

$$D\Big(g(x)\Big) = diag\left(g(-Nh), g((-N+1)h), \ldots, g(Nh)\right).$$

If the function f is defied on the real line, then for $h > 0$ the series

$$\mathrm{C}(f,h)(x) = \sum_{j=-\infty}^{\infty} f(jh)\mathrm{Sinc}\Big(\frac{x-jh}{h}\Big),$$

is called the Whittaker cardinal expansion of f where this series converges [18]. These properties are derived in the infinite strip D_d of the complex plane, where for $d > 0$, $D_d = \Big\{w = \xi + i\eta : |\eta| < d < \frac{\pi}{2}\Big\}$.

To state the decay property of functions precisely, we introduce the following function space. Let $H^1(D_d)$ be a function space defined as

$$H^1(D_d) = \{f : D_d \to C | f \text{ is analytic on } D_d \text{ and } N^1(f, D_d) < \infty\},$$

where

$$N^1(f, D_d) \equiv \lim_{\varepsilon \to 0} \int_{\partial D_d(\varepsilon)} |f(t)|\ |dt|,$$

$$D_d(\varepsilon) = \{\ t \in \mathbf{C}|\ |Re\ t| \leq 1/\varepsilon,\ |Im\ t| \leq d(1-\varepsilon)\}.$$

Theorem 1. *([21, Theorem 6.1]) Assume that the function f satisfies*
(1) $f \in H^1(D_d)$,
(2) $\forall x \in \boldsymbol{R} : \ \ | f(x) | \leq A \exp(-B\ \ \exp(\gamma\ \ | x |))$,
for positive constants A, B, γ and d where $\gamma d \leq \frac{\pi}{2}$. Then, there exists a constant C independent of N, such that:

$$\sup_{-\infty<x<\infty}\Big|\ f(x) - \sum_{k=-N}^{N} f(kh)S(k,h)(x)\Big| \leq C \exp\Big(-\frac{\pi d\gamma N}{log(\pi d\gamma N/B)}\Big),$$

where $h = \log(\pi d\gamma N/B)/(\gamma N)$.

3 DE-Sinc-collocation method for the problem

Since the Sinc function vanishes at the boundary domain, as the first step, converting of nonhomogeneous boundary conditions to homogeneous ones is needed. Therefore we replaced $v(x)$ by $y(x)-x$. By applying this change of variable problem (1) is converted to:

$$\begin{cases} L(v(x)) \equiv v''(x) - \eta \ \sinh(\eta \ v(x) + \eta \ x) = 0, \qquad 0 < x < 1, \\ v(0) = 0, \qquad v(1) = 0. \end{cases} \tag{7}$$

Due to the noncompatibility of the domain of the problem (7) with the domain of the Sinc method, the domain of problem should be transferred to $(-\infty, \infty)$, by using the suitable conformal map. For our problem with domain $(0, 1)$, the appropriate transformation is following conformal mapping:

$$x = \psi(t) = \frac{1}{2} \tanh\Big(\frac{\pi}{2} \sinh(t)\Big) + \frac{1}{2}, \tag{8}$$

$$t = \phi(x) = \psi^{-1}(x) = \log\Big[\frac{1}{\pi} \log(\frac{x}{1-x}) + \sqrt{1 + \Big(\frac{1}{\pi} \log(\frac{x}{1-x})\Big)^2}\Big], \tag{9}$$

which is known as the double exponential (DE) transformation and Sinc-collocation method based on this transformation is called DE-Sinc-collocation method. This DE transformation maps $\mathbf{R}$ to $(0, 1)$ and maps D_d onto the domain ([20, 21]):

$$\psi(D_d) = \Big\{ z\epsilon\mathcal{C} : \Big| \arg\Big(\frac{1}{\pi} \log(\frac{z}{1-z}) + \sqrt{1 + \Big(\frac{1}{\pi} \log(\frac{z}{1-z})\Big)^2}\Big)\Big| < d \Big\}.$$

By applying ψ to the problem (7), this problem is transformed to new one on $(-\infty, \infty)$ as follows:

$$\begin{cases} L(v(\psi(t))) \equiv \frac{d^2}{dx^2} v(\psi(t)) - \eta \ \sinh(\eta \ v(\psi(t)) + \eta \ \psi(t)) = 0, \\ \lim_{t \to \pm\infty} v(\psi(t)) = 0. \end{cases} \tag{10}$$

Consider $u(t) = v(\psi(t))$, so by differentiation chain rule, the first and second order derivatives are calculated as follow:

$$\frac{d}{dx} v(\psi(t)) = \frac{1}{\psi'(t)} u'(t), \tag{11}$$

$$\frac{d^2}{dx^2}(v(\psi(t))) = \Big(\frac{1}{\psi'(t)}\Big)^2 u''(t) - \frac{\psi''(t)}{(\psi'(t))^3} u'(t). \tag{12}$$

By replacing the equations (11) and (12) in the problem (10) and multiplying by $\Big(\psi'(t)\Big)^2$ we conclude:

$$\begin{cases} L(u(t)) \equiv u''(t) + \Big(- \frac{\psi''(t)}{(\psi'(t))}\Big) u'(t) \\ \qquad\qquad -\eta \ \Big(\psi'(t)\Big)^2 \sinh \Big(\eta \ u(t) + \eta \ \psi(t)\Big) = 0, \\ \lim\limits_{t\to\pm\infty} u(t) = 0. \end{cases} \tag{13}$$

The approximate solution to the problem (13) is given by

$$u_m(t) = \sum_{j=-N}^{N} c_j S_j(t), \qquad m = 2N+1, \tag{14}$$

where the bases Sinc functions $S_j(t) = S(j,h)(t)$ are defined in (6) and the unknown coefficients $\{c_j\}_{k=-N}^{N}$ need to be determined. Notice that the u_m satisfies to the boundary conditions because of $\lim_{t\to\pm\infty} S_j(t) = 0$.

To determine the coefficients of c_j's in (14), the collocation method is applied. Substituting $u_m(x)$, its first and second derivatives into (13) and then by replacing x by the collocation points $x_k = kh$, $k = -N, -N+1, \dots, N$, in which h is defined in Theorem 1, following nonlinear system is calculated

$$\sum_{j=-N}^{N} c_j \Big\{ S_k''(jh) - \Big[\frac{\psi''(jh)}{\psi'(jh)}\Big] S_k'(jh) \Big\} - \eta \ (\psi'(kh))^2 \ \sinh \Big(\eta \ c_k + \eta \ (\psi(kh))\Big) = 0,$$
$$k = -N, -N+1, \dots, N. \tag{15}$$

Notice that $\sum_{j=-N}^{N} c_j S_j(kh) = c_k$.

The notations of $\delta_{kj}^{(l)}$ in Lemma 1 are used and the system (15) is rewritten as follow:

$$\sum_{j=-N}^{N} c_j \Big\{ \frac{1}{h^2}\delta_{kj}^{(2)} - \Big[\frac{\psi''(jh)}{\psi'(jh)}\Big] \frac{1}{h}\delta_{kj}^{(1)} \Big\} - \eta \ (\psi'(kh))^2 \ \sinh \Big(\eta \ c_k + \eta \ (\psi(kh))\Big) = 0,$$
$$k = -N, -N+1, \dots, N. \tag{16}$$

By recalling the notations in Section 2, and knowing that $\delta_{kj}^{(1)} = -\delta_{jk}^{(1)}$, $\delta_{kj}^{(2)} = \delta_{jk}^{(2)}$ we can write down the nonlinear algebraic system of (16) as matrix-vector form as follow:

$$AC + B \sinh \Big(\eta \ C + \eta \ \psi\Big) = 0, \tag{17}$$

where

$$A = I^{(2)} - hD\Big(-\Big(\frac{\psi''}{\psi'}\Big)\Big)I^{(1)}, \tag{18}$$

$$B = h^2\, D\Big(-\eta\,(\psi')^2\Big), \tag{19}$$

$$C = \Big(c_{-N}, c_{-N+1}, \ldots, c_{N-1}, c_N\Big)^T,$$

$$\sinh\Big(m\;C + m\;\psi)\Big) = \begin{pmatrix} m\;c_{-N} + m\;\psi(-Nh) \\ m\;c_{-N+1} + m\;\psi((-N+1)h) \\ \vdots \\ m\;c_N + m\;\psi(Nh) \end{pmatrix}$$

Eq. (17) is a nonlinear system of $m = 2N+1$ equations and m unknown coefficients $\{c_j\}_{j=-N}^{N}$. By solving this system, unknown coefficients $\{c_j\}_{j=-N}^{N}$ are calculated and the Sinc approximation solution is obtained by (14).

Newton's method with an initial guess C_0 is applied to solve nonlinear algebraic system (17) as follows:

$$C_{k+1} = C_k - J^{-1}(C_k)\;\Big\{F(C_k)\Big\}, \tag{20}$$

where

$$F(C) = AC + B\sinh\Big(\eta\;C + \eta\;\psi\Big), \tag{21}$$

and

$$J(C) = A + \eta\;B\;D\Big(\cosh\Big(\eta\;C + \eta\;\psi\Big)\Big). \tag{22}$$

Here, C_k is the current iteration and C_{k+1} is the new iteration. Newton iteration is stopped whenever $\|C_{k+1} - C_k\| < \varepsilon$, where ε is given tolerance and the Euclidean norm is used.

4 Numerical results

In this section, DE-Sinc-collocation method (DESC) is applied to solve Troesch's Problem for several η's. Besides our results are compared with results of other methods in the literatures [1, 3–5, 13, 17, 24].

To apply the DE-Sinc-collocation method, we suppose that $d = \frac{\pi}{4}$, $\gamma = 1$, $B = \pi$ which led to $h = \log\Big(\frac{\pi N}{4}\Big)/N$. For solving system (17), the Newton's method is used. In Newton's method, we start with an initial guess C_0 as zero vector then use the Newton iteration (20).

In Table 1, and Table 2, the exact solutions for $\eta = 0.5$ and $\eta = 1$, and the numerical results of the present method are compared with several existing methods [3–5, 13, 24] in the literatures. These methods are the decomposition method approximation [3], the modified homotopy perturbation method (HPM) [5], the Sinc-Galerkin based on single exponential transformation (SESG) [24], the Sinc-collocation method based on single exponential transformation (SESC) [4] and the Sinc Galerkin based on double exponential transformation [13]. It is interesting that inaccurate tabulated exact solutions are given in [3–5, 24]. If they had used the exact solution reported here and in [7, 10] as a basis for comparison, they would have found that their approximation methods were actually much more accurate than they realized.

Table 1: Results for Troesch's Problem with $\eta = 0.5$.

x	exact solution	DESC $N = 15$	Dec.Method [3]	HPM [5]	DESG $N = 15$ [13]	SESG $N = 20$ [24]	SESC $N = 20$ [4]
0.1	0.0959443493	0.09594434932	0.0959383534	0.0959395656	0.09594434932	——-	0.0959445348
0.2	0.1921287477	0.19212874766	0.1921180592	0.1921193244	0.19212874768	0.19212882	0.1921287458
0.3	0.2887944009	0.28879440094	0.2887803297	0.2887806940	0.28879440094	——-	0.2887947251
0.4	0.3861848464	0.38618484637	0.3861687095	0.3861675428	0.38618484638	0.38618437	0.3861843754
0.5	0.4845471647	0.48454716476	0.4845302901	0.4845274183	0.48454716477	——-	0.4845471259
0.6	0.5841332484	0.58413324848	0.5841169798	0.5841127822	0.58413324848	0.58413371	0.5841336720
0.7	0.6852011483	0.68520114829	0.6851868451	0.6851822495	0.68520114831	——-	0.6852009802
0.8	0.7880165227	0.78801652269	0.7880055691	0.7880018367	0.78801652269	0.78801652	0.7880164746
0.9	0.8928542161	0.89285421615	0.8928480234	0.8928462193	0.89285421616	——-	0.8928542003
MR		$4.00\ E-011$	$1.68\ E-005$	$2.04\ E-005$	8.60 E-011	4.76 E-007	4.71 E-007

Table 2: Results for Troesch's Problem with $\eta = 1$.

x	exact solution	DESC $N = 15$	Dec.Method [3]	HPM [5]	DESG N=15 [13]	SESG $N = 20$ [24]	SESC $N = 20$ [4]
0.1	0.0846612565	0.084661256679	0.084248760	0.0843817004	0.084661256642	$-------$	0.0846618450
0.2	0.1701713582	0.170171358302	0.169430700	0.1696207644	0.170171358273	0.1701715950	0.1701714089
0.3	0.2573939080	0.257393908227	0.256414500	0.2565929224	0.257393908175	$-------$	0.2573949705
0.4	0.3472228551	0.347222855264	0.346085720	0.3462107378	0.347222855224	0.3472212611	0.3472212935
0.5	0.4405998351	0.440599835322	0.439401985	0.4394422743	0.440599835276	$-------$	0.4405997991
0.6	0.5385343980	0.538534398245	0.537365700	0.5373300622	0.538534398177	0.5385360476	0.5385359466
0.7	0.6421286091	0.642128609395	0.641083800	0.6410104651	0.642128609348	$-------$	0.6421279331
0.8	0.7526080939	0.752608094217	0.751788000	0.7517335467	0.752608094135	0.7526080999	0.7526080112
0.9	0.8713625196	0.871362520012	0.870908700	0.8708835371	0.871362519949	$-------$	0.8713623820
ME		$1.0\ E-010$	0.0012	0.0012	$3.90\ E-010$	$1.59\ E-006$	$1.56\ E-006$

In the last row of tables the maximum error (MR) of any column are listed which are computed via

$$MR = \max\{Error(x_i),\ x_i = 0.1, 0.2, \ldots, 0.9\},$$

where

$$Error(x) = |exact\_solution(x) - numerical\_solution(x)|.$$

It is shown that our results are more accurate than the results of other methods and more or less have the same accuracy.

In Table 3 and Table 4, absolute error in the solution of Troesch's Problem at $x_j = 0.1, 0.2, 0.3, \ldots, 0.9$ with $\eta = 0.5$ and $\eta = 1$ are presented respectively. In these tables, results of our method (DESC), the Sinc-collocation method based on single exponential (SESC) [4], and the Sinc Galerkin method based on double exponential transformation (DESG) [13], for $N = 5,\ 10,\ 15$ are compared. These tables show that our results are more accurate than reference [4], and more or less, has same the accuracy as the DE-Sinc Galerkin method.

Table 3: Absolute error of Troesch's Problem for $\eta = 0.5$.

x	DESC N=10	DESC N=15	DESG [13] N=10	DESG [13] N=15	SESC [4] N=10	SESC [4] N=10
0.1	1.7601 $E-08$	2.2460 $E-11$	1.8155 E-08	2.5048 $E-11$	1.2862 E-05	1.2652 E-06
0.2	1.5318 $E-08$	3.4851 $E-11$	1.6427 E-08	1.7526 $E-11$	1.5730 E-05	2.1638 E-06
0.3	2.4064 $E-08$	4.3462 $E-11$	2.3845 E-08	4.3119 $E-11$	7.3417 E-06	9.2352 E-07
0.4	1.5487 $E-08$	3.0492 $E-11$	1.7477 E-08	1.5468 $E-11$	1.8795 E-05	2.2513 E-06
0.5	2.3755 $E-08$	6.8339 $E-11$	2.5432 E-08	7.4792 $E-11$	3.9774 E-06	3.8266 E-08
0.6	3.1947 $E-08$	8.4385 $E-11$	3.1650 E-08	8.6008 $E-11$	1.1479 E-05	2.4579 E-06
0.7	2.3238 $E-08$	3.8008 $E-12$	2.4687 E-08	1.0786 $E-11$	7.2266 E-06	4.2468 E-07
0.8	3.2628 $E-08$	5.7676 $E-12$	3.4867 E-08	7.2489 $E-12$	5.2517 E-06	7.6033 E-07
0.9	3.0423 $E-08$	5.3624 $E-11$	3.0881 E-08	6.4269 $E-11$	9.8091 E-07	9.2200 E-07

In Table 5 and Table 6, the comparison of our method with $N = 5, 10, 15$ and the Sinc-Galerkin method based on single exponential transformation (SESG) [24] for $N = 5,\ 10,\ 20$ are tabulated for $\eta = 0.5,\ \eta = 1$, respectively. The numerical results of the presented method show that it is more accurate than the method of reference [24].

In Table 7, the numerical solution for $\eta = 5$ computed by the presented method (DESG), the DE-Sinc Galerkin method (DESG) [13], the SE Sinc Galerkin method (SESG) [24], the numerical approximation of the exact solutions given by a Fortran code called TWPBVP [9] are shown. Note that for $\eta > 1$ the decomposition method [3], the Laplace decomposition [8] do not yield a good approximation, but, the Sinc method without any changes gives acceptable results.

The numerical results of our method, the adaptive collocation method [9], the DE-Sinc Galerkin method [13], results of the new technique (NT) proposed by Chang et al. [1], the invariant embedding algorithm (IEA) [17] for $\eta = 10$ are tabulated in Table 8.

Table 4: Absolute error of Troesch's Problem for $\eta = 1$.

x	DESC N=10	DESC N=15	DESG [13] N=10	DESG [13] N=15	SESC [4] N=10	SESC [4] N=10
0.1	6.6560 $E-08$	1.7917 $E-10$	6.2704 $E-08$	1.4252 E-10	1.2862 E-05	4.2466 E-06
0.2	6.8279 $E-08$	1.0153 $E-10$	6.1456 $E-08$	7.3548 E-10	1.5730 E-05	7.2176 E-06
0.3	7.8965 $E-08$	2.2692 $E-10$	7.3691 $E-08$	1.7571 E-10	7.3417 E-06	2.7879 E-06
0.4	7.7083 $E-08$	1.6438 $E-10$	7.7978 $E-08$	1.2443 E-10	1.8795 E-05	8.0623 E-06
0.5	8.9098 $E-08$	2.2184 $E-10$	8.5391 $E-08$	1.7609 E-10	3.9774 E-06	3.3665 E-07
0.6	1.0187 $E-07$	2.4455 $E-10$	9.6812 $E-08$	1.7736 E-10	1.1479 E-05	9.0358 E-06
0.7	1.0203 $E-07$	2.9475 $E-10$	1.0412 $E-07$	2.4831 E-10	7.2266 E-06	8.1195 E-07
0.8	1.1963 $E-07$	3.1726 $E-10$	1.1892 $E-07$	2.3567 E-10	5.2517 E-06	3.5125 E-06
0.9	1.2546 $E-07$	4.1239 $E-10$	1.2622 $E-07$	3.4981 E-10	9.8091 E-07	3.9632 E-06

Table 5: Absolute error of Troesch's Problem for $\eta = 0.5$.

x	DESC N=5	DESC N=10	DESC N=15	SESG [24] N=5	SESG [24] N=10	SESG [24] N=20
0.2	8.7934 $E-05$	6.8279 $E-08$	1.0153 $E-10$	1.7798 $E-05$	1.7132 E-05	7.2300 E-08
0.4	9.8118 $E-05$	7.7083 $E-08$	1.6438 $E-10$	1.5723 $E-04$	1.6366 E-05	4.7640 E-07
0.6	1.1809 $E-04$	1.0187 $E-07$	2.4455 $E-10$	1.1858 $E-04$	1.5162 E-05	4.6160 E-07
0.8	1.3838 $E-04$	1.1963 $E-07$	3.1726 $E-10$	5.0917 $E-05$	5.1927 E-06	2.7000 E-09

Table 6: Absolute error of Troesch's Problem for $\eta = 1$.

x	DESC N=5	DESC N=10	DESC N=15	SESG [24] N=5	SESG [24] N=10	SESG [24] N=20
0.2	8.7934 $E-05$	6.8279 $E-08$	1.0153 $E-10$	5.9688 $E-05$	6.0723 E-05	2.3680 E-07
0.4	9.8118 $E-05$	7.7083 $E-08$	1.6438 $E-10$	5.7321 $E-04$	5.9727 E-05	1.5940 E-06
0.6	1.1809 $E-04$	1.0187 $E-07$	2.4455 $E-10$	5.2725 $E-04$	6.0328 E-05	1.6496 E-06
0.8	1.3838 $E-04$	1.1963 $E-07$	3.1726 $E-10$	1.9061 $E-04$	2.6693 E-05	6.0000 E-09

Solution of Troesch's Problem, for several η's have been shown in Figure 1 and Figure 2.

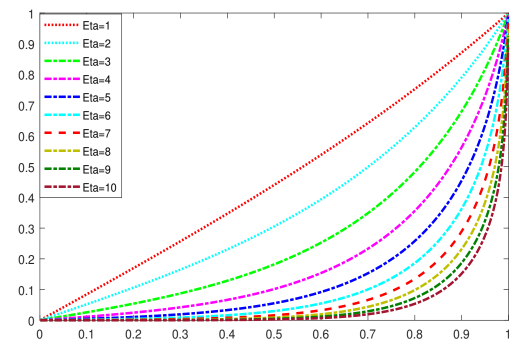

Figure 1: Approximation solution with $\eta = 1,\ 2, \ldots, 10$ for Troesch's problem.

Table 7: Sulotion of Troesch's Problem for $\eta = 5$.

x	DESC	DESG [13]	FORT.Code [4]	SESC [4]	B-Spline [9]
0.2	0.017534066	0.01075340	0.01075340	0.00762552	0.01002027
0.4	0.033200490	0.03320049	0.03320049	0.03817903	0.03099793
0.8	0.258216487	0.25821648	0.25821648	0.23252435	0.24170496
0.9	0.455060027	0.45506002	0.45506002	0.44624551	0.42461830

5 Conclusions

In this study, the Sinc collocation method based on double exponential transformation has been developed to solve a special nonlinear two point boundary value problem as Troesch's problem. It has been shown that the presented method reduces the strong nonlinear problem to a system of nonlinear algebraic equations. The matrix-vector form of the obtained system, which simplifies the solution of nonlinear system by a software, was calculated. The tabulated results demonstrated that, this method can solve the same problems effectively. The results obtained from the our method were compared with the exact solution, decomposition method, modified homotopy perturbation method, SE-Sinc Galerkin and collocation methods, and DE Sinc Galerkin method. The comparison illustrated that combination of DE transformation and Sinc method increases the speed

Table 8: Sulotion of Troesch's Problem for $\eta = 10$.

x	DESC	DESG [13]	B-Spline [9]	NT [1]	IEA [17]
0.100	4.21112870895 E-5	4.2109517178 E-5	4.2073350769 E-5	4.2111836797 E-5	4.2112817259 E-5
0.200	1.29964954884 E-4	1.2996470996 E-4	1.2985174301 E-4	1.2996392382 E-4	1.2996694887 E-4
0.300	3.58977809160 E-4	3.5897799875 E-4	3.5869050068 E-4	3.5897788554 E-4	3.5898622667 E-4
0.400	9.77899720459 E-4	9.7789976331 E-4	9.7718285202 E-4	9.7790142270 E-4	9.7792408894 E-4
0.500	2.65902039867 E-3	2.6590199596 E-3	2.6572397289 E-3	2.6590171780 E-3	2.6590784553 E-3
0.600	7.22893497357 E-2	7.2893481546 E-3	7.2245714919 E-3	7.2289246952 E-3	7.2290888288 E-3
0.700	1.96640645884 E-2	1.9664064913 E-2	1.9653516239 E-2	1.9664060256 E-2	1.9664492439 E-2
0.800	5.37303293317 E-1	5.3730329728 E-2	5.3705176342 E-2	5.3730329585 E-2	5.3731514744 E-2
0.900	1.52114082216 E-1	1.5211408177 E-1	1.5205681609 E-1	1.5211407878 E-1	1.5211772137 E-1
0.925	2.02001677397 E-1	2.0200167749 E-1	2.0199329428 E-1	2.0200168549 E-1	2.0200687498 E-1
0.950	2.76267739284 E-1	2.7626773862 E-1	2.7617352773 E-1	2.7626773490 E-1	2.7627586266 E-1
0.970	3.72264336588 E-1	3.7226433662 E-1	3.7214738437 E-1	3.7226433306 E-1	3.7227801567 E-1
0.980	4.48233035280 E-1	4.4823303439 E-1	4.4810300700 E-1	4.4823303862 E-1	4.4825330785 E-1
0.990	5.74076510633 E-1	5.7407650921 E-1	5.7394040616 E-1	5.7407649824 E-1	5.7411484073 E-1
0.995	6.90114949255 E-1	6.9011494806 E-1	6.9001577313 E-1	6.9011494173 E-1	6.9018359305 E-1
0.997	7.65769731259 E-1	7.6576972817 E-1	7.6570041051 E-1	7.6576972613 E-1	7.6586999710 E-1
0.998	4.48233035280 E-1	8.1803281864 E-1	8.1797690736 E-1	8.1803282828 E-1	8.1816295300 E-1
0.999	8.88993131730 E-1	8.8899312643 E-1	8.8894966541 E-1	8.8899311717 E-1	8.8917294622 E-1

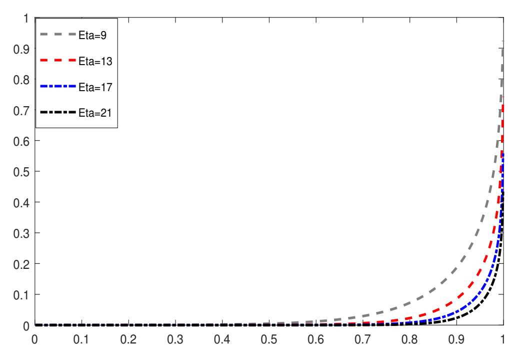

Figure 2: Approximation solution with $\eta = 9,\ 13,\ 17,\ 21$ for Troesch's problem.

of convergence. DE Sinc Galerkin and DE Sinc collocation methods have approximately accuracy, but these methods were more accurate than other methods in the literature. Moreover, our method has been used for $\eta > 1$ without any changes, and results were significant and impressive.

Acknowledgements

The authors would like to thank the referee for his/her helpful comments.

References

[1] S.H. Chang, and I.L. Chang, *A new algorithm for calculating the one dimensional differential transform of nonlinear functions*, Appl. Math. Comput. **195** (2008) 799–808.

[2] S.H. Chang, *Numerical solution of Troeschs problem by simple shooting method*, Appl. Math. Comput. **216** (2010) 3303-3306.

[3] E. Deeba, S.A. Khuri, S. Xie, *An algorithm for solving boundary value problems*, J. Comput. Phys. **159** (2000) 125-138.

[4] M. EL-Gamel, *Numerical Solution of Troeschs Problem by Sinc-collocation Method*, Appl. Math. **4** (2013) 707–712.

[5] X. Feng, L. Mei, G. He, *An efficient algorithm for solving Troesch's problem*, Appl. Math. Comput. **189** (2007) 500-507.

[6] D. Gidaspow, B.S. Baker, *A model for discharge of storage batteries*, J. Electrochem. Soc. **120** (1973) 1005-1010.

[7] I.O. Govrilyuk, I.I. Lazurchak, V.L. Mokarav, *A method with a controllable exponential convergence rate for nonlinear differential operator equations*, Comput. Meth. Appl. Math. **9** (2009) 63-78.

[8] S.A. Khuri, *A numerical algorithm for solving the Troesch's problem*, Int. J. Copmut. Math **80** (2003) 493–498.

[9] S.A. Khuri, A. Sayty, *Troesch's problem: A B-Spline collocation approach*, Math. Comput. Model. **54** (2011) 1907–1918.

[10] Y. Lin, J.A. Enszer, M.A. Stadtherr, *Enclosing all solutions of two point boundary value problems for ODEs*, Comput. Chem. Eng. **32** (2008) 1714-1725.

[11] J. Lund, K. Bowers, *Sinc methods for quadrature and differential equation*, SIAM , Philadelphia , PA, 1992.

[12] V.S. Markin, A.A. Chernenko, Y.A. Chizmadehev, Y.G. Chirkov, *Aspects of the theory of gas porous electrodes, in: V.S. Bagotskii, Y.B.*

Vasilev (Eds.), Fuel Cells: Their Electrochemical Kinetics, Consultants Bureau, New York, pp. 21-33, 1966.

[13] M. Nabati, M. Jalalvand, *Solution of Troesch's problem through double exponential Sinc Galerkin method*, Comput. Methods Differ. Equ. **5** (2017) 141–157.

[14] A. Nurmuhammad, M. Muhammad, M. Mori, M. Sugihara, *Double exponential transformation in the Sinc-collocation method for a boundary value problem with fourth-order ordinary differential equation*, J. Comp. Appl. Math. **182** (2005) 32–50.

[15] J. Rashidinia , M. Nabati, *Sinc-Galerkin and Sinc-collocation methods in the solution of nonlinear two-point boundary value problems*, Comp. Appl. Math. **32** (2013) 315–330.

[16] S.M. Roberts, J.S. Shipman, *On the closed form solution of Troesch's problem*, J. Comput. Phys. **21** (1976) 291-304.

[17] M.R. Scott, *On the conversion of boundary-value problems into stable initial-value problems via several invariant imbedding algorithm, in: A. K. Aziz(EN), Numerical Solution of Boundary Value Problems for Ordinary Differential Equations*, Academic Press, New York, 1975, 89–146.

[18] F. Stenger, *Numerical method based on Sinc and analytic function*, Springer-Verlag New York, 1993.

[19] M. Sugihara, *Near optimality of Sinc approximation*, Math. Comput. **72** (2003) 767–786.

[20] H.Takahasi, M. Mori, *Double exponential formulas for numerical integration*, Publ. Res. Inst. Math. Sci. **9** (1974) 721–741.

[21] K. Tanaka, M. Sugihara and K. Murata, *Function classes for successful DE-Sinc approximations*, Math. Comput. **78** (2009) 1553–1571.

[22] B.A. Troesch, *Intrinsic difficulties in the numerical solution of a boundary value problem*, Internal Report NN142, TRW Inc., Redondo Beach, California, 1960.

[23] E.S. Weibel, *On the confinement of a plasma by magneto static fields*, Phys. Fluids. **2** (1959) 52-56.

[24] M. Zarebnia, M. Sajjadian, *The SincGalerkin method for solving Troesch's problem*, Math. Comput. Model. **56** (2012) 218-228.