Journal of Mathematical Modeling
Vol. 5, No. 2, 2017, pp. 89-118

JMM

Numerical solution of non-planar Burgers equation by Haar wavelet method

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Abstract. In this paper, an efficient numerical scheme based on uniform Haar wavelets is used to solve the non-planar Burgers equation. The quasilinearization technique is used to conveniently handle the nonlinear terms in the non-planar Burgers equation. The basic idea of Haar wavelet collocation method is to convert the partial differential equation into a system of algebraic equations that involves a finite number of variables. The solution obtained by Haar wavelet collocation method is compared with that obtained by finite difference method and are found to be in good agreement. Shock waves are found to be formed due to nonlinearity and dissipation. We have analyzed the effects of non-planar and nonlinear geometry on shock existence. We observe that non-planar shock structures are different from planar ones. It is of interest to find that Haar wavelets enable to predict the shock structure accurately.

Keywords: Haar wavelets, non-planar Burgers equation, quasilinearization, collocation points, finite difference, cylindrical and spherical geometry.
AMS Subject Classification: 65T60.

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Received: 29 June 2017 / Revised: 24 July 2017 / Accepted: 28 July 2017.

1 Introduction

Burgers equation is the simplest nonlinear parabolic equation representing a phenomena described by a balance between nonlinear convection and linear diffusion or dissipation. Burgers equation in one dimension was first derived by Bateman [1] to describe certain viscous flows. Later, Burgers [5, 6] derived it from Navier-Stokes equation as a model for turbulence. Burgers equation and its various generalizations, often called generalized Burgers equations, are categorized as nonlinear diffusive wave equations [20]. It appears in several physical problems such as one-dimensional turbulence, sound waves in an viscous medium, waves in fluid filled viscous elastic tubes, and magnetohydrodynamic waves in a medium with finite electrical conductivity.

The study of nonlinear partial differential equations (PDEs) serves as an exceptionally productive theme of study and research in the mathematics of explosions. The motion of a spherical piston pushing out air or gas in front of it creates spherical or cylindrical expansion and this is one of the most simplest ways to simulate an explosion. This can be studied by non-planar Burgers equation which is derived by Navier-Stokes equation. The non-planar Burgers equation is given by

$$\frac{\partial u}{\partial t} + u^{\alpha}\frac{\partial u}{\partial x} + \frac{ju}{2(t+1)} = \epsilon\frac{\partial^2 u}{\partial x^2}, \tag{1}$$

where $\epsilon > 0$ is small, $\alpha \geq 1$ is an integer and $j > 0$ represents the geometry of the non-planar source. Moreover, $j = 1$ corresponds to the cylindrical geometry of the source, whereas $j = 2$ corresponds to spherical geometry. A reference may be made to [18] and [13] for the derivation of equation (1) with $\alpha = 1$ which describes the propagation of weakly nonlinear longitudinal waves in gases or liquids from a non-planar source. For $j = 0$, equation (1) reduces to the planar Burgers equation which has been solved by Ram Jiwari [12] using Haar wavelet method. Here we present the analysis of the solution of a more general form of Burgers equation.

The non-planar Burgers equation (1) has applications in nonlinear acoustics [8]. Grundy et al. [9] studied the large time solution of an initial value problem for equation (1) subject to non-negative integrable data. Sachdev et al. [21] constructed N-wave solution of equation (1). Srinivasa Rao et al. [24] studied the existence and non-existence of self similar solutions of equation (1). Sachdev et al. [22, 23] studied the large time behaviour of periodic solution of equation (1) using a perturbation technique. More recently, Srinivasa Rao [25] studied the solution of equation (1) using Hermite interpolation. Also, Srinivasa Rao [26] studied the large time behaviour of the solution of inviscid form of equation (1).

Wavelets are mathematical functions that decompose data into different frequency components and then each component is studied with a resolution matched to its scale. Wavelet theory is the result of a multidisciplinary effort that brought together mathematicians, physicists and engineers. This connection has created a flow of ideas that goes well beyond the construction of new bases or transforms. Wavelets are well-suited for approximating data with sharp discontinuities. This representation is more accurate and useful in data compression, noise removal, pattern classification and fast scientific computation.

In recent years, the wavelet approach for the solution of PDEs has become very popular. Multi-resolution analysis of wavelets capture local features efficiently as such enables to detect singularities, shocks, irregular structure and transient phenomena exhibited by the analyzed equations. Haar wavelets are based on the functions which were introduced by the Hungarian mathematician Alfred Haar in 1910.

Chen and Hsiao [7] recommended to expand into the Haar series the highest order derivatives appearing in the differential equation. This idea has been very prolific and it is being abundantly applied for the solution of PDEs. The wavelet coefficients appearing in the Haar series are calculated either using Collocation method or Galerkin method. Lepik [**?**,14–16] used Haar wavelet method to solve linear Fredholm integral equation, nonlinear Volterra integral equation, stiff differential equations, Duffing equation, diffusion equation, Burgers equation and Sine-Gordon equation. Bujurke et al. [3] have computed eigenvalues and solutions of regular Sturm-Liouville problems using Haar wavelets. More recently, Hariharan et al. [10] have solved Klein-Gordon and Sine-Gordon equations using Haar wavelet methods.

In this paper, an efficient and novel numerical scheme based on uniform Haar wavelets [17] is used to solve the non-planar Burgers equation. The quasilinearization technique is used to conveniently handle the nonlinear terms in the non-planar Burgers equation.

The paper is organized as follows. The Haar wavelet preliminaries and the function approximation are presented in Section 2. The method of solution of the non-planar Burgers equation using Haar wavelets is proposed in Section 3 and the solution using Finite difference method in Section 4. The results and discussions are presented in Section 5. The conclusions drawn are presented in Section 6. Section 7 contains the graphs and tables obtained.

2 Haar wavelets

The Haar wavelet is a sequence of rescaled "square-shaped" functions which together form a wavelet family or basis. The Haar wavelets consist of piecewise constant functions and are therefore the simplest orthonormal wavelets with a compact support. An advantage of these wavelets is the possibility to integrate them analytically arbitrary times. They are conceptually simple, fast, memory efficient and exactly reversible. They are often known as a first order Daubechies wavelet. The Haar wavelet family for $x \in [0,1]$ is defined as follows

$$h_i(x) = \begin{cases} 1, & \text{for } x \in [\xi_1, \xi_2), \\ -1, & \text{for } x \in [\xi_2, \xi_3), \\ 0, & \text{elsewhere,} \end{cases} \tag{2}$$

where $\xi_1 = k/m$, $\xi_2 = (k+0.5)/m$, and $\xi_3 = (k+1)/m$.

In the above definition $m = 2^d, d = 0, 1, \ldots, J$ indicates the level of the wavelet; $k = 0, 1, \ldots, m-1$ is the translation parameter and J is the maximum level of resolution. The index i in equation (2) is calculated by the formula $i = m + k + 1$. In the case of minimum values $m = 1, k = 0$ we have $i = 2$. The maximum value of i is $i = 2M = 2^{J+1}$. For $i = 1$, $h_1(x)$ is assumed to be the scaling function which is defined as follows:

$$h_1(x) = \begin{cases} 1, & \text{for } x \in [0, 1), \\ 0, & \text{elsewhere.} \end{cases} \tag{3}$$

In order to solve PDEs of any order, we need the following integrals:

$$p_i(x) = \int_0^x h_i(x)dx = \begin{cases} x - \xi_1, & \text{for } x \in [\xi_1, \xi_2), \\ \xi_3 - x, & \text{for } x \in [\xi_2, \xi_3), \\ 0, & \text{elsewhere,} \end{cases}$$

$$q_i(x) = \int_0^x p_i(x)dx = \begin{cases} \dfrac{(x-\xi_1)^2}{2}, & \text{for } x \in [\xi_1, \xi_2), \\ \dfrac{1}{4m^2} - \dfrac{(\xi_3 - x)^2}{2}, & \text{for } x \in [\xi_2, \xi_3), \\ \dfrac{1}{4m^2}, & \text{for } x \in [\xi_3, 1], \\ 0, & \text{elsewhere.} \end{cases}$$

3 Multi-resolution analysis

The best way to understand wavelets is through multi-resolution analysis. A multi-resolution analysis (MRA) of $L^2(\mathbb{R})$ is defined as a sequence of closed subspaces $V_d \in L^2(\mathbb{R})$, $d \in \mathbb{Z}$ with the following properties.

(i) $\ldots \subset V_{-1} \subset V_0 \subset V_1 \subset \ldots$

(ii) The spaces V_d satisfy $\bigcup_{d\in\mathbb{Z}} V_d$ is dense in $L^2(\mathbb{R})$ and $\bigcap_{d\in\mathbb{Z}} V_d = 0$.

(iii) If $f(x) \in V_0$, then $f(2^d x) \in V_d$, i.e., the spaces V_d are scaled versions of the central space V_0.

(iv) If $f(x) \in V_0$, then $f(2^d x - k) \in V_d$, i.e., all the V_d are invariant under translation.

(v) There exists $\phi \in V_0$ such that $\{\phi(x-k); k \in \mathbb{Z}\}$ is a Riesz basis in V_0.

The space V_d is used to approximate general functions by defining approximate projection of these functions onto these spaces. Since the union of all the V_d is dense in $L^2(\mathbb{R})$, so it guarantees that any function in $L^2(\mathbb{R})$ can be approximated arbitrarily close by such projections. As an example, the space V_d can be defined like

$$V_d = W_{d-1} \oplus V_{d-1} = W_{d-1} \oplus W_{d-2} \oplus V_{d-2} = \ldots = \bigoplus_{d=1}^{J+1} W_d \oplus V_0,$$

then the scaling function $h_1(x)$ generates an MRA for the sequence of spaces $\{V_d; d \in \mathbb{Z}\}$ by translation and dilation as defined in equations (2), (3). For each d, the space W_d serves as an orthogonal complement of V_d in V_{d+1}. The space W_d includes all the functions in V_{d+1} that are orthogonal to all those in V_d under some chosen inner product. The set of functions which forms a basis for the space W_d are called wavelets.

Given a function $f \in L^2(\mathbb{R})$, the MRA of $L^2(\mathbb{R})$ produces a sequence of subspaces $V_d, V_{d+1}, \ldots$ such that the projections of f onto these spaces give finer and finer approximations of the function f as $J \to \infty$ [19].

4 Function approximation

Any function $f(x)$ which is square integrable on $(0,1)$ can be expressed as an infinite sum of Haar wavelets as

$$f(x) = \sum_{i=1}^{\infty} a(i) h_i(x), \tag{4}$$

where $a(i) = \int_0^1 f(x) h_i(x) dx$. If $f(x)$ is approximated as piecewise constant during each subinterval, then equation (4) will be terminated at finite terms, i.e. $f(x) = \sum_{i=1}^{2M} a(i)h_i(x)$, where the wavelet coefficients $a(i),\ i = 1, 2, \ldots, 2M$ are to be determined.

5 Method of solution

5.1 Haar wavelet collocation method

Consider an initial boundary value problem (IBVP) for the non-planar Burgers equation [25],

$$u_t + u^\alpha u_x + \frac{ju}{2(t+1)} = \epsilon u_{xx},\ 0 < x < 1,\ t > 0, \tag{5}$$

with the initial and boundary conditions

$$u(x, 0) = \sin(\pi x),\ 0 \le x \le 1, \tag{6}$$

$$u(0, t) = 0,\ u(1, t) = 0,\ t \ge 0. \tag{7}$$

Divide the interval $[0, T]$ into N equal parts of length $\Delta t = \frac{T}{N}$ and denote $t_s = (s-1)\Delta t,\ s = 1, 2, \ldots, N$.

We assume a Haar wavelet solution for equation (5) in the form

$$\dot{u}''(x, t) = \sum_{i=1}^{2M} a_s(i)h_i(x), \tag{8}$$

where the dot and prime denote differentiation with respect to t and x respectively and the row vector a_s is constant in the subinterval $t \in [t_s, t_{s+1}]$. Integrating equation (8) with respect to t in the limits $[t_s, t]$ leads to

$$u''(x, t) = (t - t_s)\sum_{i=1}^{2M} a_s(i)h_i(x) + u''(x, t_s). \tag{9}$$

Repeatedly integrating equation (9) with respect to x in the limits $[0, x]$ gives

$$u'(x, t) = (t - t_s)\sum_{i=1}^{2M} a_s(i)p_i(x) + u'(x, t_s) + u'(0, t) - u'(0, t_s), \tag{10}$$

$$\begin{aligned} u(x, t) = (t - t_s)\sum_{i=1}^{2M} a_s(i)q_i(x) + u(x, t_s) + u(0, t) - u(0, t_s) \\ + x[u'(0, t) - u'(0, t_s)]. \end{aligned} \tag{11}$$

By using the boundary conditions (7), we obtain

$$u(0,t_s)=0, \quad u(1,t_s)=0. \tag{12}$$

Putting $x=1$ in equation (11) and using the conditions in (7) and (12), we arrive at

$$u'(0,t)-u'(0,t_s)=-(t-t_s)\sum_{i=1}^{2M}a_s(i)q_i(1), \tag{13}$$

Substituting equations (7), (12) and (13) into equations (10) and (11), we have

$$u'(x,t)=(t-t_s)\sum_{i=1}^{2M}a_s(i)[p_i(x)-q_i(1)]+u'(x,t_s), \tag{14}$$

$$u(x,t)=(t-t_s)\sum_{i=1}^{2M}a_s(i)[q_i(x)-xq_i(1)]+u(x,t_s). \tag{15}$$

Differentiating equation (15) with respect to t leads to

$$\dot{u}(x,t)=\sum_{i=1}^{2M}a_s(i)[q_i(x)-xq_i(1)]. \tag{16}$$

The wavelet collocation points are defined as

$$x_l \ = \ \frac{l-0.5}{2M},\ l=1,2,\ldots,2M.$$

Taking the collocation points $x\to x_l$ and $t\to t_{s+1}$ in equations (9), (14), (15) and (16), we get

$$u''(x_l,t_{s+1}) \ = \ \Delta t\sum_{i=1}^{2M}a_s(i)h_i(x_l)+u''(x_l,t_s), \tag{17}$$

$$u'(x_l,t_{s+1}) \ = \ \Delta t\sum_{i=1}^{2M}a_s(i)[p_i(x_l)-q_i(1)]+u'(x_l,t_s), \tag{18}$$

$$u(x_l,t_{s+1}) \ = \ \Delta t\sum_{i=1}^{2M}a_s(i)[q_i(x_l)-x_lq_i(1)]+u(x_l,t_s), \tag{19}$$

$$\dot{u}(x_l,t_{s+1}) \ = \ \sum_{i=1}^{2M}a_s(i)[q_i(x_l)-x_lq_i(1)]. \tag{20}$$

Using the well known quasilinearization technique [2] to handle the nonlinearity in equation (5), we obtain the following approximation scheme

$$\dot{u}(x,t_{s+1}) + u^{\alpha}(x,t_s)u'(x,t_{s+1}) - \epsilon u''(x,t_{s+1}) + \Big[\alpha u^{\alpha-1}(x,t_s)u'(x,t_s) + \frac{j}{2(t_{s+1}+1)}\Big]u(x,t_{s+1}) = \alpha u^{\alpha}(x,t_s)u'(x,t_s),\ s=0,1,2,\dots$$

which leads us from the layer t_s to t_{s+1}.

Taking the collocation points x_l in (5.1) and using equations (17) - (20), the following $2M$ system of equations are obtained:

$$\sum_{i=1}^{2M} A(x_l,t_s)a_s(i) = f(x_l,t_s),\ l=1,2,\dots,2M,\ s=0,1,2,\dots$$

where

$$\begin{aligned} A(x_l,t_s) = &\left\{1 + \frac{j\Delta t}{2(t_{s+1}+1)} + \alpha\Delta t u^{\alpha-1}(x_l,t_s)u'(x_l,t_s)\right\} q_i(x_l) \\ &+ \Delta t u^{\alpha}(x_l,t_s)p_i(x_l) - \epsilon\Delta t h_i(x_l) - \left\{x_l + \frac{j\Delta t x_l}{2(t_{s+1}+1)}\right. \\ &\left. + \alpha\Delta t x_l u^{\alpha-1}(x_l,t_s)u'(x_l,t_s) + \Delta t u^{\alpha}(x_l,t_s)\right\} q_i(1), \end{aligned}$$

$$f(x_l,t_s) = \epsilon u''(x_l,t_s) - u^{\alpha}(x_l,t_s)u'(x_l,t_s) - \frac{j}{2(t_{s+1}+1)}u(x_l,t_s).$$

Using the initial conditions (6), we have

$$u(x_l,0) = \sin(2\pi x_l),\ u'(x_l,0) = 2\pi\cos(2\pi x_l),\ u''(x_l,0) = -4\pi^2\sin(2\pi x_l).$$

The wavelet coefficients $a_s(i)$, $i = 1,2,\dots,2M$ can be successively calculated by solving the $2M$ system of equations in equation (17). This process is started with the conditions in equation (5.1). The $2M$ system of equations are solved in MATLAB using the Symbolic Math Toolbox which takes care of the sparseness of the coefficient matrix. These coefficients are then substituted in equations (17)-(19) to obtain the approximate solutions at different time levels.

5.2 Finite difference method

The Finite difference method [11] is used to solve the quasilinearized nonplanar Burgers equation (5.1) with initial condition (6) and boundary conditions (7).

Let Δx and Δt be the step sizes with respect to x and t respectively, $r = \frac{1}{\Delta x}$ be the number of subintervals with respect to x, $N = \frac{T}{\Delta t}$ be the number of time levels where $t \in [0, T]$, $x_l = l\Delta x$ for $l = 0, 1, 2, \ldots, r$ and $t_s = s\Delta t$ for $s = 0, 1, 2, \ldots, N$.

The backward difference approximation for the first order derivative with respect to t is

$$\dot{u}(x_l, t_{s+1}) = \frac{u(x_l, t_{s+1}) - u(x_l, t_s)}{\Delta t}. \tag{21}$$

The forward difference approximation for the first order derivative with respect to x at s and $s+1$ time levels are respectively given by

$$u'(x_l, t_s) = \frac{u(x_{l+1}, t_s) - u(x_l, t_s)}{\Delta x}, \tag{22}$$

$$u'(x_l, t_{s+1}) = \frac{u(x_{l+1}, t_{s+1}) - u(x_l, t_{s+1})}{\Delta x}. \tag{23}$$

The centered difference approximation for the second order derivative with respect to x will be

$$u''(x_l, t_{s+1}) = \frac{u(x_{l+1}, t_{s+1}) - 2u(x_l, t_{s+1}) + u(x_{l-1}, t_{s+1})}{(\Delta x)^2}. \tag{24}$$

Substituting equations (21)-(24) into equation (5.1), we obtain the following tridiagonal system of equations,

$$\begin{aligned} -\frac{\epsilon}{(\Delta x)^2} u(x_{l-1}, t_{s+1}) &+ \left[\frac{\alpha}{\Delta x} u^{\alpha-1}(x_l, t_s) u(x_{l+1}, t_s) - \frac{(\alpha+1)}{\Delta x} u^{\alpha}(x_l, t_s) \right. \\ &\left. + \frac{j}{2(t_{s+1}+1)} + \frac{2\epsilon}{(\Delta x)^2} + \frac{1}{\Delta t}\right] u(x_l, t_{s+1}) + \left[\frac{1}{\Delta x} u^{\alpha}(x_l, t_s) \right. \\ &\left. - \frac{\epsilon}{(\Delta x)^2}\right] u(x_{l+1}, t_{s+1}) = \frac{1}{\Delta t} u(x_l, t_s) + \frac{\alpha}{\Delta x}\left[u(x_{l+1}, t_s) \right. \\ &\left. - u(x_l, t_s)\right] u^{\alpha}(x_l, t_s), \; l = 1, 2, \ldots, r-1, \; s = 1, 2, \ldots, N, \end{aligned} \tag{25}$$

with the initial and boundary conditions

$$u(x_l, t_0) = \sin(\pi x_l), \; l = 0, 1, 2, \ldots, r,$$

$$\left. \begin{aligned} u(x_0, t_s) = 0 \\ u(x_r, t_s) = 0 \end{aligned} \right\} \; s = 0, 1, 2, \ldots, N.$$

We use Thomas algorithm [4] to solve these tridiagonal system of equations (25) using the above conditions. The obtained results are compared with the Haar wavelet collocation method solution.

6 Error analysis

In this section, the error analysis for the Haar wavelet method has been discussed.

Lemma 1. *Let $f(x) \in L^2(\mathbb{R})$ be a continuous function in $(0,1)$ with $|f'(x)| \leq K$; $\forall x \in (0,1)$; $K > 0$ and $f(x) = \sum_{i=1}^{\infty} a_i h_i(x)$. Then $|a_i| \leq 2^{-(3d+2)/2}K$.*

Proof. According to the one-dimensional MRA,

$$f(x) = \sum_{i=1}^{\infty} a_i h_i(x), \tag{26}$$

where

$$h_i(x) = 2^{d/2} h(2^d x - k)dx; \quad k = 1, 2, \ldots, 2^d - 1, \quad d = 0, 1, \ldots J, \tag{27}$$

$$a_i = \int_0^1 f(x)h_i(x)dx = \int_0^1 2^{d/2} f(x)h(2^d x - k)dx. \tag{28}$$

We have,

$$h(2^d x - k) = \begin{cases} 1 & \text{if } k2^{-d} \leq x \leq (k+\frac{1}{2})2^{-d} \\ -1 & \text{if } (k+\frac{1}{2})2^{-d} \leq x \leq (k+1)2^{-d} \\ 0 & \text{elsewhere} \end{cases} \tag{29}$$

Using equation (29) in equation (28), we obtain

$$\begin{aligned} a_i &= 2^{d/2}\left[\int_{k2^{-d}}^{(k+\frac{1}{2})2^{-d}} f(x)dx - \int_{(k+\frac{1}{2})2^{-d}}^{(k+1)2^{-d}} f(x)dx\right] \\ &= 2^{d/2}\left[\left\{\left(k+\frac{1}{2}\right)2^{-d} - k2^{-d}\right\} f(\eta_1) - \left\{(k+1)2^{-d}\right.\right. \\ &\qquad \left.\left. - \left(k+\frac{1}{2}\right)2^{-d}\right\} f(\eta_2)\right] \\ &\qquad \text{where } \eta_1 \in \left(k2^{-d}, (k+\frac{1}{2})2^{-d}\right),\ \eta_2 \in \left((k+\frac{1}{2})2^{-d}, (k+1)2^{-d}\right) \\ &= 2^{-(d+2)/2}[f(\eta_1) - f(\eta_2)] \\ &= 2^{-(d+2)/2}(\eta_1 - \eta_2)f'(\eta) \quad \text{where } \eta \in (\eta_1, \eta_2) \ \text{[by Mean Value Theorem.]} \end{aligned}$$

Since $|f'(\eta)| \leq K$, we get

$$|a_i| \leq 2^{-(d+2)/2} 2^{-d} K = 2^{-(3d+2)/2} K.$$

□

Theorem 1. *If $u(x, t_{s+1})$ is the exact solution and $u_{2M}(x, t_{s+1})$ is the Haar wavelet solution at $t = t_{s+1}$, then*

$$\|E_J\| = \|u(x, t_{s+1}) - u_{2M}(x, t_{s+1})\| \leq \frac{2^{-\frac{1}{2}(J+3)} \Delta t K \sqrt{C}}{1 - 2^{-\frac{1}{2}}},$$

where $C, K > 0$, J is the level of resolution of the wavelet and $M = 2^J$.

Proof. From equation (15), the Haar wavelet solution at $t = t_{s+1}$ is given by

$$u_{2M}(x, t_{s+1}) = \Delta t \sum_{i=1}^{2M} a_s(i)[q_i(x) - xq_i(1)] + u(x, t_s).$$

Taking the asymptotic expansion of the above equation, we get

$$u(x, t_{s+1}) = \Delta t \sum_{i=1}^{\infty} a_s(i)[q_i(x) - xq_i(1)] + u(x, t_s).$$

The error estimation at J^{th} level of resolution is

$$\|E_J\| = \|u(x, t_{s+1}) - u_{2M}(x, t_{s+1})\| = \left| \Delta t \sum_{i=2M+1}^{\infty} a_s(i)[q_i(x) - xq_i(1)] \right|,$$

$$\begin{aligned} \|E_J\|^2 &= \left| \int_{-\infty}^{\infty} \left\langle \Delta t \sum_{i=2M+1}^{\infty} a_s(i)[q_i(x) - xq_i(1)], \right.\right. \\ &\qquad \left.\left. \Delta t \sum_{l=2M+1}^{\infty} a_s(l)[q_l(x) - xq_l(1)] \right\rangle dx \right| \\ &= \left| (\Delta t)^2 \sum_{i=2M+1}^{\infty} \sum_{l=2M+1}^{\infty} a_s(i) a_s(l) \int_0^1 [q_i(x) - xq_i(1)][q_l(x) \right. \\ &\qquad \left. - xq_l(1)] dx \right| \\ &\leq (\Delta t)^2 \sum_{i=2M+1}^{\infty} \sum_{l=2M+1}^{\infty} |a_s(i)||a_s(l)| C, \end{aligned}$$

where $C = \sup_{i,l} \int_0^1 [q_i(x) - xq_i(1)][q_l(x) - xq_l(1)]dx$. Thus, we obtain

$$\|E_J\|^2 \leq C(\Delta t)^2 \sum_{i=2M+1}^{\infty} |a_s(i)| \sum_{l=2M+1}^{\infty} |a_s(l)|. \tag{30}$$

Using Lemma 1, we have

$$\begin{aligned} \sum_{i=2M+1}^{\infty} |a_s(i)| &\leq \sum_{i=2M+1}^{\infty} 2^{-(3d+2)/2} K \\ &= K \sum_{d=J+1}^{\infty} \sum_{i=2^d+1}^{2^{d+1}} 2^{-(3d+2)/2} \\ &= K \sum_{d=J+1}^{\infty} 2^{-(3d+2)/2} 2^d = K \sum_{d=J+1}^{\infty} 2^{-(d+2)/2}. \end{aligned} \tag{31}$$

Hence, we get

$$\sum_{i=2M+1}^{\infty} |a_s(i)| \leq \frac{2^{-\frac{1}{2}(J+3)} K}{1 - 2^{-\frac{1}{2}}}. \tag{32}$$

Similarly,

$$\sum_{l=2M+1}^{\infty} |a_s(l)| \leq \frac{2^{-\frac{1}{2}(J+3)} K}{1 - 2^{-\frac{1}{2}}}. \tag{33}$$

Substituting equations (32) and (33) into equation (30), we obtain

$$\|E_J\|^2 \leq \frac{2^{-(J+3)} (\Delta t)^2 K^2 C}{(1 - 2^{-\frac{1}{2}})^2}. \tag{34}$$

Therefore,

$$\|E_J\| \leq \frac{2^{-\frac{1}{2}(J+3)} \Delta t K \sqrt{C}}{1 - 2^{-\frac{1}{2}}}. \tag{35}$$

□

It is clear from equation (35) that the error bound $\|E_J\| \to 0$ as $J \to \infty$. Hence the accuracy of the Haar wavelet method improves as the level of resolution J of the Haar wavelet is increased.

7 Results and discussions

Equation (5) is a Burgers equation modified by an extra term $\frac{ju}{2(t+1)}$ arising due to the effect of the non planar cylindrical ($j = 1$) or spherical ($j = 2$) geometry. The Haar wavelet collocation method (HWCM) solutions are obtained for $\alpha = 1, 2$, $j = 1, 2$ and $\epsilon = 0.1, 0.05, 0.01, 0.005$ with $\Delta t = 0.001$, $M = 64$. Lagrange's interpolation is used to find the solution at specified points. The results are compared with the solutions obtained from Finite difference method (FDM) with $\Delta t = 0.001$, $\Delta x = 0.001$ and are presented in Tables 1-4. The solutions obtained from both the methods are found to be in good agreement. It can be noted that the solution of non-planar Burgers equation by FDM requires nearly 1000 grid points whereas the solution by HWCM is approximately accurate using only 128 grid points. Srinivasa Rao [25] has solved Equation (5) using Hermite interpolation. The HWCM solution obtained by us are in good agreement with the solution of Srinivasa Rao [25]. But we observe that HWCM solution reveals shock structure which are not explored by Srinivasa Rao [25].

The exact solution of equation (5) for $\alpha = 1$ and $j = 0$ (planar geometry) obtained using Hopf-Cole transformation [20] is given by

$$u(x,t) = \frac{2\pi\epsilon \sum_{n=1}^{\infty} A_n n e^{-n^2\pi^2\epsilon t} \sin(n\pi x)}{A_0 + \sum_{n=1}^{\infty} A_n e^{-n^2\pi^2\epsilon t} \cos(n\pi x)},$$

where

$$A_0 = \int_0^1 \exp\left(-\left(\frac{1-\cos(\pi x)}{2\pi\epsilon}\right)\right) dx,$$

$$A_n = 2\int_0^1 \exp\left(-\left(\frac{1-\cos(\pi x)}{2\pi\epsilon}\right)\right) \cos(n\pi x)\, dx.$$

The HWCM solution and the exact solution are found to be in good agreement and are presented in Table 5.

In order to measure the accuracy of the solutions obtained by HWCM, we define the error estimate at $t = t_s$ by

$$\mu(t_s) = \frac{1}{2M} \left\| u(x,t_s) - u_{ex}(x,t_s) \right\|,$$

where $u_{ex}(x,t_s)$ is the exact solution at $t = t_s$. The L_2 and L_∞ error norms are calculated for $\epsilon = 0.1, 0.05, 0.01, 0.005$ with $\Delta t = 0.001$, $M = 64$ and are presented in Table 6.

The HWCM solution for $\alpha = 1$, $\epsilon = 0.1$, $j = 1$ (cylindrical symmetry) and $j = 2$ (spherical symmetry) are presented in Figure 1 and for $\epsilon = 0.005$ in Figure 4. The figures are depicted up to time $t \leq 1$. We observe that the graphs are becoming steep as ϵ decreases for both cylindrical as well as spherical geometry. The HWCM solution for $\alpha = 2$, $\epsilon = 0.1$, $j = 1$ (cylindrical symmetry) and $j = 2$ (spherical symmetry) are presented in Figure 7 and for $\epsilon = 0.005$ in Figure 10. The steepening of the graphs continues with decreasing ϵ for both cylindrical as well as spherical geometry even as the nonlinearity increases. The physical behaviour of the HWCM solution in contour and 3D for $\alpha = 1$, $\epsilon = 0.1$, $j = 1$ (cylindrical symmetry) and $j = 2$ (spherical symmetry) are presented in Figures 2 and 3 and for $\epsilon = 0.005$ in Figures 5 and 6. The physical behaviour of the HWCM solution in contour and 3D for $\alpha = 2$, $\epsilon = 0.1$, $j = 1$ (cylindrical symmetry) and $j = 2$ (spherical symmetry) are presented in Figures 8 and 9 and for $\epsilon = 0.005$ in Figures 11 and 12. The steepening of the graphs as ϵ decreases can also be observed in these contour and 3D graphs.

7.1 Shock analysis

We have studied the effects of planar ($j = 0$), cylindrical ($j = 1$) and spherical ($j = 2$) geometries on the time dependent shock waves. It is observed that the shock structures exist in both planar and non-planar systems. The formation of shock structures depends on the geometry of the source. The effects of the geometry on the shock amplitude have been explicitly observed. The HWCM solution reveals that for a large value of t the cylindrical and spherical shock structures are similar to one dimensional planar ones. This is because for large t the term $\frac{ju}{2(t+1)}$ which is due to the effect of cylindrical and spherical geometry becomes negligible. However as t decreases the term $\frac{ju}{2(t+1)}$ becomes important, and both the cylindrical and spherical shocks significantly differ from one dimensional planar ones.

For $\alpha = 1$, it is observed that shocks exist in the range $x \in (0.9, 1)$ at time $t = 0.32$ for planar geometry, at $t = 0.35$ for cylindrical geometry and at $t = 0.38$ for spherical geometry (see Figures 13, 14 and 15). It is observed that the shocks are delayed from planar to cylindrical and cylindrical to spherical geometries. Similarly, for $\alpha = 2$, that is, when the nonlinearity increases, it is observed that shocks exist in the range $x \in (0.8, 1)$ at time $t = 0.36$ for planar geometry, at $t = 0.45$ for cylindrical geometry and at $t = 0.6$ for spherical geometry (see Figures 16, 17 and 18). Here the shocks are much more delayed from planar to cylindrical and cylindrical to spherical geometries. It is found that as the value of t decreases, the amplitude of these shocks increases and it is also found that the amplitude of the cylindrical shock is larger than that of the spherical ones.

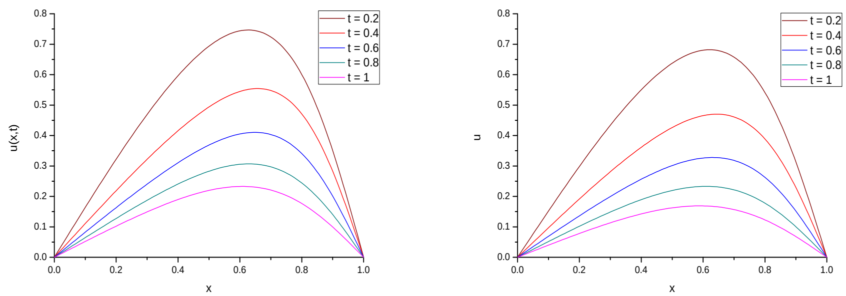

Figure 1: HWCM solution for $\alpha = 1$, $j = 1$ (left), $j = 2$ (right) and $\epsilon = 0.1$ at different times t with $\Delta t = 0.001$.

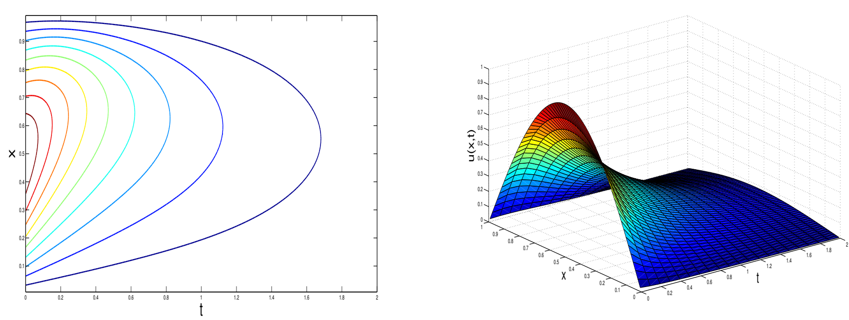

Figure 2: Physical behaviour of the HWCM solution for $\alpha = 1$, $j = 1$ and $\epsilon = 0.1$.

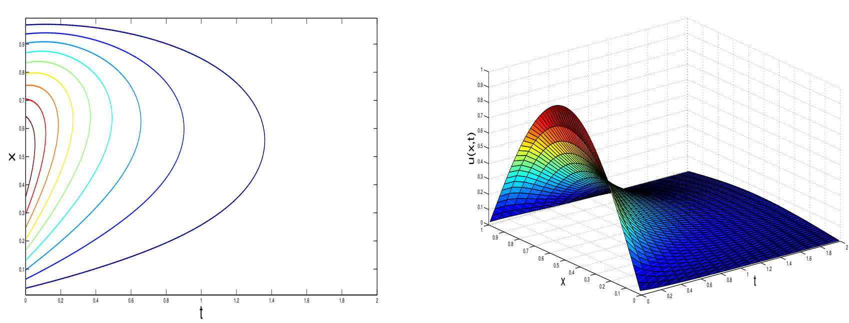

Figure 3: Physical behaviour of the HWCM solution for $\alpha = 1$, $j = 2$ and $\epsilon = 0.1$.

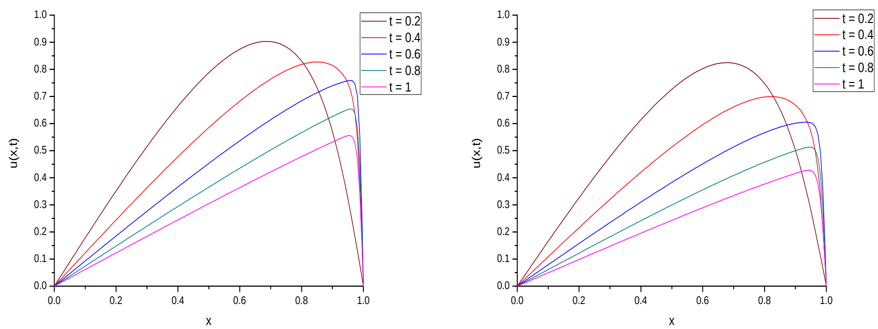

Figure 4: HWCM solution for $\alpha = 1$, $j = 1$ (left), $j = 2$ (right) and $\epsilon = 0.005$ at different times t with $\Delta t = 0.001$.

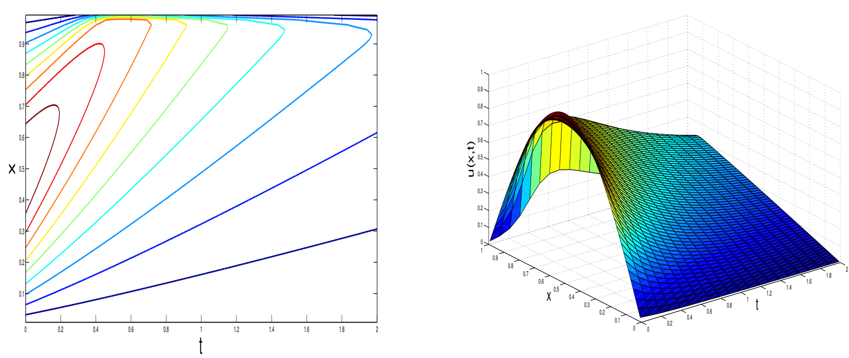

Figure 5: Physical behaviour of the HWCM solution for $\alpha = 1$, $j = 1$ and $\epsilon = 0.005$.

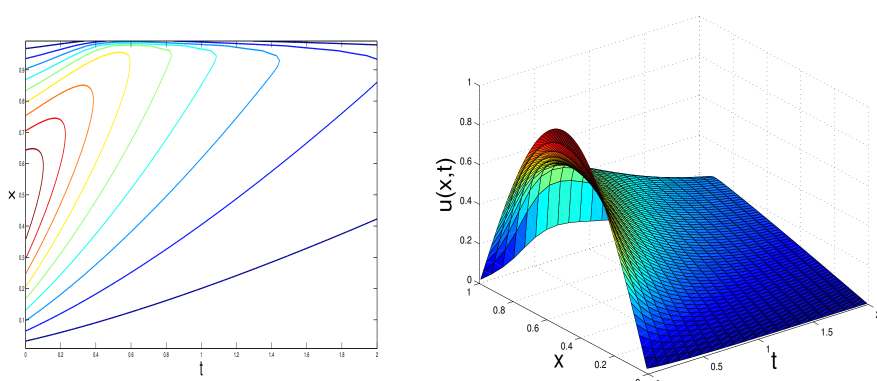

Figure 6: Physical behaviour of the HWCM solution for $\alpha = 1$, $j = 2$ and $\epsilon = 0.005$.

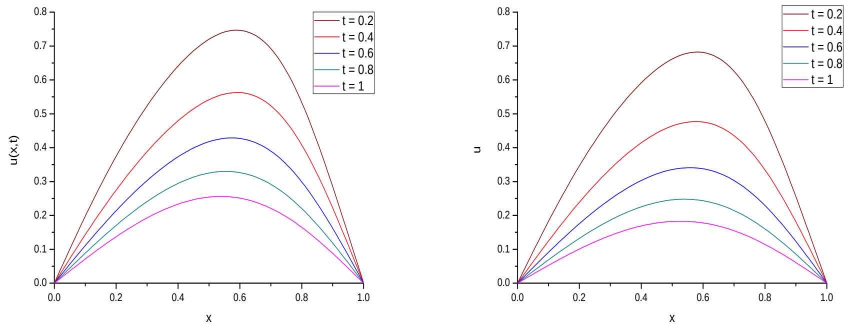

Figure 7: HWCM solution for $\alpha = 2$, $j = 1$ (left), $j = 2$ (right) and $\epsilon = 0.1$ at different times t with $\Delta t = 0.001$.

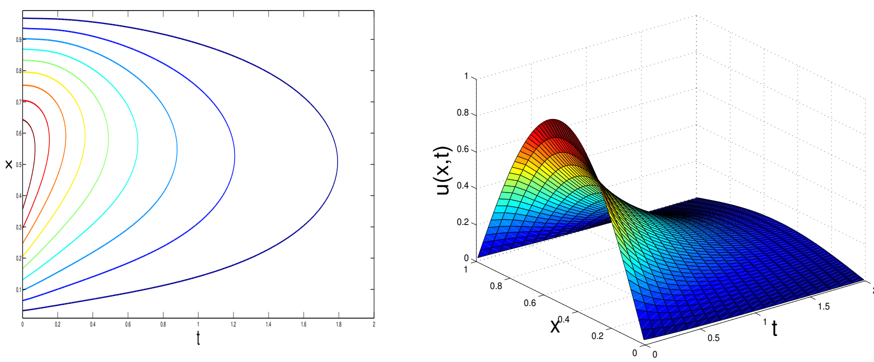

Figure 8: Physical behaviour of the HWCM solution for $\alpha = 2$, $j = 1$ and $\epsilon = 0.1$.

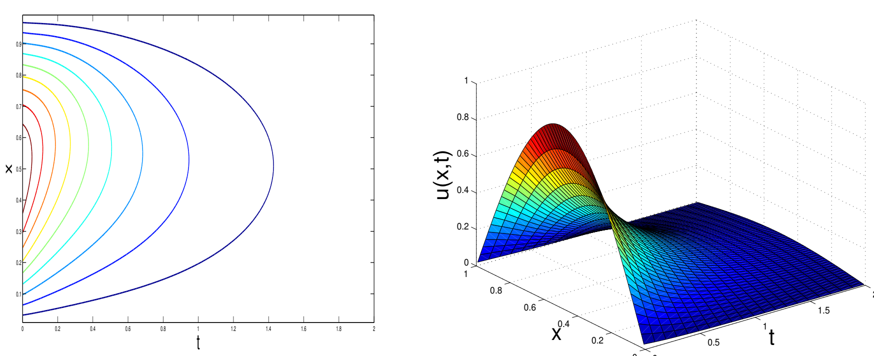

Figure 9: Physical behaviour of the HWCM solution for $\alpha = 2$, $j = 2$ and $\epsilon = 0.1$.

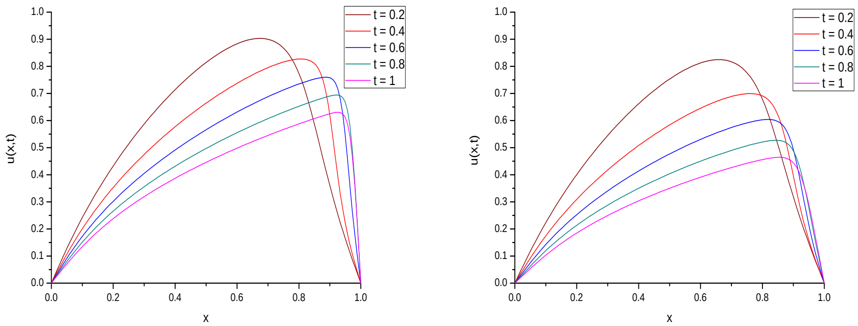

Figure 10: HWCM solution for $\alpha = 2$, $j = 1$ (left), $j = 2$ (right) and $\epsilon = 0.005$ at different times t with $\Delta t = 0.001$.

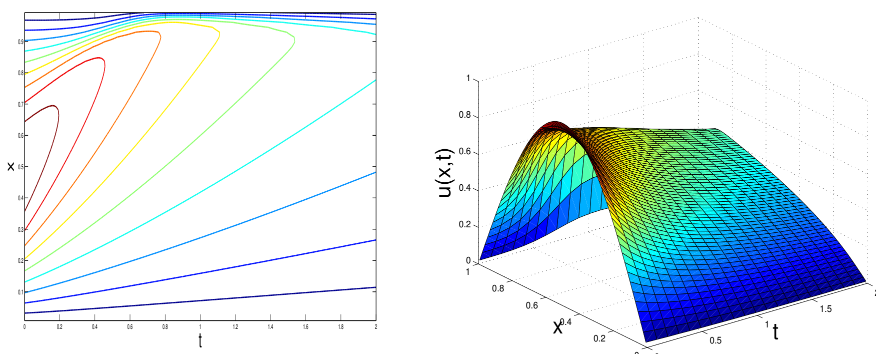

Figure 11: Physical behaviour of the HWCM solution for $\alpha = 2$, $j = 1$ and $\epsilon = 0.0.005$.

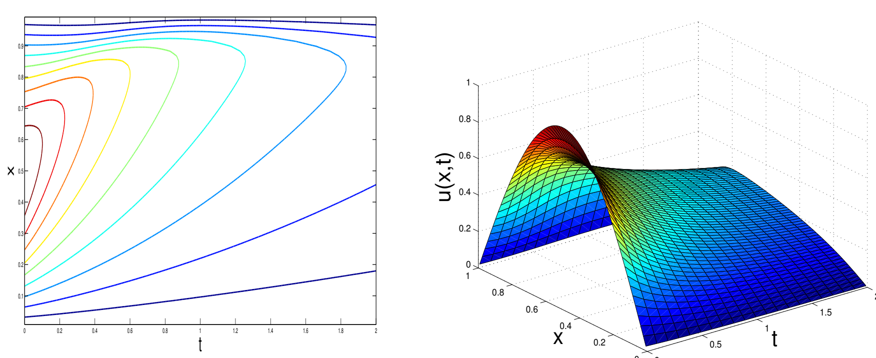

Figure 12: Physical behaviour of the HWCM solution for $\alpha = 2$, $j = 2$ and $\epsilon = 0.005$.

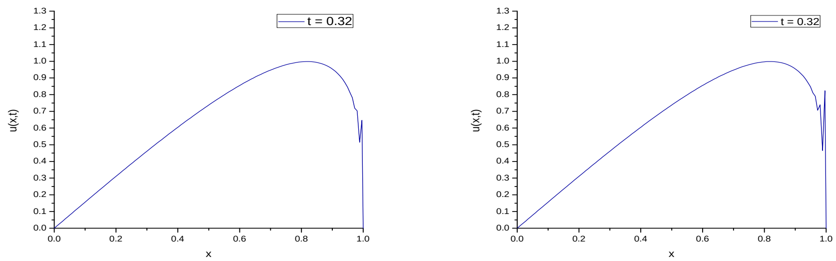

Figure 13: HWCM solution for $\alpha = 1$, $j = 0$, $\epsilon = 0.0001$ (left) and $\epsilon = 0.0000001$ (right) with $\Delta t = 0.001$.

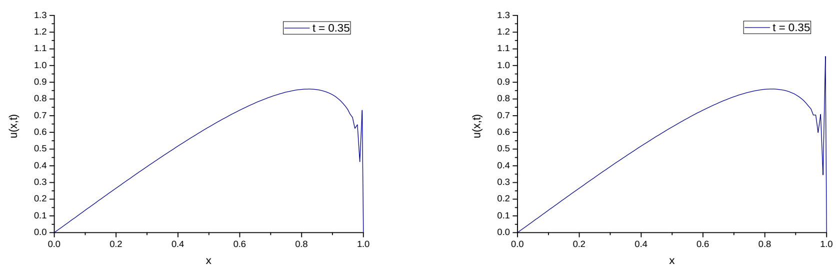

Figure 14: HWCM solution for $\alpha = 1$, $j = 1$, $\epsilon = 0.0001$ (left) and $\epsilon = 0.0000001$ (right) with $\Delta t = 0.001$.

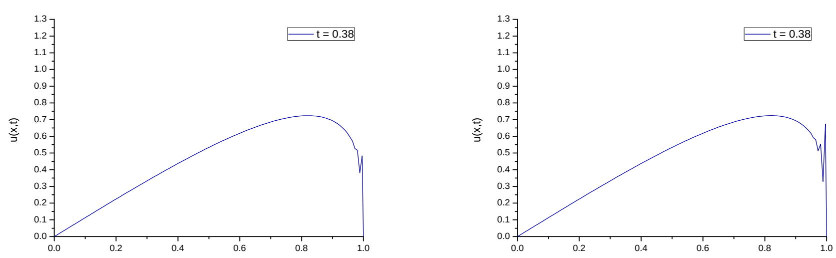

Figure 15: HWCM solution for $\alpha = 1$, $j = 2$, $\epsilon = 0.0001$ (left) and $\epsilon = 0.0000001$ (right) with $\Delta t = 0.001$.

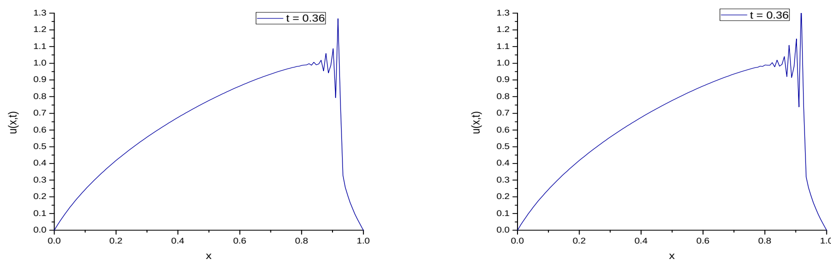

Figure 16: HWCM solution for $\alpha = 2$, $j = 0$, $\epsilon = 0.0001$ (left) and $\epsilon = 0.0000001$ (right) with $\Delta t = 0.001$.

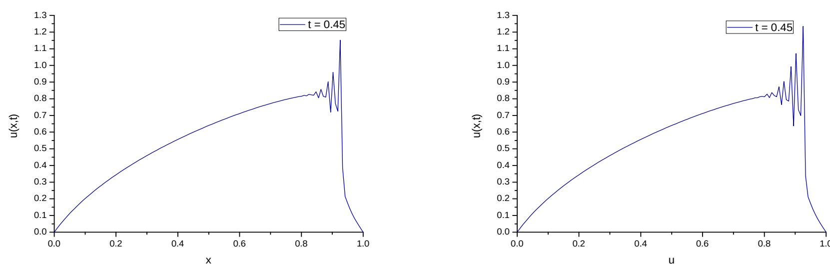

Figure 17: HWCM solution for $\alpha = 2$, $j = 1$, $\epsilon = 0.0001$ (left) and $\epsilon = 0.0000001$ (right) with $\Delta t = 0.001$.

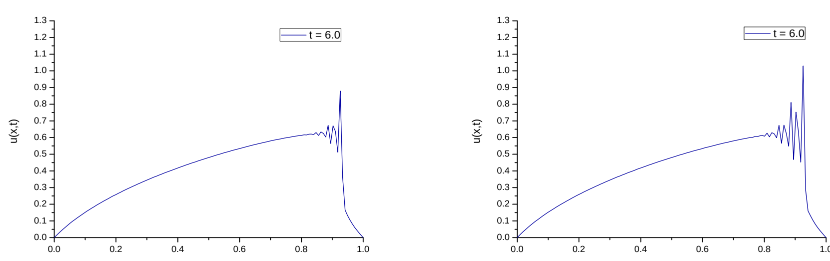

Figure 18: HWCM solution for $\alpha = 2$, $j = 2$, $\epsilon = 0.0001$ (left) and $\epsilon = 0.0000001$ (right) with $\Delta t = 0.001$.

Table 1: Comparison of the HWCM solution and FDM solution for $\alpha = 1$, $j = 1$ and $\epsilon = 0.1, 0.05, 0.01, 0.005$ at different times t and x.

x	t	$u(x,t)$			
		HWCM	FDM	HWCM	FDM
		$\epsilon = 0.1$		$\epsilon = 0.05$	
0.25	0.2	0.3977430	0.3978635	0.4179511	0.4180571
	0.4	0.2712137	0.2713318	0.2903930	0.2904786
	0.6	0.2022805	0.2023968	0.2203689	0.2204376
	0.8	0.1579861	0.1581049	0.1764700	0.1765276
	1.0	0.1262658	0.1263862	0.1464795	0.1465306
	2.0	0.0438310	0.0439029	0.0749411	0.0749972
	3.0	0.0149376	0.0149572	0.0437530	0.0438102
	4.0	0.0050802	0.0050770	0.0256629	0.0257030
	5.0	0.0017500	0.0017383	0.0149237	0.0149450
0.50	0.2	0.6933896	0.6937845	0.7442324	0.7446282
	0.4	0.4937264	0.4941208	0.5487110	0.5490260
	0.6	0.3686424	0.3690213	0.4252621	0.4255058
	0.8	0.2826228	0.2829794	0.3433262	0.3435345
	1.0	0.2196262	0.2199482	0.2852889	0.2854852
	2.0	0.0670473	0.0671717	0.1372345	0.1374253
	3.0	0.0216801	0.0217097	0.0729007	0.0730319
	4.0	0.0072467	0.0072421	0.0399176	0.0399910
	5.0	0.0024821	0.0024654	0.0222753	0.0223100
0.75	0.2	0.6771242	0.6778301	0.7806894	0.7815773
	0.4	0.5217762	0.5226810	0.6807172	0.6818740
	0.6	0.3810644	0.3818584	0.5542190	0.5553021
	0.8	0.2768852	0.2775048	0.4471790	0.4481625
	1.0	0.2030481	0.2035122	0.3620212	0.3628951
	2.0	0.0516072	0.0517144	0.1391561	0.1395295
	3.0	0.0157433	0.0157655	0.0626479	0.0628012
	4.0	0.0051689	0.0051656	0.0313354	0.0314023
	5.0	0.0017603	0.0017484	0.0166717	0.0166998
		$\epsilon = 0.01$		$\epsilon = 0.005$	
0.25	0.2	0.4332243	0.4333149	0.4350776	0.4351662
	0.4	0.3028405	0.3028985	0.3042578	0.3043122
	0.6	0.2299961	0.2300353	0.2310289	0.2310645
	0.8	0.1841292	0.1841581	0.1849055	0.1849311
	1.0	0.1528202	0.1528428	0.1534242	0.1534442
	2.0	0.0805381	0.0805488	0.0807827	0.0807928
	3.0	0.0536871	0.0536938	0.0538226	0.0538298
	4.0	0.0399197	0.0399242	0.0400072	0.0400130
	5.0	0.0316177	0.0316207	0.0316817	0.0316865
0.50	0.2	0.7828017	0.7831855	0.7874634	0.7878451
	0.4	0.5815200	0.5817585	0.5850879	0.5853183
	0.6	0.4504626	0.4506043	0.4529476	0.4530813
	0.8	0.3636685	0.3637600	0.3654691	0.3655540
	1.0	0.3031158	0.3031797	0.3044801	0.3045386
	2.0	0.1607260	0.1607461	0.1612459	0.1612646
	3.0	0.1072681	0.1072780	0.1075528	0.1075630
	4.0	0.0797637	0.0797703	0.0799782	0.0799847
	5.0	0.0630571	0.0630648	0.0633471	0.0633516
0.75	0.2	0.8714380	0.8724533	0.8828280	0.8838540
	0.4	0.7860245	0.7868398	0.7954457	0.7962190
	0.6	0.6444616	0.6448931	0.6501302	0.6505281
	0.8	0.5313554	0.5316013	0.5350542	0.5352779
	1.0	0.4472070	0.4473609	0.4498250	0.4499636
	2.0	0.2400135	0.2400551	0.2410078	0.2410351
	3.0	0.1593601	0.1594267	0.1610796	0.1610882
	4.0	0.1157131	0.1158172	0.1198160	0.1198219
	5.0	0.0876478	0.0877654	0.0947297	0.0947425

Table 2: Comparison of the HWCM solution and FDM solution for $\alpha = 1$, $j = 2$ and $\epsilon = 0.1, 0.05, 0.01, 0.005$ at different times t and x.

x	t	$u(x,t)$			
		HWCM	FDM	HWCM	FDM
		$\epsilon = 0.1$		$\epsilon = 0.05$	
0.25	0.2	0.3679063	0.3680170	0.3871136	0.3872115
	0.4	0.2371068	0.2372115	0.2552672	0.2553448
	0.6	0.1685392	0.1686392	0.1858973	0.1859596
	0.8	0.1258408	0.1259382	0.1437897	0.1438427
	1.0	0.0962326	0.0963252	0.1157586	0.1158065
	2.0	0.0272142	0.0272510	0.0517684	0.0518155
	3.0	0.0079283	0.0079310	0.0264189	0.0264535
	4.0	0.0024009	0.0023917	0.0137083	0.0137250
	5.0	0.0007587	0.0007456	0.0071904	0.0071942
0.50	0.2	0.6372362	0.6375924	0.6851780	0.6855391
	0.4	0.4261469	0.4264855	0.4780113	0.4782969
	0.6	0.3012237	0.3015312	0.3549454	0.3551711
	0.8	0.2193689	0.2196382	0.2761681	0.2763652
	1.0	0.1623760	0.1626013	0.2217554	0.2219406
	2.0	0.0406150	0.0406753	0.0907497	0.0908850
	3.0	0.0113807	0.0113848	0.0419491	0.0420181
	4.0	0.0034101	0.0033971	0.0205472	0.0205752
	5.0	0.0010744	0.0010558	0.0104695	0.0104756
0.75	0.2	0.6125294	0.6131342	0.7063654	0.7071321
	0.4	0.4348874	0.4355837	0.5728822	0.5738464
	0.6	0.2970322	0.2975899	0.4414657	0.4423518
	0.8	0.2039637	0.2043659	0.3390805	0.3398462
	1.0	0.1424893	0.1427709	0.2621613	0.2627969
	2.0	0.0303882	0.0304370	0.0842191	0.0844210
	3.0	0.0081698	0.0081729	0.0337275	0.0337956
	4.0	0.0024218	0.0024126	0.0154441	0.0154673
	5.0	0.0007607	0.0007475	0.0076276	0.0076323
		$\epsilon = 0.01$		$\epsilon = 0.005$	
0.25	0.2	0.4016669	0.4017512	0.4034348	0.4035172
	0.4	0.2671096	0.2671630	0.2684587	0.2685089
	0.6	0.1951803	0.1952164	0.1961720	0.1962049
	0.8	0.1513412	0.1513677	0.1520955	0.1521192
	1.0	0.1222063	0.1222270	0.1228010	0.1228193
	2.0	0.0583247	0.0583340	0.0585820	0.0585908
	3.0	0.0363930	0.0363984	0.0365445	0.0365503
	4.0	0.0257537	0.0257569	0.0258585	0.0258627
	5.0	0.0195999	0.0196019	0.0196897	0.0196929
0.50	0.2	0.7217743	0.7221274	0.7262087	0.7265602
	0.4	0.5096391	0.5098601	0.5130789	0.5132927
	0.6	0.3801237	0.3802578	0.3825668	0.3826935
	0.8	0.2974718	0.2975600	0.2992745	0.2993563
	1.0	0.2413887	0.2414510	0.2427774	0.2428345
	2.0	0.1161169	0.1161368	0.1166851	0.1167032
	3.0	0.0725761	0.0725862	0.0729200	0.0729291
	4.0	0.0512751	0.0512838	0.0516357	0.0516409
	5.0	0.0387059	0.0387164	0.0393314	0.0393345
0.75	0.2	0.7898351	0.7907277	0.8004290	0.8013336
	0.4	0.6749730	0.6757356	0.6844310	0.6851563
	0.6	0.5353687	0.5358003	0.5413735	0.5417684
	0.8	0.4294112	0.4296684	0.4334433	0.4336737
	1.0	0.3526947	0.3528614	0.3556061	0.3557529
	2.0	0.1721416	0.1722090	0.1736876	0.1737197
	3.0	0.1052453	0.1053407	0.1088740	0.1088884
	4.0	0.0701635	0.0702622	0.0769100	0.0769291
	5.0	0.0489550	0.0490357	0.0579139	0.0579440

Table 3: Comparison of the HWCM solution and FDM solution for $\alpha = 2$, $j = 1$ and $\epsilon = 0.1, 0.05, 0.01, 0.005$ at different times t and x.

x	t	$u(x,t)$			
		HWCM	FDM	HWCM	FDM
		$\epsilon = 0.1$		$\epsilon = 0.05$	
0.25	0.2	0.4528947	0.4530076	0.4843711	0.4844762
	0.4	0.3340788	0.3342115	0.3749329	0.3750457
	0.6	0.2604422	0.2605847	0.3069631	0.3070730
	0.8	0.2075421	0.2076866	0.2600208	0.2601290
	1.0	0.1664093	0.1665460	0.2251935	0.2253036
	2.0	0.0542853	0.0543373	0.1246904	0.1248154
	3.0	0.0177153	0.0177229	0.0706156	0.0707061
	4.0	0.0059262	0.0059182	0.0396061	0.0396562
	5.0	0.0020284	0.0020150	0.0222823	0.0223051
0.50	0.2	0.7199254	0.7202749	0.7736452	0.7739859
	0.4	0.5413433	0.5417245	0.6185650	0.6188992
	0.6	0.4175751	0.4179407	0.5153752	0.5157088
	0.8	0.3250077	0.3253298	0.4399164	0.4402609
	1.0	0.2540243	0.2542924	0.3805254	0.3808806
	2.0	0.0776199	0.0776969	0.1953196	0.1955939
	3.0	0.0250847	0.0250955	0.1038516	0.1040015
	4.0	0.0083820	0.0083708	0.0567600	0.0568347
	5.0	0.0028686	0.0028497	0.0316487	0.0316816
0.75	0.2	0.6264105	0.6269270	0.7323974	0.7332051
	0.4	0.4756745	0.4762611	0.6440543	0.6451882
	0.6	0.3510169	0.3514890	0.5387584	0.5398593
	0.8	0.2604508	0.2607999	0.4455024	0.4464743
	1.0	0.1958994	0.1961535	0.3681514	0.3689670
	2.0	0.0554990	0.0555556	0.1556991	0.1559931
	3.0	0.0177598	0.0177675	0.0764906	0.0766130
	4.0	0.0059278	0.0059199	0.0406777	0.0407332
	5.0	0.0020284	0.0020150	0.0224764	0.0225001
		$\epsilon = 0.01$		$\epsilon = 0.005$	
0.25	0.2	0.5105786	0.5106768	0.5138261	0.5139236
	0.4	0.4121276	0.4122331	0.4170524	0.4171569
	0.6	0.3497798	0.3498836	0.3561311	0.3562347
	0.8	0.3056630	0.3057624	0.3132191	0.3133200
	1.0	0.2723769	0.2724709	0.2809017	0.2809991
	2.0	0.1796682	0.1797364	0.1903981	0.1904765
	3.0	0.1358182	0.1358691	0.1466789	0.1467417
	4.0	0.1098848	0.1099247	0.1202931	0.1203445
	5.0	0.0925553	0.0925898	0.1024592	0.1025021
0.50	0.2	0.8100523	0.8103532	0.8141415	0.8144371
	0.4	0.6600078	0.6602336	0.6641056	0.6643237
	0.6	0.5616494	0.5618314	0.5658449	0.5660202
	0.8	0.4925377	0.4926943	0.4969601	0.4971111
	1.0	0.4409447	0.4410846	0.4456593	0.4457944
	2.0	0.2992386	0.2993367	0.3055537	0.3056512
	3.0	0.2318738	0.2319559	0.2395078	0.2395875
	4.0	0.1905133	0.1906009	0.1997534	0.1998213
	5.0	0.1608477	0.1609491	0.1726899	0.1727498
0.75	0.2	0.8505220	0.8518072	0.8675212	0.8688478
	0.4	0.8053124	0.8061938	0.8159282	0.8166348
	0.6	0.7064960	0.7069569	0.7128052	0.7131723
	0.8	0.6255118	0.6258176	0.6304414	0.6306904
	1.0	0.5624665	0.5627032	0.5668848	0.5670779
	2.0	0.3832456	0.3835065	0.3902102	0.3903163
	3.0	0.2865587	0.2869737	0.3066685	0.3067717
	4.0	0.2185100	0.2189222	0.2548806	0.2550426
	5.0	0.1692750	0.1696113	0.2167176	0.2169456

Table 4: Comparison of the HWCM solution and FDM solution for $\alpha = 2$, $j = 2$ and $\epsilon = 0.1, 0.05, 0.01, 0.005$ at different times t and x.

x	t	$u(x,t)$			
		HWCM	FDM	HWCM	FDM
		$\epsilon = 0.1$		$\epsilon = 0.05$	
0.25	0.2	0.4187385	0.4188396	0.4485032	0.4485973
	0.4	0.2902983	0.2904100	0.3280608	0.3281571
	0.6	0.2134813	0.2135922	0.2560097	0.2561017
	0.8	0.1606390	0.1607409	0.2076148	0.2077046
	1.0	0.1218622	0.1219492	0.1724709	0.1725601
	2.0	0.0318867	0.0319062	0.0780948	0.0781649
	3.0	0.0089768	0.0089727	0.0374784	0.0375134
	4.0	0.0026898	0.0026785	0.0185771	0.0185895
	5.0	0.0008463	0.0008325	0.0094973	0.0094975
0.50	0.2	0.6615450	0.6618527	0.7133077	0.7136176
	0.4	0.4638673	0.4641711	0.5375510	0.5378512
	0.6	0.3352542	0.3355144	0.4248405	0.4251313
	0.8	0.2457882	0.2459944	0.3445913	0.3448717
	1.0	0.1820340	0.1821905	0.2835205	0.2837832
	2.0	0.0453533	0.0453818	0.1177835	0.1179153
	3.0	0.0127013	0.0126956	0.0540526	0.0541069
	4.0	0.0038041	0.0037881	0.0264171	0.0264352
	5.0	0.0011969	0.0011773	0.0134515	0.0134519
0.75	0.2	0.5635353	0.5639462	0.6559812	0.6566117
	0.4	0.3923610	0.3927623	0.5269125	0.5276935
	0.6	0.2706703	0.2709609	0.4097215	0.4104027
	0.8	0.1901735	0.1903723	0.3186327	0.3191753
	1.0	0.1365276	0.1366636	0.2501433	0.2505615
	2.0	0.0322543	0.0322750	0.0892260	0.0893458
	3.0	0.0089857	0.0089816	0.0389860	0.0390277
	4.0	0.0026900	0.0026787	0.0187830	0.0187963
	5.0	0.0008464	0.0008325	0.0095260	0.0095263
		$\epsilon = 0.01$		$\epsilon = 0.005$	
0.25	0.2	0.4732031	0.4732905	0.4762631	0.4763498
	0.4	0.3615783	0.3616651	0.3659733	0.3660592
	0.6	0.2932951	0.2933751	0.2986737	0.2987531
	0.8	0.2464687	0.2465412	0.2525810	0.2526537
	1.0	0.2121445	0.2122100	0.2187750	0.2188414
	2.0	0.1225112	0.1225520	0.1297743	0.1298186
	3.0	0.0841735	0.0842019	0.0908736	0.0909052
	4.0	0.0630663	0.0630895	0.0691458	0.0691694
	5.0	0.0495985	0.0496202	0.0553596	0.0553779
0.50	0.2	0.7490511	0.7493305	0.7530769	0.7533516
	0.4	0.5794545	0.5796626	0.5834873	0.5836873
	0.6	0.4725852	0.4727485	0.4765814	0.4767366
	0.8	0.3993662	0.3995019	0.4033976	0.4035257
	1.0	0.3458978	0.3460149	0.3500042	0.3501141
	2.0	0.2063116	0.2063888	0.2110088	0.2110746
	3.0	0.1446658	0.1447430	0.1506666	0.1507140
	4.0	0.1082401	0.1083210	0.1166734	0.1167125
	5.0	0.0834973	0.0835726	0.0946236	0.0946608
0.75	0.2	0.7604571	0.7615018	0.7763880	0.7774979
	0.4	0.6837758	0.6847965	0.6988899	0.6997210
	0.6	0.5805158	0.5811994	0.5912271	0.5916892
	0.8	0.4975600	0.4980881	0.5065717	0.5068804
	1.0	0.4333128	0.4337788	0.4420488	0.4422819
	2.0	0.2499374	0.2503959	0.2686685	0.2688179
	3.0	0.1582619	0.1586006	0.1892567	0.1894588
	4.0	0.1063545	0.1065700	0.1402960	0.1405077
	5.0	0.0753124	0.0754506	0.1069869	0.1071662

Table 5: Comparison of the HWCM solution and exact solution for $\alpha = 1$, $j = 0$ and $\epsilon = 0.1, 0.05, 0.01, 0.005$ at different times t and x.

x	t	$u(x,t)$			
		HWCM	FDM	HWCM	FDM
		$\epsilon = 0.1$		$\epsilon = 0.05$	
0.25	0.2	0.1627137	0.1625649	0.1823093	0.1821455
	0.4	0.0683010	0.0682061	0.1033053	0.1032236
	0.6	0.0272652	0.0272023	0.0671050	0.0670443
	0.8	0.0104604	0.0104201	0.0439536	0.0439027
	1.0	0.0039526	0.0039245	0.0282842	0.0282411
	2.0	0.0005678	0.0005481	0.0111740	0.0111433
	3.0	0.0000463	0.0000284	0.0026285	0.0026068
	4.0	0.0000217	0.0000039	0.0009961	0.0009767
	5.0	0.0000180	0.0000002	0.0002410	0.0002228
0.50	0.2	0.2921908	0.2919160	0.3592228	0.3589297
	0.4	0.1080589	0.1078901	0.1964300	0.1962525
	0.6	0.0403025	0.0402049	0.1180458	0.1179171
	0.8	0.0150437	0.0149846	0.0719243	0.0718280
	1.0	0.0056252	0.0055849	0.0439108	0.0438373
	2.0	0.0008037	0.0007758	0.0163844	0.0163377
	3.0	0.0000655	0.0000402	0.0037487	0.0037174
	4.0	0.0000307	0.0000056	0.0014132	0.0013855
	5.0	0.0000254	0.0000003	0.0003410	0.0003153
0.75	0.2	0.2877731	0.2874744	0.4851991	0.4848389
	0.4	0.0867329	0.0865786	0.2201513	0.2198857
	0.6	0.0298481	0.0297721	0.1110480	0.1108935
	0.8	0.0108207	0.0107774	0.0603844	0.0602892
	1.0	0.0040030	0.0039741	0.0344180	0.0343542
	2.0	0.0005688	0.0005491	0.0120285	0.0119929
	3.0	0.0000463	0.0000284	0.0026733	0.0026508
	4.0	0.0000217	0.0000039	0.0010025	0.0009828
	5.0	0.0000180	0.0000002	0.0002413	0.0002231
		$\epsilon = 0.01$		$\epsilon = 0.005$	
0.25	0.2	0.1883804	0.1881940	0.1889774	0.1887881
	0.4	0.1074645	0.1073814	0.1076840	0.1076005
	0.6	0.0751607	0.0751141	0.0752724	0.0752266
	0.8	0.0577881	0.0577582	0.0578548	0.0578265
	1.0	0.0469385	0.0469174	0.0469824	0.0469634
	2.0	0.0341160	0.0341030	0.0341473	0.0341375
	3.0	0.0240792	0.0240695	0.0242216	0.0242168
	4.0	0.0199246	0.0199150	0.0202894	0.0202858
	5.0	0.0153857	0.0153752	0.0163106	0.0163076
0.50	0.2	0.3747644	0.3744200	0.3760744	0.3757228
	0.4	0.2147271	0.2145581	0.2151822	0.2150117
	0.6	0.1502777	0.1501790	0.1505061	0.1504083
	0.8	0.1155605	0.1154948	0.1157006	0.1156373
	1.0	0.0938445	0.0937961	0.0939645	0.0939201
	2.0	0.0679596	0.0679265	0.0682987	0.0682728
	3.0	0.0469168	0.0468901	0.0484369	0.0484214
	4.0	0.0378185	0.0377930	0.0405376	0.0405245
	5.0	0.0278352	0.0278103	0.0324509	0.0324388
0.75	0.2	0.5564758	0.5560507	0.5588247	0.5583839
	0.4	0.3215393	0.3212820	0.3222837	0.3220245
	0.6	0.2249789	0.2248112	0.2256548	0.2254976
	0.8	0.1718995	0.1717713	0.1735189	0.1734119
	1.0	0.1374126	0.1373069	0.1409118	0.1408316
	2.0	0.0936860	0.0936092	0.1021658	0.1021091
	3.0	0.0568555	0.0568039	0.0711782	0.0711338
	4.0	0.0420047	0.0419626	0.0581392	0.0580996
	5.0	0.0276024	0.0275691	0.0441670	0.0441329

Table 6: Error values of the solution of the non-planar Burgers equation for $\alpha = 1$, $j = 0$ and $\epsilon = 0.1, 0.05, 0.01, 0.005$ at different times t.

t	$\mu(t)$			
	L_2	L_∞	L_2	L_∞
	$\epsilon = 0.1$		$\epsilon = 0.05$	
1.0	1.912E-05	2.462E-06	2.185E-05	2.815E-06
2.0	1.094E-05	10398E-06	1.470E-05	2.074E-06
3.0	6.131E-06	7.691E-07	9.288E-06	1.250E-06
4.0	3.694E-06	4.619E-07	6.398E-06	8.286E-07
5.0	2.522E-06	3.152E-07	4.074E-06	5.967E-07
7.0	1.746E-06	2.182E-07	2.931E-06	3.671E-07
10.0	1.581E-06	1.976E-07	1.953E-06	2.441E-07
12.0	1.571E-06	1.963E-07	1.731E-06	2.163E-07
15.0	1.569E-06	1.961E-07	1.612E-06	2.015E-07
	$\epsilon = 0.01$		$\epsilon = 0.005$	
1.0	4.569E-05	1.704E-05	5.612E-05	4.413E-05
2.0	2.279E-05	5.866E-06	3.302E-05	1.455E-05
3.0	1.323E-05	2.881E-06	1.714E-05	5.894E-06
4.0	8.821E-06	1.742E-06	1.076E-05	3.185E-06
5.0	6.433E-06	1.189E-06	7.532E-06	1.991E-06
7.0	4.069E-06	6.876E-07	4.486E-06	1.021E-06
10.0	2.659E-06	4.104E-07	2.725E-06	5.429E-07
12.0	2.232E-06	3.285E-07	2.183E-06	4.091E-07
15.0	1.895E-06	2.637E-07	1.739E-06	3.035E-07

8 Conclusion

The present work is the most general study for non-planar Burgers equation. We obtain single hump solution for $\epsilon = 0.1, 0.05, 0.01, 0.005$ and shocks are observed for smaller ϵ for both cylindrical and spherical geometries. We recover the work of Ram Jiwari [12] for $j = 0$ and $\alpha = 1$. They have not identified shocks whereas we have gone beyond and identified discontinuities for $j = 0$ and $\alpha = 1, 2$. The work of Ram Jiwari [12] was a particular case of our work. In the limiting cases, we have also compared the HWCM solution with the existing exact solution and they are found to be in good agreement. We conclude that Haar wavelet collocation method is more efficient to capture discontinuities accurately with less number of grid points whereas this observation is not reported by other conventional methods.

Acknowledgements

The second author is thankful to the Vision Group of Science and Technology (VGST) for providing financial assistance through GRD 105, CISE.

Appendix 1: Quasilinearization Technique

In order to solve the individual or systems of nonlinear ordinary and partial differential equations, Bellman and Kalaba [2] introduced the quasilinearization approach as a generalization of the Newton-Raphson method. This technique has quadratic rate of convergence.

Equation (5) can be rewritten in the form

$$u_t + \frac{ju}{2(t+1)} = \epsilon u_{xx} - g(u, u_x), \tag{36}$$

where $g(u, u_x) = u^\alpha u_x$.

The following approximation scheme is obtained by application of quasilinearization technique to equation (36):

$$\begin{aligned}(u_{s+1})_t + \left(\frac{ju}{2(t+1)}\right)_{s+1} &= \epsilon(u_{s+1})_{xx} - g(u_s, (u_s)_x) - (u_{s+1} \\ &-u_s)g_u(u_s, (u_s)_x) - ((u_{s+1})_x - (u_s)_x)g_{u_x}(u_s, (u_s)_x).\end{aligned} \tag{37}$$

Thus the nonlinear non-planar Burgers equation (5) followed by quasilinearization leads to equation (5.1).

Appendix 2: Thomas Algorithm

Thomas algorithm [4] is a computational procedure to solve tridiagonal system of equations. This algorithm is a simplified form of Gaussian elimination. For each $s = 1, 2, \ldots, N$, the tridiagonal system of equations in equation (25) can be rewritten in the form

$$\begin{gathered} b_1 u(x_1, t_{s+1}) + c_1 u(x_2, t_{s+1}) = w_1, \\ v_l u(x_{l-1}, t_{s+1}) + b_l u(x_l, t_{s+1}) + c_l u(x_{l+1}, t_{s+1}) = w_l; \quad l = 2, 3, \ldots, r-2, \\ v_{r-1} u(x_{r-2}, t_{s+1}) + b_{r-1} u(x_{r-1}, t_{s+1}) = w_{r-1}, \end{gathered} \tag{38}$$

where

$$\begin{aligned} b_l = &\frac{\alpha}{\Delta x} u^{\alpha-1}(x_l, t_s) u(x_{l+1}, t_s) - \frac{(\alpha+1)}{\Delta x} u^\alpha(x_l, t_s) + \frac{j}{2(t_{s+1}+1)} \\ &+ \frac{2\epsilon}{(\Delta x)^2} + \frac{1}{\Delta t}, \quad l = 1, 2, \ldots, r-1, \end{aligned} \tag{39}$$

$$v_l = -\frac{\epsilon}{(\Delta x)^2}, \quad l = 2, 3, \ldots, r-1, \tag{40}$$

$$c_l = \frac{1}{\Delta x} u^{\alpha}(x_l, t_s) - \frac{\epsilon}{(\Delta x)^2}, \quad l = 1, 2, \ldots, r-2, \tag{41}$$

$$w_1 = \frac{1}{\Delta t} u(x_1, t_s) + \frac{\alpha}{\Delta x} \left[u(x_2, t_s) - u(x_1, t_s)\right] u^{\alpha}(x_1, t_s) + \frac{\epsilon}{(\Delta x)^2} u(x_0, t_{s+1}), \tag{42}$$

$$w_l = \frac{1}{\Delta t} u(x_l, t_s) + \frac{\alpha}{\Delta x} \left[u(x_{l+1}, t_s) - u(x_l, t_s)\right] u^{\alpha}(x_l, t_s), \quad l = 2, 3, \ldots, r-2, \tag{43}$$

$$w_{r-1} = \frac{1}{\Delta t} u(x_{r-1}, t_s) + \frac{\alpha}{\Delta x} \left[u(x_r, t_s) - u(x_{r-1}, t_s)\right] u^{\alpha}(x_{r-1}, t_s) - \left[\frac{1}{\Delta x} u^{\alpha}(x_{r-1}, t_s) - \frac{\epsilon}{(\Delta x)^2}\right] u(x_r, t_{s+1}). \tag{44}$$

The coefficient matrix of the system of equation in (38) is

$$B = \begin{bmatrix} b_1 & c_1 & 0 & \cdots & 0 \\ v_2 & b_2 & c_2 & \ddots & \vdots \\ 0 & v_3 & b_3 & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & c_{r-2} \\ 0 & \cdots & 0 & v_{r-1} & b_{r-1} \end{bmatrix}. \tag{45}$$

The Thomas algorithm to solve the tridiagonal system of equations in (38) can be described in three steps:

1. $\gamma_1 = b_1,$
   $\gamma_l = b_l - \dfrac{v_l c_{l-1}}{\gamma_{l-1}}; \quad l = 2, 3, \ldots, r-1.$
2. $\beta_1 = \dfrac{w_1}{b_1},$
   $\beta_l = \dfrac{w_l - v_l \beta_{l-1}}{\gamma_l} \quad l = 2, 3, \ldots, r-1.$
3. $u(x_{r-1}, t_{s+1}) = \beta_{r-1},$
   $u(x_l, t_{s+1}) = \beta_l - \dfrac{c_l u(x_{l+1}, t_{s+1})}{\gamma_l}; \quad l = r-2, r-3, \ldots, 1.$

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