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PROPERTIES OF EXTENDED IDEAL BASED ZERO DIVISOR GRAPH OF A COMMUTATIVE RING

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Abstract. This paper deals with some results concerning the notion of extended ideal based zero divisor graph $\overline{\Gamma}_I(R)$ for an ideal I of a commutative ring R and characterize its bipartite graph. Also, we study the properties of an annihilator of $\overline{\Gamma}_I(R)$.

1. Preliminaries

Throughout this paper, we consider R as a commutative ring with identity. The zero divisor graph of a ring R, denoted by $\Gamma(R)$, is the simple graph associated to R such that its vertex set consists of all its nonzero divisors and that two distinct vertices are joined by an edge if and only if the product of these two vertices is zero. The idea of associating graphs with algebraic structures goes back to beck [6], where he was mainly interested in colourings. In his work, all elements of the ring were vertices of the graph.

In 1999, Anderson and Livingston [4], introduced the zero divisor graph of ring and initiated the study of the relation between ring-theoretic properties and graph theoretic properties. They proved that $\Gamma(R)$ is connected with $diam(\Gamma(R)) \leq 3$ and $gr(\Gamma(R)) \in \{3, 4, \infty\}$. Then the zero divisor graphs of commutative rings have attracted the attention of several researchers [1, 2, 3, 5].

In 2016, Driss Bennis and et.al. [7], studied an extension of the zero divisor graph of a commutative ring R, denoted by $\overline{\Gamma}(R)$, which we call

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the extended zero divisor graph of R, such that its vertex set consists of all its nonzero zero divisors and that two distinct vertices x and y are joined by an edge if and only if there exists two non negative integers n and m such that $x^n y^m = 0$ with $x^n \neq 0$ and $y^m \neq 0$. It is clear that $\Gamma(R)$ is a subgraph of $\overline{\Gamma}(R)$ and $\overline{\Gamma}(R)$ is the empty graph if and only if R is an integral domain.

In this paper, we have studied an extended zero divisor graph for an ideal I of R, denoted by $\overline{\Gamma}_I(R)$, which we call the extended ideal based zero divisor graph of R, whose vertices are the set $\{x \in R\backslash I \mid x^n y^m \in I \ for \ some \ y \in R\backslash I\}$ with distinct vertices x and y are adjacent if and only $x^n y^m \in I$ with $x^n \notin I$ and $y^m \notin I$ for some positive integers n and m. It is clear that $\Gamma(R)$ is a subgraph of $\overline{\Gamma}(R)$, and $\overline{\Gamma}(R)$ is a subgraph of $\overline{\Gamma}_I(R)$, and $\overline{\Gamma}_I(R)$ is the empty graph if and only if R is an integral domain.

An element x of R is called nilpotent if there exists some positive integer n such that $x^n = 0$. The nilradical of R is defined as $N = \{x \in R \mid x \text{ is nilpotent }\}$. Let $X \subseteq R$. Then the radical of X (with respect to R) is defined as $\sqrt{X} = \{z \in R \mid z^n \in X \text{ such that } n \geq 1\}$.

If X is a subset of R and I is an ideal of R, then the extended annihilator of X is defined as $\overline{Ann}(X) = \{y \in R \mid x^n y^m \in I$ for every $x \in X$ and for some positive integers $n, m\}$. The set of extended associated prime of a ring R, is denoted by $\overline{Ass(R)}$, and it is the set of prime ideals p of R such that there exists $x \in R$ with $p = \overline{Ann}(x)$.

2. Main results

In this section, we have studied some properties of $\overline{\Gamma}_I(R)$ and extended annihilator of a ring R.

Theorem 2.1. *Let R be a commutative ring. Then $\overline{\Gamma}_I(R)$ is connected and $diam(\overline{\Gamma}_I(R)) \leq 3$.*

Theorem 2.2. *Let R be a commutative ring and $a \in \overline{\Gamma}_I(R)$, adjacent to every vertex of $\overline{\Gamma}_I(R)$. Then $\overline{Ann}(a) \cup I$ is a maximal element of the set $\{\overline{Ann}(x) \cup I \mid x \in R\}$. Moreover, $\overline{Ann}(a) \cup I$ is a prime ideal.*

Proof. Suppose that $\overline{Ann}(a) \subset \overline{Ann}(x)$ for some $x \in R$. Then there exists $t \in \overline{Ann}(x) \backslash \overline{Ann}(a)$ such that $t^{n_1} x^{m_1} \in I$ for some positive integers n_1, m_2 and $t^n a^m \notin I$ for all positive integers n, m. Since $t^{n_1} x^{m_1} \in I$, we have $t \in V(\overline{\Gamma}_I(R))$. Since a is adjacent to every vertex of $\overline{\Gamma}_I(R)$, we have $t^p a^q \in I$ for some positive integers p, q, a contradiction.

Let $xy \in \overline{Ann}(a) \cup I$ and $x, y \notin \overline{Ann}(a) \cup I$. Then $(xy)^n a^m \in I$ for some positive integers n, m, and $x^r a^s \notin I$, $y^r a^s \notin I$ for all positive integers r, s. If $y^n a^m \notin I$, then $x \in \overline{Ann}(ya)$. Clearly $\overline{Ann}(a) \subseteq \overline{Ann}(ya)$

and so $\overline{Ann}(a) = \overline{Ann}(ya)$. Thus $x \in \overline{Ann}(a)$, a contradiction. Thus $\overline{Ann}(a) \cup I$ is a prime ideal. $\square$

Lemma 2.3. *Let R be a commutative ring and let $\overline{Ann}(a) \cup I$ be a maximal element of $\{\overline{Ann}(x) \cup I \mid x \in R \backslash I\}$. Then $\overline{Ann}(a) \cup I$ is a prime ideal.*

Proof. Clearly $\overline{Ann}(a) \cup I$ is an ideal of R. Let $uv \in \overline{Ann}(a)$ and $u, v \notin \overline{Ann}(a)$. Then $(uv)^n a^m \in I$ for some positive integers n, m and $u^{n_1}a^{m_1} \notin I$, $v^{n_2}a^{m_2} \notin I$ for all positive integers n_1, n_2, m_1, m_2. Now we have to claim that $\overline{Ann}(a) \subset \overline{Ann}(ua)$. Let $s \in \overline{Ann}(a)$. Then $s^n a^m \in I$ some positive integers n, m. Since $s^n(ua)^m \in I$, we have $s \in \overline{Ann}(ua)$. So $\overline{Ann}(a) \subset \overline{Ann}(ua)$. Since $\overline{Ann}(a)$ is maximal, we have $\overline{Ann}(a) = \overline{Ann}(ua)$. So $v \in \overline{Ann}(ua) = \overline{Ann}(a)$, a contraction. Thus either $u \in \overline{Ann}(a)$ or $v \in \overline{Ann}(a)$. $\square$

Theorem 2.4. *Let R be a commutative ring and I be a non zero ideal of R. Then the following hold:*

(i) If p_1 and p_2 are prime ideals of R and $I = p_1 \cap p_2 \neq \{0\}$, then $\overline{\Gamma}_I(R)$ is a complete bipartite graph.

(ii) If $I \neq 0$ is an ideal of R for which $I = \sqrt{I}$, then $\overline{\Gamma}_I(R)$ is a complete bipartite graph, if and only if there exist prime ideals p_1 and p_2 of R such that $p_1 \cap p_2 = I$

Proof. (i) Let $a, b \in R \backslash I$ with $ab \in I$. Then $ab \in p_1$ and $ab \in p_2$. Since p_1 and p_2 are prime, we have $a \in p_1$ or $b \in p_1$ and $a \in p_2$ or $b \in p_2$. Thus $\overline{\Gamma}_I(R)$ is a complete bipartite graph with parts $p_1 \backslash p_2$ and $p_2 \backslash p_1$.

(ii) Suppose that $\overline{\Gamma}_I(R)$ is a complete bipartite graph with parts V_1 and V_2. Set $p_1 = V_1 \cup I$ and $p_2 = V_2 \cup I$. Now $p_1 \cap p_2 = I$. We now prove that p_1 is an ideal of R. Let $a, b \in p_1$. Then $a, b \in V_1 \cup I$.

Case(1): If $a, b \in I$, then $a - b \in p_1$.

Case(2): If $a, b \in V_1$, then there exists $c \in V_2$ such that $c^{n_1}a^{m_1} \in I$ and $c^{n_2}a^{m_2} \in I$ for some positive integers n_1, n_2, m_1, m_2. So for $n = n_1 + n_2$, $c^n(a^{m_1} - b^{m_2}) \in I$. If $a^{m_1} - b^{m_2} \in I$, then $a^{m_1} - b^{m_2} \in p_1$. Otherwise $a^{m_1} - b^{m_2} \in V_1$ which implies $a^{m_1} - b^{m_2} \in p_1$.

Case (3): If $a \in V_1$ and $b \in I$, then $a - b \notin I$, so there is $c \in V_2$ such that $c^{n_1}(a - b)^{m_1} \in I$, for some positive integers n_1, m_1 which implies that $a - b \in V_1$ and so $a - b \in p_1$.

Now let $r \in R$ and $a \in p_1$.

Case (1): If $a \in I$, then $r^n a^m \in I$ for some positive integers n, m and so $ra \in p_1$.

Case (2): If $a \in V_1$, then there exists $c \in V_2$ such that $a^{n_1}c^{m_2} \in I$ for some positive integers n_1, m_2. So, $c^{m_2}(ra)^{n_1} \in I$. If $ra \in I$, then

$ra \in p_1$. Otherwise $ra \notin I$. Then $ra \in V_1$ which implies $ra \in p_1$. Thus p_1 is an ideal of R.

We now claim that p_1 is prime. Suppose that $ab \in p_1$ and $a, b \notin p_1$. Since $p_1 = V_1 \cup I$, we have $ab \in I$ or $ab \in V_1$, and so in any case there exists $c \in V_2$ such that $c^n(ab)^m \in I$ for some positive integers n, m. Thus $a^m(c^n b^m) \in I$. If $c^n b^m \in I$, then $b \in V_1$, a contradiction. Hence $c^n b^m \notin I$ and so $c^n b^m \in V_1$. Thus $c^{2n} b^m \in I$. Since $I = \sqrt{I}$, $c^{2n} \notin I$. Here $c^{2n} \in V_2$. So $b \in V_1$, a contradiction and hence p_1 is a prime ideal of R. □

Theorem 2.5. *Let R be a commutative ring. Then the following hold:*
(i) If $|\overline{Ass}(R)| \geq 2$ and $p = \overline{Ann}(x)$, $q = \overline{Ann}(y)$ are two distinct elements of $\overline{Ass}(R)$, then x is adjacent to y in $\overline{\Gamma}_I(R)$.
(ii) If $|\overline{Ass}(R)| \geq 3$, then $gr(\overline{\Gamma}_I(R)) = 3$.
(iii) If $|\overline{Ass}(R)| \geq 5$, then $\overline{\Gamma}_I(R)$ is non-planar.

Proof. (i) Let us assume that $r \in p \backslash q$. Then $r^{n_1} x^{m_1} \in I$ for some positive integers n_1, m_1 which implies $r^{n_1} x^{m_1} \in q$ and so $x^{m_1} \in q = \overline{Ann}(y)$ which implies $(x^{m_1})^{m_2} y^{n_2} \in I$ some positive integers n_2, m_2 and hence x is adjacent to y in $\overline{\Gamma}_I(R)$.
(ii) Let $p_1 = \overline{Ann}(x_1)$, $p_2 = \overline{Ann}(x_2)$, $p_3 = \overline{Ann}(x_3) \in \overline{Ass}(R)$. Then by(i), $x_1 - x_2 - x_3 - x_1$ is a cycle of length of 3 and hence $gr(\overline{\Gamma}_I(R)) = 3$.
(iii) Since $|\overline{Ass}(R)| \geq 5$, K_5 is a subgraph of $\overline{\Gamma}_I(R)$, and hence by Kuratowski's theorem, $\overline{\Gamma}_I(R)$ is non-planar. □

Theorem 2.6. *If $\overline{Ass}(R) = \{p_1, p_2\}$, $|p_i| \geq 3$ for $i = 1, 2$ and $p_1 \cap p_2 = I$, then $gr(\overline{\Gamma}_I(R)) = 4$.*

Proof. Let $p_i = \overline{Ann}(x_i)$, $i = 1, 2$ and $a \in p_1 \backslash (\{x_2\} \cup I)$ and $b \in p_2 \backslash (\{x_1\} \cup I)$. Since $ab \in p_1 \cap p_2 = I$, we have $a - x_1 - x_2 - b - a$, and so $gr(\overline{\Gamma}_I(R)) \leq 4$. Since $\overline{\Gamma}_I(R)$ is bipartite, we have $gr(\overline{\Gamma}_I(R)) = 4$. □

Theorem 2.7. *Let R be a commutative ring. If $\overline{\Gamma}_I(R)$ is a complete r-partite graph, $r \geq 3$, then at most one of the part has more than one vertex.*

Proof. Assume that $V_1, V_2, \cdots, V_r$ are parts of $\overline{\Gamma}_I(R)$. Let V_t and V_s have more than one element. Choose $x \in V_t$ and $y \in V_s$. Let V_l be a part of $\overline{\Gamma}_I(R)$ such that $V_l \neq V_t$ and $V_l \neq V_s$. Let $z \in V_l$. Since $\overline{\Gamma}_I(R)$ is a complete r-partite graph, $\overline{Ann}(x) = \cup_{1 \leq i \leq r,\ i \neq t} V_i \cup I$, $\overline{Ann}(y) = \cup_{1 \leq i \leq r,\ i \neq s} V_i \cup I$ and $\overline{Ann}(z) = \cup_{1 \leq i \leq r,\ i \neq l} V_i \cup I$. Thus $\overline{Ann}(z) \subseteq \overline{Ann}(x) \cup \overline{Ann}(y)$ which implies $\overline{Ann}(z) \subseteq \overline{Ann}(x)$ or $\overline{Ann}(z) \subseteq \overline{Ann}(y)$.

Suppose $\overline{Ann}(z) \subseteq \overline{Ann}(x)$. Choose $x^{'} \in V_t$ such that $x^{'} \neq x$. Then $x^{'^n} z^m \in I$ for some positive integers n, m and $x^{'^r} x^s \notin I$ for all positive integers r, s and so $x^{'} \in \overline{Ann}(z) \setminus \overline{Ann}(x)$, a contradiction.

Suppose $\overline{Ann}(z) \subseteq \overline{Ann}(y)$. Choose $y^{'} \in V_s$ such that $y^{'} \neq y$. Then $y^{'^n} z^m \in I$ for some positive integers n, m and $y^{'^r} y^s \notin I$ for all positive integers r, s and so $y^{'} \in \overline{Ann}(z) \setminus \overline{Ann}(y)$, a contradiction. □

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