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Robust portfolio selection with polyhedral ambiguous inputs

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Abstract. Ambiguity in the inputs of the models is typical especially in portfolio selection problem where the true distribution of random variables is usually unknown. Here we use robust optimization approach to address the ambiguity in conditional-value-at-risk minimization model. We obtain explicit models of the robust conditional-value-at-risk minimization for polyhedral and correlated polyhedral ambiguity sets of the scenarios. The models are linear programs in the both cases. Using a portfolio of USA stock market, we apply the buy-and-hold strategy to evaluate the model's performance. We found that the robust models have almost the same out-of-sample performance, and outperform the nominal model. However, the robust model with correlated polyhedral results in more conservative solutions.

Keywords: data ambiguity, conditional value-at-risk, polyhedral ambiguity set, robust optimization.

AMS Subject Classification: 91G10, 62H12, 90C22.

1 Introduction

The application of robust optimization in finance and especially the portfolio selection problem is popular. There are three major approaches to develop the robust counterpart of a model. The first approach was introduced by Soyster where he considered the box ambiguity set for con-

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structing the robust counterpart [23]. The resulting portfolios however, are extremely conservative in the sense that the objective values of the corresponding portfolios are too expensive. In particular, the optimal portfolio of the robust model in the case of risk minimization, suggests higher value of risk. Then Ben-Tal and Nemirovski suggested the intersection of the ellipsoid and box ambiguity sets by which the robust optimal portfolios are less conservative, and the robust model accepts computationally tractable formulation [3, 4]. Further, Bertsimas and Sim proposed the intersection of polyhedral and box [6]. The resulting optimal portfolios are less conservative and its formulation is as computationally expensive as the nominal (non-robust) model. These are important merits of polyhedral ambiguity sets that have been ignored in portfolio selection theory.

There are a few studies that have used the polyhedral ambiguity sets [5, 11] but most of the carried out studies focused on the application of box and ellipsoidal ambiguity sets [9, 13, 16–20, 25].

Now the question is: How ellipsoidal ambiguity set and not polyhedral one becomes standard in the portfolio selection theory? First, ellipsoidal ambiguity set was introduced much earlier than polyhedral, and by that time the experts in the area of finance had noticed the effect of estimation error in mean returns and its consequences on the optimal portfolio selection [7, 10]. Also, the confidence regions of the statistical estimators such as mean, turned out to be ellipsoids [22]. Hence, most of the studies focused on reducing the effect of estimation error in the optimal portfolio by applying the robust optimization approach where there is an ellipsoidal ambiguity set on mean returns [2, 8, 12].

However, when the ambiguous parameters are highly correlated, the proposed polyhedral ambiguity set can not capture this issue and as a result, some of the realizations might be out of the ambiguity set. Recently Jalilvand-Nejad et al. introduced the correlated polyhedral ambiguity set in which they follow the same approach as Bertsimas and Sim, but they incorporated the correlation coefficients in the formulation of the ambiguity set [14]. This could be helpful especially in finance where usually the correlation matrix can be estimated using huge amount of scenarios available.

The conditional value-at-risk (CVaR) is a well-known risk metric that has been embedded in Basel III for more conservative regularization objective after failing the famous value-at-risk (VaR). The CVaR belongs to the class of coherent risk measure introduced by Artzner [1], however, it is fragile in the sense that its estimate suffers from the sensitivity to inputs [15]. There are several studies that address this issue using robust optimization in portfolio selection problem where CVaR is used as risk measure. For a comprehensive review of the literature, the readers can refer to [16, 18].

El-Ghaouie et al. developed the worst-case VaR minimization under different ambiguity sets including the polyhedral [11]. Further, Bertsimas and Pachamanov studied the application of polyhedral ambiguity set in the multi-periods portfolio optimization where the objective is the final wealth maximization [5].

Recently, the robust counterpart of CVaR optimization problem when the ambiguity set for scenarios are interval and ellipsoid types, is investigated in [13] and [16]. To the best of our knowledge, there is no other study of robust CVaR (RCVaR) portfolio selection problem with polyhedral ambiguity set proposed in [6] and this is where we place our contribution. We show that the RCVaR minimization with correlated polyhedral ambiguity set is a linear program (LP) and evaluate its performance against the RCVaR minimization with polyhedral ambiguity set suggested in [6]. We use buy-and-hold strategy and test the models with real market data.

The rest of the paper is organized as follows. First, in Section 2 we formulate the RCVaR minimization model under polyhedral and correlated polyhedral ambiguity sets. Then in Section 3 we illustrate the performance of all methods on a portfolio from USA stock market by using the buy-and-hold strategy. Finally, Section 4 will conclude the paper.

2 Robust CVaR portfolio selection

The mean of α-tail distribution of portfolio loss X, $\mathrm{CVaR}_\alpha(X)$, and its minimization formula were developed in [21]. According to Theorem 1 of this paper, the CVaR of the loss function is the solution of

$$\mathrm{CVaR}_\alpha(x) = \min_{\gamma\in\mathbb{R}} F_\alpha(x,\gamma), \tag{1}$$

where

$$F_\alpha(x,\gamma) = \gamma + \frac{1}{1-\alpha}\mathbb{E}\{[f(x,\xi)-\gamma]^+\}$$

in which $f(x,\xi)$ is the loss function associated with portfolio x in $\mathbb{X}$, the set of all feasible portfolios. To simplify our discussion, we let x be in $\mathbb{X}$ denoted by the set $\{x \in \mathbb{R}^n | x \geq 0, \sum_{i=1}^n x_i = 1\}$. Then the CVaR portfolio minimization becomes:

$$\min_{\gamma\in\mathbb{R}, x\in\mathbb{X}} F_\alpha(x,\gamma). \tag{2}$$

We use $\alpha = 0.95$ whenever we do numerical experiments throughout the paper.

Consider an investor operating in a market with n risky assets and no short-selling. The n risky assets have rates of return denoted by random vector ξ. The loss function associated with decision variable $x \in \mathbb{R}^n$ of proportionate allocations to the risky assets is given by $f(x,\xi) = -x^\top \xi$. Let a collection of S historical observations $\{R_{(1)}, \ldots, R_{(S)}\}$ are available and all of them are equally probabilistic, then the empirical distribution is used to estimate the portfolio optimization problems (2). The CVaR minimization model can be posed as follows [21]:

$$\begin{aligned} \min_{x\in\mathbb{X},\, u\in\mathbb{R}^S,\, \gamma\in\mathbb{R}} \quad & \gamma + \frac{1}{S(1-\alpha)}\pi^\top u \\ s.t. \quad & -Rx - u - \gamma e \leq 0, \\ & u \geq 0. \end{aligned} \tag{3}$$

Note that the sample space of random vector ξ in problem (2) may be given by a set of scenarios. We define the ambiguity in the scenarios as follows:

Definition 1. *The ambiguity in the scenarios of the discretized random variable ξ is defined as follows:*

I. Polyhedral ambiguity set

$$\begin{aligned} P_R = \{R \in \mathbb{R}^{S\times n} | R_{ij} = \overline{R}_{ij} + \xi_{ij}\hat{R}_{ij}, \sum_{j=1}^{N} |\xi_{ij}| \leq \Gamma_i, |\xi_{ij}| \leq 1, \\ i = 1, \ldots, S, j = 1, \ldots, N\}. \end{aligned} \tag{4}$$

II. Correlated polyhedral ambiguity set

$$\begin{aligned} CP_R = \{ & R \in \mathbb{R}^{S\times n} | R_{ij} = \overline{R}_{ij} + \xi_{ij}\hat{R}_{ij}, \\ & |\xi_{ij}| + \sum_{\substack{k=1 \\ k\neq j}}^{N} \left[1 - \frac{N-\Gamma_i}{N-1} |\rho_{ijk}|\right] |\xi_{ik}| \leq \Gamma_i, |\xi_{ij}| \leq 1, \\ & i = 1, \ldots, S, j = 1, \ldots, N\}, \end{aligned}$$

where ρ_{ijk} shows the correlation between coefficients R_{ij} and R_{ik}, Γ_i, $i = 1, \ldots, S$ are the ambiguity set's parameters, and $\hat{R}_{ij} \geq 0$, $i = 1, \ldots, S, j = 1, \ldots, N$.

Interval, ellipsoidal and polyhedral ambiguity sets are the most often used ambiguity sets in the robust optimization framework [11,12,24]. With interval and polyhedral ambiguity sets, the robust counterpart of CVaR

$R_{(i)}$ denotes the ith rows of the matrix R called observation matrix.

minimization problems is an LP while with ellipsoidal ambiguity set, the associated optimization problem, belongs to much harder class of optimization problems. The RCVaR minimization (optimization) problem, when the ambiguity sets for scenarios are interval or ellipsoid types, is already investigated in [13] and [16]. Here we extend their work to the case where there is a polyhedral ambiguity set on the scenarios.

Theorem 1. *The robust counterpart of (3) for ambiguity set (4) is as follows:*

$$
\begin{aligned}
\min_{x\in\mathbb{X},\, u,p\in\mathbb{R}^S, q\in\mathbb{R}^{S\times N},\, \gamma\in\mathbb{R}} \quad & \gamma + \frac{1}{S(1-\alpha)}\sum_{i=1}^{S} u_i \\
s.t. \quad & -\sum_{j=1}^{N}\overline{R}_{ij}x_j + p_i\Gamma_i + \sum_{j=1}^{N} q_{ij} - u_i - \gamma \le 0, \\
& -\hat{R}_{ij}x_j + p_i + q_{ij} \ge 0, \\
& \sum_{j=1}^{N} x_j = 1,\ x_j \ge 0,\ u_i \ge 0,\ p_i \ge 0, q_{ij} \ge 0, \\
& j = 1,\ldots,N,\ i = 1,\ldots,S.
\end{aligned}
\tag{5}
$$

Proof. The robust counterpart of (3) under polyhedral ambiguity set is as follows:

$$
\begin{aligned}
\min_{x\in\mathbb{X},\, u\in\mathbb{R}^S,\, \gamma\in\mathbb{R}} \quad & \gamma + \frac{1}{S(1-\alpha)}\sum_{i=1}^{S} u_i \\
s.t. \quad & -\sum_{j=1}^{N}\overline{R}_{ij}x_j + \left[\max_{\xi_{(i)}\in EP_i}\sum_{j=1}^{N} -\hat{R}_{ij}\xi_{ij}x_j\right] - u_i - \gamma \le 0, \\
& \sum_{j=1}^{N} x_j = 1,\ x_j \ge 0,\ u_i \ge 0, \\
& j = 1,\ldots,N,\ i = 1,\ldots,S,
\end{aligned}
\tag{6}
$$

where $EP_i = \{\xi_{ij} | \sum_{j=1}^{N} |\xi_{ij}| \le \Gamma_i, |\xi_{ij}| \le 1,\ j = 1,\ldots,N\},\ i = 1,\ldots,S$. To find the explicit formulation of (6), we need to replace the inner maximizations in the constraints with their optimal values. Using the set of variable transformations $\xi_{ij} = \delta_{ij} - \mu_{ij},\ |\xi_{ij}| = \delta_{ij} + \mu_{ij}, \delta_{ij},\ \mu_{ij} \ge 0$, one

By CVaR optimization we mean that there is a minimum return target constraint while CVaR minimization refer to the case where there is not any.

can easily check that the inner maximization is equivalent to the following optimization problem:

$$
\begin{aligned}
\max_{\lambda_{(i)} \in \mathbb{R}^N} \quad & \sum_{j=1}^{N} \hat{R}_{ij}\lambda_{ij}x_j \\
s.t. \quad & \sum_{j=1}^{N} \lambda_{ij} \leq \Gamma_i, \\
& 0 \leq \lambda_{ij} \leq 1, \\
& j = 1, \dots, N,
\end{aligned}
\tag{7}
$$

and thus the dual of it can be written as follows:

$$
\begin{aligned}
\min_{p_i \in \mathbb{R}, q_{(i)} \in \mathbb{R}^N} \quad & p_i\Gamma_i + \sum_{j=1}^{N} q_{ij} \\
s.t. \quad & p_i + q_{ij} \geq \hat{R}_{ij}x_j, \; j = 1, \dots, N, \\
& p_{ij} \geq 0, q_{ij} \geq 0, \\
& j = 1, \dots, N.
\end{aligned}
\tag{8}
$$

Considering that the primal and dual are both feasible, we can replace the maximization problem with the corresponding dual's objective function and add its constraints to the original problem's constraints. Then we get the required formulation. □

Theorem 2. *The robust counterpart of (3) for ambiguity set (5) is as follows:*

$$
\begin{aligned}
\min_{x \in \mathbb{X},\, u \in \mathbb{R}^S, p,q \in \mathbb{R}^{S \times N},\, \gamma \in \mathbb{R}} \quad & \gamma + \frac{1}{S(1-\alpha)} \sum_{i=1}^{S} u_i \\
s.t. \quad & -\sum_{j=1}^{N} \overline{R}_{ij}x_j + \sum_{j=1}^{N} p_{ij}\Gamma_i + \sum_{j=1}^{N} q_{ij} - u_i - \gamma \leq 0, \\
& -\hat{R}_{ij}x_j + p_{ij} + \sum_{\substack{k=1 \\ k \neq j}}^{N} (1 - \frac{N-\Gamma_i}{N-1} |\rho_{ijk}|) p_{ik} + q_{ij} \geq 0, \\
& \sum_{j=1}^{N} x_j = 1, \; x_j \geq 0, \; u_i \geq 0, \; p_{ij} \geq 0, q_{ij} \geq 0, \\
& j = 1, \dots, N, \; i = 1, \dots, S.
\end{aligned}
\tag{9}
$$

Proof. The robust counterpart of (3) under correlated polyhedral ambiguity set is as follows:

$$\begin{aligned}
\min_{x\in\mathbb{X},\, u\in\mathbb{R}^S,\, \gamma\in\mathbb{R}} \quad & \gamma + \frac{1}{S(1-\alpha)}\sum_{i=1}^{S} u_i \\
s.t. \quad & -\sum_{j=1}^{N} \overline{R}_{ij} x_j + \left[\max_{\xi_{(i)}\in EC_i} \sum_{j=1}^{N} -\hat{R}_{ij}\xi_{ij}x_j\right] - u_i - \gamma \le 0, \\
& \sum_{j=1}^{N} x_j = 1,\ x_j \ge 0,\ u_i \ge 0, \\
& j = 1,\dots,N,\ i = 1,\dots,S,
\end{aligned} \tag{10}$$

where $EC_i = \{\xi_{ij} |\, |\xi_{ij}| + \sum_{k=1,k\neq j}^{N} \left[1 - \frac{N-\Gamma_i}{N-1}\,|\rho_{ijk}|\right] |\xi_{ik}| \le \Gamma_i, |\xi_{ij}| \le 1,\ j = 1,\dots,N\}$, $i = 1,\dots,S$. Following an argument similar to the proof of Theorem 1, the inner maximization in optimization problem (10) can be equivalently written as:

$$\begin{aligned}
\max_{\lambda_{(i)}\in\mathbb{R}^N} \quad & \sum_{j=1}^{N} \hat{R}_{ij}\lambda_{ij}x_j \\
s.t. \quad & \lambda_{ij} + \sum_{k=1,k\neq j}^{N} \left[1 - \frac{N-\Gamma_i}{N-1}\,|\rho_{ijk}|\right]\lambda_{ik} \le \Gamma_i, \\
& 0 \le \lambda_{ij} \le 1, \\
& j = 1,\dots,N.
\end{aligned} \tag{11}$$

The dual of (11) can be written as follows:

$$\begin{aligned}
\min_{p_{(i)},q_{(i)}\in\mathbb{R}^N} \quad & \sum_{j=1}^{N} p_{ij}\Gamma_i + \sum_{j=1}^{N} q_{ij} \\
s.t. \quad & p_{ij} + \sum_{k=1,k\neq j}^{N} (1 - \frac{N-\Gamma_i}{N-1}\,|\rho_{ijk}|)p_{ik} + q_{ij} \ge \hat{R}_{ij}x_j, \\
& p_{ij} \ge 0, q_{ij} \ge 0, \\
& j = 1,\dots,N.
\end{aligned}$$

Since both primal and dual are feasible, then the duality gap is equal to zero, hence we can replace the objective of the dual and add its constraint to the original problem's constraints to obtain the robust model (9). $\square$

3 Empirical tests

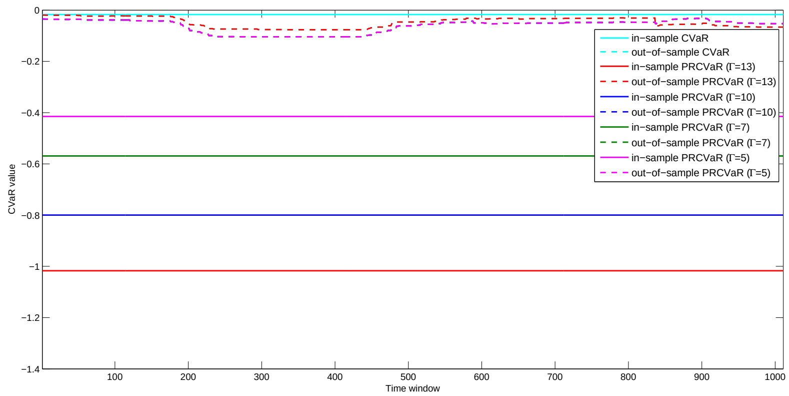
Here we test the nominal and robust models under different ambiguity sets. The thirteen stocks for investment are selected from USA stock market, all having moderate to high correlation coefficients. The daily prices of these stocks are obtained from 3 January 2006 to 28 December 2011. To assess the out-of-sample performance of the solutions of the proposed correlated polyhedral against the polyhedral proposed in [6], we apply the buy-and-hold strategy as described in [17].

We develop nominal and robust models based on the scenarios observed between 3 January 2006 and 28 December 2007, the pre-crisis period. Then we obtain the associated asset allocations, calculate the risk value and record it (in-sample). As the new information is observed and the market moves into 2008 crisis period and later, the recorded allocations is used to compute out-of-sample portfolio performance. That is we compute the CVaR value with the new time window for which we dropped the oldest observation so that CVaR is computed on a constant size window of recent data. This procedure is repeated until the time window meets the last observation.

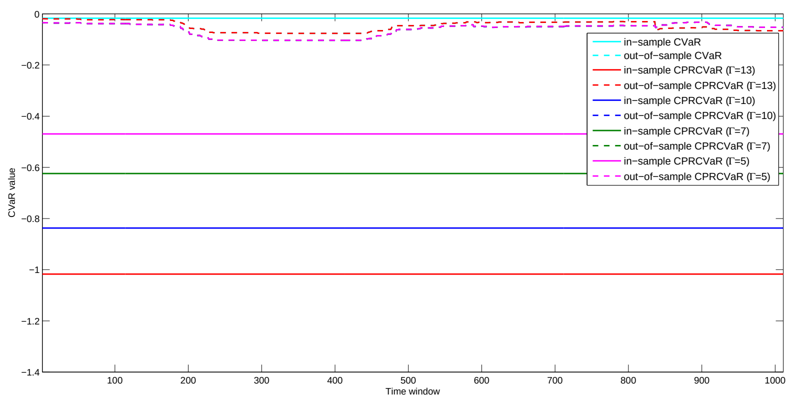
We consider correlation between the assets singly and calculate the correlation matrix based on the observations used for in-sample calculation. Further, $\hat{R}$ is assumed to be a matrix of all ones. All Γ_i's, $i = 1, \dots, S$ are assumed to be equal and are denoted by Γ in the sequel. The results are shown for different values of Γ in Figures 1, panels (a) and (b). We use solid and dotted lines to illustrate the in-sample and out-of-sample risk values, respectively.

We observe that out-of-sample risk measure for the nominal portfolios is worse than the in-sample value, but this is not the case for the the robust portfolios. Also, $\Gamma = N$ results in the most conservative solutions as it can be observed in the gap between in-sample and out-of-sample in both robust models. This is the case when all ambiguity sets reduce to the interval one. One interesting observation however, is that the application of correlated polyhedral results in more conservative portfolios compared to the polyhedral proposed by Bertsimas and Sim [6]. Further, the resulting portfolios are as efficient as the polyhedral ambiguity set suggested by Bertsimas and Sim in terms of out-of-sample.

We consider pure perturbation.



(a) Polyhedral ambiguity set



(b) Correlated polyhedral ambiguity set

Figure 1: Out-of-sample performance of buy-and-hold (PRCVaR and CPRCVaR denote the RCVaR with polyhedral and correlated polyhedral ambiguity set, respectively).

4 Conclusions

This paper develops equivalent models for robust CVaR under polyhedral and correlated polyhedral ambiguity sets when there is ambiguity in the scenarios. RCVaR minimization models for both type of polyhedral ambiguity sets are as computationally complex as the CVaR minimization model. Using a portfolio of stocks chosen from USA market, the models were tested over the highly volatile period that covered the 2008 crisis and later. The results of buy-and-hold strategy show that the robust models outperform the nominal model in terms of out-of-sample, however, the robust model with correlated polyhedral ambiguity set results in more conservative solutions.

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