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ON TWO GENERALIZATIONS OF SEMI-PROJECTIVE MODULES: SGQ-PROJECTIVE AND π -SEMI-PROJECTIVE

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ABSTRACT. Let R be a ring and M a right R-module with $S = End_R(M)$. A module M is called semi-projective if for any epimorphism $f: M \to N$, where N is a submodule of M, and for any homomorphism $g: M \to N$, there exists $h: M \to M$ such that fh = g. In this paper, we study SGQ-projective and π -semi-projective modules as two generalizations of semi-projective modules. A module M is called an SGQ-projective module if for any $\phi \in S$, there exists a right ideal X_{ϕ} of S such that $D_S(\text{Im}\phi) = \phi S \oplus X_{\phi}$ as right S-modules. We call M a π -semi-projective module if for any $0 \neq s \in S$, there exists a positive integer n such that $s^n \neq 0$ and any R-homomorphism from M to $s^n M$ can be extended to an endomorphism of M. Some properties of this class of modules are investigated.

1. INTRODUCTION

Throughout this paper R will denote an associative ring with identity, M a unitary right R-module and $S = End_R(M)$ the ring of all Rendomorphisms of M. If N is a submodule of M, then we will use the notation $N \ll M$ to indicate that N is small in M (i.e. $\forall L \leq$ $M, L + N \neq M$). The notation $N \leq^{\oplus} M$ denotes that N is a direct summand of M. We also denote $r_M(I) = \{x \in M \mid Ix = 0\}$, for $I \subseteq S$; $D_S(N) = \{\phi \in S \mid \operatorname{Im} \phi \subseteq N\}$, for $N \subseteq M$. An M-projective module

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M is called *self-projective*. In fact an R-module M is self-projective if and only if $D_S(\operatorname{Im}\phi) = \phi S$, where $S = End_R(M)$. A module M is called *semi-projective* if for any epimorphism $f: M \to N$, where Nis a submodule of M, and for any homomorphism $g: M \to N$, there exists $h: M \to M$ such that fh = g. Obviously, M is semi-projective if and only if $fS = Hom_R(M, fM)$ for every $f \in End_R(M) = S$. The semi-projective modules are studied by different authors (see [3], [5] and [12]).

In [4], Kaleboğaz, Keskin-Tütüncü and Smith introduced SGQ-proj ective modules and studied some results about this class of modules. Let M be a right R-module with $S = End_R(M)$. Then M is called an SGQ-projective module if for any $\phi \in S$, there exists a right ideal X_{ϕ} of S such that $D_S(\operatorname{Im}\phi) = \phi S \oplus X_{\phi}$ as right S-modules. It is clear that the notion of SGQ-projective modules is a generalization of semi-projective modules. In Section 2 of this paper we investigate more properties of SGQ-projective modules.

A right *R*-module *M* is called *quasi-principally injective* if every homomorphism from an *M*-cyclic submodule of *M* can be extended to an endomorphism of *M*. The quasi principally-injective modules were first studied by Wisbauer in [13] under the terminology of semi-injective modules. In [15], Zhu Zhanmin generalized quasi principally-injective modules to the general quasi-principally injective modules. In Section 3 of this note, we introduce the dual notion of such modules and call them π -semi-projective modules. We call *M* a π -semi-projective module if for any $0 \neq s \in S$, there exists a positive integer *n* such that $s^n \neq 0$ and any *R*-homomorphism from *M* to $s^n M$ can be extended to an endomorphism of *M*. Obviously, every semi-projective module is π -semi-projective module. Section 3 contains the results on π -semi-projective modules. First, we give a characterization of π -semiprojective modules. We prove the following main result:

Let M_R be a finitely generated π -semi-projective retractable module with $S = End_R(M)$. Then the mappings

$$K \to KM$$
 and $T \to D_S(T)$

are mutually inverse bijections between the set of all minimal right ideals K of S and the set of all minimal submodules T of M. In particular, we have:

(1) $D_S(KM) = K$ for all minimal right ideal K of S.

(2) $D_S(T)M = T$ for all minimal submodule T of M.

2. SGQ-PROJECTIVE MODULES

Let M be a right R-module with $S = End_R(M)$. Then M is called SGQ-projective if for any $\phi \in S$, there exists a right ideal X_{ϕ} of S such that $D_S(\operatorname{Im}\phi) = \phi S \oplus X_{\phi}$ as right S-modules.

Example 2.1. (cf. [4, Example 2.6]) (1) The \mathbb{Z} -module $\mathbb{Q}_{\mathbb{Z}}$ is semiprojective and hence SGQ-projective, but it is not self-projective.

(2) Let R be any integral domain with quotient field $F \neq R$. Then $M_R = F \oplus R$ is semi-projective and so SGQ-projective, but in general not self-projective. This can be easily seen from the fact that $End_R(M) = \begin{pmatrix} F & F \\ 0 & R \end{pmatrix}$.

Recall that an *R*-module *M* is called *coretractable* if, for any proper submodule *K* of *M*, there exists a nonzero endomorphism $f \in S$ with f(K) = 0, that is, $Hom_R(M/K, M) \neq 0$ [1].

It is well known that if M is a self-projective module, then $J(S) = \nabla(M)$, where $\nabla(M) = \{s \in S \mid \text{Im} s \ll M\}$ (see [13, 22.2]). In the following theorem, we prove the similar result for SGQ-projective modules.

Theorem 2.2. Let M be a right R-module with $S = End_R(M)$. If M is an SGQ-projective coretractable module, then $J(S) = \nabla(M)$ where $\nabla(M) = \{\phi \in S \mid Im\phi \ll M\}.$

Proof. Since SGQ-projective modules are semi-Hopfian, by [2, 4.28], we have $\nabla(M) \subseteq J(S)$. Conversely, let $s \in J(S)$. Then we will show that $s \in \nabla(M)$. If not, then there exists a proper submodule K of M such that $\operatorname{Im} s + K = M$. Since M is coretractable, $Hom_R(M/K, M) \neq 0$. Thus there exists $0 \neq t \in S$ such that $K \subseteq$ Kert. Hence we have $\operatorname{Im} s + \operatorname{Ker} t = M$. So $\operatorname{Im} ts = \operatorname{Im} t$ and $ts \neq 0$. Since M is SGQ-projective, $D_S(\operatorname{Im} ts) = (ts)S \oplus X_{ts}$ as right S-modules. As $t \in D_S(\operatorname{Im} t) = D_S(\operatorname{Im} ts) = (ts)S \oplus X_{ts}$, we can write t = tsu + vfor some $u \in S$ and $v \in X_{ts}$. Then $ts - tsus = vs \in X_{ts} \cap (ts)S = 0$, and so ts(1 - us) = 0. Since 1 - us is right invertible, ts = 0, a contradiction. \Box

Recall that a module M_R is said to satisfy the D_2 -condition if whenever N is a submodule of M and M/N is isomorphic to a direct summand of M, then N is a direct summand of M [9]. We mention that the following results are dual of some results in [14].

Theorem 2.3. If M_R is an SGQ-projective module, then it satisfies the D_2 -condition.

Proof. Let $N \leq M$ and $M/N \cong eM$ for some $e^2 = e \in S$. Then $N = \operatorname{Ker}h$ with h = es for some $s \in S$ and $\operatorname{Im}e = \operatorname{Im}es$. Since M is SGQ-projective, $e \in D_S(\operatorname{Im}e) = D_S(\operatorname{Im}h) = hS \oplus X_h$ where X_h is a right S-module. Then e = ht + x with $t \in S$ and $x \in X_h$. Hence h = eh = hth + xh and thus $h - hth = xh \in X_h \cap hS = 0$, so h = hth. Let f = th, then $f^2 = f$ and $N = \operatorname{Ker}h = (1 - f)M$.

Dual Rickart modules are defined by Lee, Rizvi and Roman in [7]. The module M is called *dual Rickart* if for any $f \in S$, Imf = eM for some $e^2 = e \in S$.

Corollary 2.4. Let M be a module and $S = End_R(M)$. Then S is a von Neumann regular ring if and only if M is an SGQ-projective and dual Rickart module.

Proof. It is easy to see by [10, Theorem 4] and [4, Corollary 3.3]. \Box

Lemma 2.5. Let M_R be a module with $S = End_R(M)$. Given a set $\{X_s \mid s \in S\}$ of right ideals of S, the following are equivalent:

(1) M is SGQ-projective;

(2) $D_S(Kert + Ims) = (X_{ts} : t)_r + sS$ and $(X_{ts} : t)_r \cap sS \subseteq r_S(t)$ for all $s, t \in S$, where $(X_{ts} : t)_r = \{x \in S \mid tx \in X_{ts}\}.$

Proof. (1) ⇒ (2) Let $x \in D_S(\operatorname{Kert} + \operatorname{Im} s)$. Then $\operatorname{Im}(tx) \subseteq \operatorname{Im}(ts)$ and so $tx \in D_S(\operatorname{Im} tx) \subseteq D_S(\operatorname{Im} ts) = (ts)S \oplus X_{ts}$. Write $tx = tss_1 + y$, where $s_1 \in S$ and $y \in X_{ts}$, then $t(x-ss_1) = y \in X_{ts}$ and hence $x-ss_1 \in (X_{ts}:t)_r$. Thus $x \in (X_{ts}:t)_r + sS$. Clearly $sS \subseteq D_S(\operatorname{Kert} + \operatorname{Im} s)$. If $z \in (X_{ts}:t)_r$, then $tz \in X_{ts} \subseteq D_S(\operatorname{Im} ts)$. Let $y \in \operatorname{Im} z$, then y = z(m)for some $m \in M$. Hence $ty = tz(m) \in \operatorname{Im} tz \subseteq \operatorname{Im} ts$, thus ty = ts(m')for some $m' \in M$ and so t(y - sm') = 0. Then $y - sm' \in \operatorname{Ker} t$. This implies that $\operatorname{Im} z \subseteq \operatorname{Im} s + \operatorname{Ker} t$ and so $z \in D_S(\operatorname{Im} s + \operatorname{Ker} t)$. Therefore $D_S(\operatorname{Im} s + \operatorname{Ker} t) = (X_{ts}: t)_r + sS$. If $s' \in (X_{ts}: t)_r \cap sS$, then $ts' \in X_{ts} \cap (ts)S = 0$ and hence $s' \in r_S(t)$. (2) ⇒ (1) Let t = 1.

A nonzero R-module M is called *hollow* if every proper submodule is small in M [9].

Lemma 2.6. Let M_R be an SGQ-projective module with $S = End_R(M)$ and an index set $\{X_s \mid s \in S\}$ of ideals such that $X_{st} = X_{ts}$ for all $s, t \in S$. Define $M_u = \{s \in S \mid Ims + Keru \neq M\}$, where $0 \neq u \in S$. If M/Keru is a hollow factor module of M, then M_u is the unique maximal right ideal of S which contains $\sum_{s \in S} (X_{us} : u)_r$.

Proof. Since M/Keru is a hollow factor of M, M_u is a right ideal of S. Let $t \in (X_{us} : u)_r$, then $ut \in X_{us}$ and so $sut \in X_{us} \cap (su)S =$

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 $X_{su} \cap (su)S$ since $X_{su} = X_{us}$ is an ideal. Hence sut = 0 and so $t \in M_u$ if $su \neq 0$. If su = 0, then $D_S(\operatorname{Im}(su)) = 0$, and hence $X_{us} = X_{su} = 0$. This implies that ut = 0 and so $t \in M_u$. Therefore $(X_{us} : u)_r \subseteq M_u$ for all $s \in S$. Now if $s \notin M_u$, then $\operatorname{Im} s + \operatorname{Ker} u = M$, and hence $S = (X_{us} : u)_r + sS$ by Lemma 2.5, so $S = M_u + sS$, this shows that M_u is a maximal right ideal. Finally, let I be a right ideal of S such that $\sum_{s \in S} (X_{us} : u)_r \subseteq I \neq M_u$. Then, as above, $S = (X_{us} : u)_r + sS$ for any $s \in I - M_u$. Consequently, I = S.

Proposition 2.7. Let M_R be an SGQ-projective module with $S = End_R(M)$ and an index set $\{X_s \mid s \in S\}$ of ideals such that $X_{st} = X_{ts}$ for all $s, t \in S$. Assume that $W = Keru_1 \oplus Keru_2 \oplus \cdots \oplus Keru_n$ is a direct sum of submodules of M where $0 \neq u_i \in S$ and $M/Keru_i$ is a nonzero hollow factor module of M. If $T \subseteq S$ is a maximal right ideal not of the form M_u , for any $u \in S$, for which M/Keru is hollow, then there exists $t \in T$ such that $\frac{Im(1-t)+W}{W} \ll \frac{M}{W}$.

Proof. Since $T \neq M_{u_1}$, there exists $a \in T$ such that $\operatorname{Im} a + \operatorname{Ker} u_1 = M$. Then $\operatorname{Im} u_1 \subseteq \operatorname{Im} u_1 a$ and hence $u_1 \in D_S(\operatorname{Im} u_1) \subseteq D_S(\operatorname{Im} u_1 a) = (u_1 a)S \oplus X_{u_1 a}$. Thus there exists $s \in S$ such that $u_1(1 - as) \in X_{u_1 a}$ and so $1 - as \in (X_{u_1 a} : u_1)_r \subseteq M_{u_1}$. Set $a_1 = as$. If $1 - a_1 \in M_{u_i}$ for all i, the proof is complete since $M/\operatorname{Ker} u_i$ is hollow. If, say, $1 - a_1 \notin M_{u_2}$, then $\overline{\frac{M}{\operatorname{Ker} u_2(1-a_1)}}$ is hollow since $\overline{\frac{M}{\operatorname{Ker} u_2(1-a_1)}} \cong \overline{\frac{M}{\operatorname{Ker} u_2}}$. Hence, as above, $(1 - a') \in M_{u_2}(1-a_1)$ for some $a' \in T$. Let $a_2 = a_1 + a' - a'a_1$, then $1 - a_2 \in M_{u_1} \cap M_{u_2}$, continue in this way to obtain $t \in S$, such that $\operatorname{Im}(1 - t) + \operatorname{Ker} u_i \neq M$ for each i. Therefore $\overline{\frac{\operatorname{Im}(1-t)+W}{W}} \ll \frac{M}{W}$.

An *R*-module M is said to have *finite hollow dimension* if there exists a small epimorphism from M to a finite direct sum of n hollow factor modules [8].

Theorem 2.8. Let M_R be a coretractable module with finite hollow dimension. If M is SGQ-projective with an index set $\{X_s \mid s \in S\}$ of right ideals of S such that $X_{st} = X_{ts}$ for all $s, t \in S$, then

(1) If T is a maximal right ideal of S, then $T = M_u$, for some $u \in S$, and for which M/Keru is a nonzero hollow factor module of M.

(2) $J(S) = M_{u_1} \cap \cdots \cap M_{u_n}$ for some $n \in \mathbb{N}$ and $u_1, \cdots, u_n \in S$, where $M/Keru_i$ is a nonzero hollow factor module of M.

Proof. (1) Since M has finite hollow dimension, there exists a small epimorphism $f: M \to \bigoplus_{i=1}^{n} M/A_i$, where $n \in \mathbb{N}$ and M/A_i is a nonzero hollow factor module of M for each $1 \leq i \leq n$. Note that $Hom_R(M/A_i, M) \neq 0$, for each $1 \leq i \leq n$, since M is coretractable. Thus there exists $0 \neq s_i \in S$ such that $A_i \subseteq \text{Ker} s_i$. If T is not

of the form M_u for some $u \in S$ such that M/Keru is hollow, then, by the proof of Proposition 2.7, there exists some $t \in T$ such that $\text{Im}(1-t) + \text{Ker}s_i \neq M$, thus $\text{Im}(1-t) + A_i \neq M$. By [11, 3.5], $\text{Im}(1-t) \ll M$. By Theorem 2.2, $1-t \in J(S) \subseteq T$, a contradiction.

(2) Let $s \in M_{u_1} \cap \cdots \cap M_{u_n}$, then $\operatorname{Im} s + \operatorname{Ker} u_i \neq M$ for each $1 \leq i \leq n$. Similar to the proof of (1), $\operatorname{Im} s \ll M$. Hence $s \in J(S)$, this shows that $M_{u_1} \cap \cdots \cap M_{u_n} \subseteq J(S)$. The reverse inclusion is obvious. Thus $J(S) = M_{u_1} \cap \cdots \cap M_{u_n}$.

3. π -Semi-projective modules

Definition 3.1. Let M be a right R-module with $S = End_R(M)$. We call M a π -semi-projective module if for any $0 \neq s \in S$, there exists a positive integer n such that $s^n \neq 0$ and any R-homomorphism from M to $s^n M$ can be extended to an endomorphism of M.

It is clear that every self-projective module is π -semi-projective. Note that there is the π -semi-projective module which is not self-projective (see Example 2.1).

The following theorem is a characterization of π -semi-projective modules.

Theorem 3.2. Let M be a right R-module. Then the following conditions are equivalent:

(1) M is π -semi-projective.

(2) For any $0 \neq s \in S$ there exists a positive integer n such that $s^n \neq 0$ and $D_S(Ims^n) = s^n S$.

(3) For any $s,t \in S$ with $ts \neq 0$, there exists a positive integer n such that $(ts)^n \neq 0$ and $D_S(Ims(ts)^{n-1} + Kert) = s(ts)^{n-1}S + r_S(t)$.

Proof. (1) \Rightarrow (3) For any $s, t \in S$ with $ts \neq 0$, since M is π -semiprojective, there exists a positive integer n such that $(ts)^n \neq 0$ and any R-homomorphism from M to $(ts)^n M$ extends to an endomorphism of M. Let $a \in D_S[\operatorname{Ims}(ts)^{n-1} + \operatorname{Kert}]$. Now we define $f : M \rightarrow$ $(ts)^n M$ by $x \mapsto ta(x)$. It is easy to see that f is an R-homomorphism. Hence f extends to an endomorphism of M, i.e., there exists $b \in S$ such that $ta = (ts)^n b$. This shows that $a - s(ts)^{n-1}b \in r_S(t)$ and so $a \in s(ts)^{n-1}S + r_S(t)$. Consequently, $D_S[\operatorname{Ims}(ts)^{n-1} + \operatorname{Kert}] \subseteq$ $s(ts)^{n-1}S + r_S(t)$. The reverse inclusion is obvious.

 $(3) \Rightarrow (2)$ By taking t = 1, then (2) follows from (3).

 $(2) \Rightarrow (1)$ Let $0 \neq s \in S$. Then, by (2), there exists a positive integer n such that $s^n \neq 0$ and $D_S(\operatorname{Im} s^n) = s^n S$. If $f: M \to s^n M$ is any R-homomorphism, by taking $u = s^n f$, then we have $u \in D_S(\operatorname{Im} s^n) = s^n S$. Thus $u = s^n g$ for some $g \in S$, and so g extends f. \Box

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We again recall that if M is a self-projective module, then $J(S) = \nabla(M)$, where $\nabla(M) = \{s \in S \mid \text{Im} s \ll M\}$. In the following theorem, we prove the similar result for π -semi-projective modules.

Theorem 3.3. Let M be a right R-module and $S = End_R(M)$. If M is a coretractable π -semi-projective module, then $J(S) = \nabla(M)$.

Proof. Let $s \in J(S)$. Then we will show that $s \in \nabla(M)$. If it is not, then there exists a proper submodule K of M such that $\operatorname{Im} s + K = M$. Since M is coretractable, $Hom_R(M/K, M) \neq 0$. Then there exists $0 \neq t \in S$ such that $K \subseteq \operatorname{Kert}$. Hence $\operatorname{Im} s + \operatorname{Ker} t = M$. Clearly, $\operatorname{Im} ts = \operatorname{Im} t$ and so $ts \neq 0$. Since M is π -semi-projective, there exists a positive integer n such that $(ts)^n \neq 0$ and $D_S(\operatorname{Im}(ts)^n) = (ts)^n S$. Let $a = (ts)^n$ and $b = (ts)^{n-1}t$. If $y \in \operatorname{Im} b$, then $y = b(x) = (ts)^{n-1}t(x)$ for some $x \in M$. Note that $t(x) \in \operatorname{Im} t = \operatorname{Im} ts$, thus t(x) = ts(x') for some $x' \in M$. Hence $y = (ts)^{n-1}t(x) = (ts)^{n-1}ts(x') \in \operatorname{Im} a$. This follows that $\operatorname{Im} a = \operatorname{Im} b$. Now since $b \in D_S(\operatorname{Im} b) = D_S(\operatorname{Im} a) = aS$, there exists an $u \in S$ such that b = au. Thus, $0 = b - au = (ts)^{n-1}t - (ts)^n u =$ $(ts)^{n-1}t(1 - su) = b(1 - su)$. Since $s \in J(S)$, 1 - su is invertible, and hence b = 0. Thus a = bs = 0, a contradiction.

Conversely, let $s \in \nabla(M)$, then for any $t \in S$, $st \in \nabla(M)$. Clearly, Im(st) + Im(1 - st) = M and so Im(1 - st) = M. Note that for any positive integer k, we have $(1 - st)^k = 1 - h_k$ for some $h_k \in \nabla(M)$, hence Im $(1 - st)^k = M$. Since M is π -semi-projective, there exists a positive integer n such that $(1 - st)^n S = D_S(\text{Im}(1 - st)^n)$. Hence $(1 - st)^n S = D_S(M) = S$. and so 1 - st is right invertible. Therefore $s \in J(S)$.

Lemma 3.4. Let M be a π -semi-projective right R-module with $S = End_R(M)$. Then for any $0 \neq u \in S$ such that M/Keru is hollow, the set $M_u = \{s \in S \mid Ims + Keru \neq M\}$ is a maximal right ideal of S containing $D_S(Keru)$.

Proof. Clearly, $0 \in M_u$. Since $M/\operatorname{Ker} u$ is hollow, M_u is a right ideal. It is easy to see that $D_S(\operatorname{Ker} u) \subseteq M_u \neq S$. Now we show that M_u is a maximal right ideal. In fact, for any $s \in S \setminus M_u$, we have $\operatorname{Im} s + \operatorname{Ker} u = M$, this means that $\operatorname{Im} us = \operatorname{Im} u$ and $us \neq 0$. As M is a π -semi-projective module, there exists a positive integer n such that $(us)^n \neq 0$ and $D_S(\operatorname{Im}(us)^n) = (us)^n S$. If $y \in \operatorname{Im}(us)^{n-1}u$, then $y = (us)^{n-1}u(x)$ for some $x \in M$. Note that $u(x) \in \operatorname{Im} u = \operatorname{Im} us$, thus u(x) = us(x') for some $x' \in M$. Hence $y = (us)^{n-1}u(x) = (us)^{n-1}u(s(x')) \in \operatorname{Im}(us)^n$. This follows that $\operatorname{Im}(us)^n = \operatorname{Im}(us)^{n-1}u$. Thus $(us)^{n-1}u \in D_S(\operatorname{Im}(us)^{n-1}u) = D_S(\operatorname{Im}(us)^n) = (us)^n S$. Suppose that $(us)^{n-1}u = (us)^n t$ for some $t \in S$. Hence $(us)^{n-1}u(1-st) = 0$ and

so $1 - st \in r_S((us)^{n-1}u)$. This shows that $S = sS + r_S((us)^{n-1}u)$ and so $S = sS + M_u$ since $r_S((us)^{n-1}u) \subseteq M_u$. Therefore M_u is maximal in S.

Corollary 3.5. If M is a coretractable π -semi-projective and hollow module, then S is local.

Proof. By hypothesis and Theorem 3.3, $J(S) = \nabla(M) = \{s \in S \mid \text{Im} s \ll M\} = \{s \in S \mid \text{Im} s \neq M\} = M_1 \text{ and so } S \text{ is local.}$

Lemma 3.6. Let M be a right π -semi-projective module with $S = End_R(M)$ and let $W = \bigoplus_{i=1}^n Keru_i$ be a direct sum of submodules of M where $0 \neq u_i \in S$ and $M/Keru_i$ is a nonzero hollow factor module for all $1 \leq i \leq n$. If $I \subseteq S$ is a maximal right ideal not of the form M_u for some $u \in S$ such that M/Keru is hollow, then there exists $\phi \in I$ such that $\frac{Im(1-\phi)+W}{W} \ll \frac{M}{W}$.

Proof. Let $a \in I \setminus M_{u_1}$. Then $\operatorname{Im} a + \operatorname{Ker} u_1 = M$. By the argument in Lemma 3.4, there exist a positive integer n and an element $t \in S$ such that $(u_1a)^{n-1}u_1 \neq 0$ and $1 - at \in r_S((u_1a)^{n-1}u_1)$. Set $\phi_1 = at$. Then $\phi_1 \in I$ and $\operatorname{Im}(1 - \phi_1) + \operatorname{Ker} u_1 \subseteq \operatorname{Im}(1 - \phi_1) + \operatorname{Ker}(u_1a)^{n-1}u_1 =$ $\operatorname{Ker}(u_1a)^{n-1}u_1 \neq M$. If $\operatorname{Im}(1 - \phi_1) + \operatorname{Ker} u_i \neq M$ for all $i \geq 2$, then we are done since $M/\operatorname{Ker} u_i$ is hollow for all $1 \leq i \leq n$. If, say, $\operatorname{Im}(1 - \phi_1) + \operatorname{Ker} u_2 = M$, then $\overline{\operatorname{Ker} u_2(1 - \phi_1)}$ is hollow since $\overline{\frac{M}{\operatorname{Ker} u_2(1 - \phi_1)}} \cong \frac{M}{\operatorname{Ker} u_2}$. Thus, as above, $(1 - \phi_2) \in M_{u_2(1 - \phi_1)}$ for some $\phi_2 \in I$. Let $\phi_3 = \phi_1 + \phi_2 - \phi_2 \phi_1$, then $\phi_3 \in I$ and $\operatorname{Im}(1 - \phi_3) + \operatorname{Ker} u_i \neq M$, i = 1, 2. Continue in this way, we obtain that $\phi \in I$ such that $\operatorname{Im}(1 - \phi) + \operatorname{Ker} u_i \neq M$ for each i. Therefore $\frac{\operatorname{Im}(1 - \phi) + W}{W} \ll \frac{M}{W}$.

Theorem 3.7. Let M_R be a coretractable π -semi-projective module which has finite hollow dimension. Then the following statements hold:

(1) If I is a maximal right ideal of S, then $I = M_u$, for some $u \in S$, for which M/Keru is a nonzero hollow factor module of M.

(2) $J(S) = M_{u_1} \cap \cdots \cap M_{u_n}$, for some $u_1, \cdots, u_n \in S$, for which $M/Keru_i$ is a nonzero hollow factor module of M.

Proof. It can be proved by using the Theorem 3.3 and Lemmas 3.4 and 3.6 with an argument similar to the proof of Theorem 2.8.

We recall that a subset X of a ring R is right t-nilpotent if, for every sequence x_1, x_2, \cdots of elements in X, there is a $k \in \mathbb{N}$ with $x_1x_2\cdots x_k = 0$. Similarly left t-nilpotent is defined [13]. We also recall that factor modules of M are called M-cyclic modules.

Lemma 3.8. Let M be a right R-module. If Rad(M) satisfies DCC on M-cyclic submodules, then $\nabla(M)$ is right t-nilpotent.

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Proof. For any subset I of $\nabla(M)$, let $I = \{\overline{s} \in \text{Hom}_R(M, Rad(M)) \mid s = i\overline{s}, s \in I\}$, where $i : Rad(M) \to M$ is the inclusion map. For any $s_1, s_2, \dots \in \nabla(M)$, since $\text{Im}\overline{s_1} \supseteq \text{Im}\overline{s_1s_2} \supseteq \text{Im}\overline{s_1s_2s_3} \supseteq \dots$ and by using the hypothesis, there exists a positive integer m such that $\text{Im}(\overline{s_1s_2\cdots s_m}) = \text{Im}(\overline{s_1s_2\cdots s_m}\cdots s_k)$ for all $k \ge m$. Thus $\text{Im}(s_1s_2\cdots s_{m+1}) = \text{Im}(s_1s_2\cdots s_{m+2})$. This shows that $\text{Im}s_{m+2} + \text{Ker}(s_1s_2\cdots s_{m+1}) = M$. Since $\text{Im}s_{m+2} \ll M$, $\text{Ker}(s_1s_2\cdots s_{m+1}) = M$. Therefore $\nabla(M)$ is right t-nilpotent. \Box

Corollary 3.9. Let M_R be a coretractable π -semi-projective R-module. If Rad(M) satisfies DCC on M-cyclic submodules, then J(S) is right t-nilpotent.

Proof. By Theorem 3.3 and Lemma 3.8.

A module M_R is called *retractable* if $Hom_R(M, N) \neq 0$ for all nonzero R-submodules N of M [6].

Theorem 3.10. Let M_R be a finitely generated π -semi-projective retractable module with $S = End_R(M)$. Then the mappings

$$K \to KM$$
 and $T \to D_S(T)$

are mutually inverse bijections between the set of all minimal right ideals K of S and the set of all minimal submodules T of M. In particular, we have:

- (1) $D_S(KM) = K$ for all minimal right ideal K of S.
- (2) $D_S(T)M = T$ for all minimal submodule T of M.

Proof. We first prove (1). Let K be any minimal right ideal of S. Then K = sS for some $0 \neq s \in S$. Since M is π -semi-projective, there exists a positive integer n such that $s^n \neq 0$ and $s^n S = D_S(\text{Im}s^n)$ by Theorem 3.2. Note that K is minimal, hence $K = s^n S$. Thus $D_S(KM) = K$.

To prove (2), we know that $D_S(T)M \subseteq T$ always holds and $D_S(T)M \neq 0$ since M is retractable. Therefore $D_S(T)M = T$ as T is minimal. The proof is completed by establishing the following claims:

Claim 1: KM is minimal for all minimal right ideals K of S.

Proof. Let $T \subseteq KM$, where T is a nonzero submodule of M_R . Then $D_S(T) \subseteq D_S(KM) = K$ by (1), thus $D_S(T) = K$ since M_R is retractable. Hence $KM = D_S(T)M = T$ by (2).

Claim 2: $D_S(T)$ is minimal for all minimal submodule T of M_R .

Proof. Since M_R is retractable, $D_S(T) \neq 0$. For any $0 \neq s \in D_S(T)$, there exists a positive integer n such that $s^n \neq 0$ and $s^n S = D_S(s^n M)$ by general semi-projectivity of M_R . Then $0 \neq \text{Im} s^n \subseteq D_S(T)M = T$

and hence $\operatorname{Im} s^n = T$. Thus $D_S(T) = D_S(s^n M) = s^n S$. Clearly, $s^n S \subseteq sS \subseteq D_S(T) = s^n S$, hence $D_S(T) = sS$. It follows that $D_S(T)$ is a minimal right ideal.

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