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An efficient numerical method for singularly perturbed second order ordinary differential equation

Jugal Mohapatra* **and Manas kumar Mahalik**

Department of Mathematics, National Institute of Technology Rourkela, India
Emails: jugal@nitrkl.ac.in, 513MA1002@nitrkl.ac.in

Abstract. In this paper an exponentially fitted finite difference method is presented for solving singularly perturbed two-point boundary value problems with the boundary layer. A fitting factor is introduced and the model equation is discretized by a finite difference scheme on an uniform mesh. Thomas algorithm is used to solve the tri-diagonal system. The stability of the algorithm is investigated. It is shown that the proposed technique is of first order accurate and the error constant is independent of the perturbation parameter. Several problems are solved and numerical results are presented to support the theoretical error bounds established.

Keywords: Singular perturbation problems, boundary layers, Thomas algorithm, exponential fitting factor, uniform convergence.
AMS Subject Classification: 65L10, 65L12.

1 Introduction

Singularly perturbed boundary value problems often arise in applied sciences and engineering, typical examples include high Reynolds number flow in fluid dynamics, modelling the problems in mathematical biology and semi-conductor devices where the edge effects are important. These problems depend on a small positive parameter ε known as the singular perturbation parameter. These problems have been received a significant

*Corresponding author.
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amount of attention in past and recent years. A well known fact is that the solution of such problems display sharp boundary or interior layers when ε is very small, *i.e.*, the solution varies rapidly in some parts and varies slowly in some other parts. Typically there are thin transition layers where the solutions can jump abruptly, while away from the layers the solution behaves regularly and vary slowly. So the treatment of singularly perturbed problems present severe difficulties that have to be addressed to ensure accurate numerical solutions. Thus more efficient but simpler computational techniques are required to solve singular perturbation problems. For a good analytical discussion on singular perturbation, one may refer the books: Doolan et. al. [4], Bender et. al. [2], Kevorkian and Cole [10], O'Malley [14]. Also, for some numerical methods and their convergence analysis, one may refer to recent books: Farrell et. al. [5], Miller et. al. [11], Roos et. al. [18], Shishkin et. al. [19] and the references therein. In the articles [1, 3, 6–8, 15, 20], many researchers have followed different numerical approaches combining fitted mesh methods and fitted operator methods for solving singular perturbation problems where as [9] gives an erudite outline on the numerical methods for singular perturbation problems. In [12, 13, 16] efficient numerical methods are used for singularly perturbed differential equations with a delay (or shift) term. Recently, Reddy et. al [17] have developed an exponential finite difference method for solving model equation of the form (1). But Most of these available numerical techniques are constructed on fitted operator techniques or by the use of reasonable apriori information about the solutions which is a limitation of this kind of approach.

In this paper, we introduce a simple exponentially fitted finite difference method for solving singularly perturbed two-point boundary value problems with the boundary layer at one end (left or right) point. A fitting factor is introduced and the model equation is discretized by a finite difference scheme on an uniform mesh. Thomas algorithm is used to solve the tri-diagonal system. The stability of the algorithm is investigated. Several linear and nonlinear problems are solved to demonstrate the applicability of the method. It is observed that the present method approximates the exact solution very well.

The rest of the paper is organized as follows: Section 2 recalls pertinent properties of the solution $y(x)$ of (1). In Section 3, we describe the finite difference scheme, followed by a brief discussion on Thomas algorithm and its stability analysis. We discuss the convergence analysis of the numerical solution obtained by the exponential scheme in Section 4. Finally, Section 5 gives some numerical examples that confirm the theoretical error estimates. Also, we apply the proposed scheme on some nonlinear problems and problems with right end boundary layer.

Throughout this paper C denotes a generic positive constant independent of the grid points x_j and the parameters ε and N (the number of mesh intervals) which can take different values at different places, even in the same argument. A subscripted C (*i.e.,* C_1) is a constant that is independent of ε and of the nodal points x_j, but whose value is fixed. Whenever we write $\phi = \mathcal{O}(\psi)$, we mean that $|\phi| \leq C|\psi|$. To simplify the notation, we set $g_j = g(x_j)$ for any function g, while g_j^N denotes an approximation of g at x_j. Also, we assume that $\varepsilon \leq CN^{-1}$ as is generally the case of discretization of convection-diffusion problems. It is worthwhile to mention that this assumption is not a restriction in practical situation.

2 Continuous Problem

In this article, we consider the following singularly perturbed boundary value problem (SPBVP):

$$\begin{cases} Ly(x) \equiv \varepsilon y''(x) + a(x)y'(x) + b(x)y(x) = f(x), \; x \in \Omega = (0,1), \\ B_0 \equiv y(0) = \alpha, \qquad B_1 \equiv y(1) = \beta, \end{cases} \tag{1}$$

where $0 < \varepsilon \ll 1$ is a small singular perturbation parameter, the functions $a(x), b(x), f(x)$ are sufficiently smooth and α, β are given constants. Further, we assume that $a(x) \geq 2M > 0$ and $b(x) \leq 0$. Under these assumptions, the above problem (1) has a unique solution which exhibits a boundary layer at $x = 0$.

From the theory of singular perturbations ([14]) and using Taylor's series expansion for $a(x)$ about $x = 0$ and restriction to their first terms, we get

$$y(x) = y_0(x) + (\alpha - y_0(0)) \exp\left(-\frac{a(0)}{\varepsilon} x \right) + \mathcal{O}(\varepsilon), \tag{2}$$

where $y_0(x)$ is the solution of the reduced problem of (1), given by

$$a(x)y_0'(x) + b(x)y_0(x) = f(x) \quad \text{with } y_0(1) = \beta. \tag{3}$$

First, the interval $[0, 1]$ is divided into N equal number of subintervals, each of length h. Let $0 = x_0 < x_1 < x_2 < \cdots < x_N = 1$ be the points such that $x_i = ih$ for $i = 0, 1, \cdots, N$. From (2) as $h \to 0$, we have

$$\lim_{h \to 0} y(ih) = y_0(0) + (\alpha - y_0(0)) \exp\big(-\frac{a(0)}{\varepsilon} ih \big). \tag{4}$$

Let $\rho = \frac{h}{\varepsilon}$. Now the equation becomes

$$\lim_{h\to 0} y(ih) = y_0(0) + (\alpha - y_0(0)) \exp\big(-i\rho a(0)\big). \tag{5}$$

Now introducing an exponentially fitting factor $\sigma(\rho)$ in (1), we get

$$\varepsilon\sigma(\rho)y''(x) + a(x)y'(x) + b(x)y(x) = f(x), \tag{6}$$

with boundary conditions $B_0 \equiv y(0) = \alpha$, and $B_1 \equiv y(1) = \beta$. The fitting factor $\sigma(\rho)$ is to be determined in such a way that the solution of (6) converges uniformly to the solution of (1).

Lemma 1. *(Maximum Principle) Let v be a smooth function satisfying $v(0) \geq 0,\ v(1) \geq 0$ and $Lv(x) \leq 0,\ \forall x \in\ \Omega,$ then $v(x) \geq 0,\ \forall x \in\ \overline{\Omega}$.*

Proof. We can prove the above lemma by method of contradiction. Let $x^* \in \overline{\Omega}$ be such that $v(x^*) = \min v(x), x \in \overline{\Omega}$ and assume that $v(x^*) < 0$. Clearly $x^* \notin \{0,1\}$ and $v'(x^*) = 0$ and $v''(x^*) \geq 0$. Now consider

$$Lv(x^*) \equiv \varepsilon v''(x^*) + a(x^*)v'(x^*) + b(x^*)v(x^*) > 0$$

which is a contradiction to our assumption. Hence $v(x) \geq 0, \forall x \in\ \overline{\Omega}$. □

An immediate consequence of the maximum principle is the following stability estimate.

Lemma 2. *If u is the solution of the boundary value problem (1), then*

$$\|u\| \ \leq\ M^{-1}\|f\| + \max\{|\alpha|, |\beta|\}. \tag{7}$$

Proof. Consider the following barrier function

$$\psi^{\pm}(x) = \left[\left(\frac{1-x}{M}\right)\|f\| + \max\big\{|\alpha|, |\beta|\big\}\right] \pm u(x).$$

It is easy to check that $\psi^{\pm}(x) \geq 0$ at $x = 0, 1$. Now from (1)

$$\begin{aligned}
L\psi^{\pm}(x) &= \varepsilon\big(\psi^{\pm}(x)\big)'' + a(x)\big(\psi^{\pm}(x)\big)' + b(x)\psi^{\pm}(x) \\
&= \frac{-a(x)}{M}\|f\| + b(x)\left[\left(\frac{1-x}{M}\right)\|f\| + \max\{|\alpha|, |\beta|\}\right] \pm Lu(x) \\
&\leq \big[-\|f\| \pm f(x)\big] + b(x)\left[\left(\frac{1-x}{M}\right)\|f\| + \max\{|\alpha|, |\beta|\}\right] \leq 0.
\end{aligned}$$

Thus by applying the maximum principle (Lemma 1), we can conclude that $\psi^{\pm}(x) \geq 0, \forall x \in \overline{\Omega}$, which is the required result. □

Lemma 3. *The solution $u(x)$ and its derivatives of the BVP (1) satisfy the following bounds:*

$$|u^{(k)}(x)| \le C\Big(1 + \varepsilon^{-k}\exp(-Mx/\varepsilon)\Big), \quad k = 0, 1, 2, 3, \qquad x \in \overline{\Omega}. \tag{8}$$

Proof. One can prove this lemma by following the method of proof as given in [11]. □

3 Discrete Problem

Consider the difference approximation of (1) on a uniform grid $\overline{\Omega}^N = \{x_j\}_{j=0}^N$ and denote $h = x_{j+1} - x_j$. For a mesh function Z_j, we define the following difference operators:

$$D^+Z_j = \frac{Z_{j+1} - Z_j}{h}, \quad D^-Z_j = \frac{Z_j - Z_{j-1}}{h},$$

$$D^0Z_j = \frac{Z_{j+1} - Z_{j-1}}{2h}, \quad D^+D^-Z_j = \frac{Z_{j+1} - 2Z_j + Z_{j-1}}{h^2}.$$

The upwind finite difference scheme for (6) takes the form

$$\begin{cases} \varepsilon\sigma(\rho)D^+D^-y(x_i) + a(x_i)D^0y(x_i) + b(x_i)y(x_i) = f(x_i)\,, & 1 \le i \le N-1, \\ y_0 = y(x_0) = \alpha, \quad y_N = y(x_N) = \beta\ . \end{cases} \tag{9}$$

Using the above difference operators, we have

$$L^N y(x_i) = \begin{cases} \varepsilon\sigma(\rho)\dfrac{y(x_{i+1}) - 2y(x_i) + y(x_{i-1})}{h^2} + a(x_i)\dfrac{y(x_{i+1}) - y(x_{i-1})}{2h} + \\ +b(x_i)y(x_i) = f(x_i), \quad 1 \le i \le N-1, \\ y_0 = y(x_0) = \alpha, \quad y_N = y(x_N) = \beta\ . \end{cases} \tag{10}$$

Multiplying (10) by h and taking the limit $h \to 0$, we get

$$\lim_{h\to 0}\frac{\sigma(\rho)}{\rho}\Big(y(x_{i+1}) - 2y(x_i) + y(x_{i-1})\Big) + \frac{a(x_i)}{2}\Big(y(x_{i+1}) - y(x_{i-1})\Big) = 0, \tag{11}$$

where $f(x_i) - b(x_i)y(x_i)$ is bounded. Substituting (4) in (11) and then simplifying, we get

$$\sigma(\rho) = \frac{\sigma a(0)}{2}\coth\left[\frac{\sigma a(0)}{2}\right]. \tag{12}$$

Hence, (9) takes the form

$$\begin{cases} \sigma(\rho)y''(x_i) + p(x_i)D^0y(x_i) + q(x_i)y(x_i) = r(x_i)\,, \quad 1 \le i \le N-1, \\ y_0 = y(x_0) = \alpha, \quad y_N = y(x_N) = \beta\ , \end{cases} \tag{13}$$

where $p(x) = a(x)/\varepsilon$, $q(x) = b(x)/\varepsilon, r(x) = f(x)/\varepsilon$.

Let δ be a small deviating argument such that $0 < \delta \ll 1$. By using Talylor's expansion about the point $x = x_i$ up to the second order approximation, we have

$$y(x_i - \delta) = y(x_i) - \delta y'(x_i) + \frac{\delta^2}{2}y''(x_i).$$

Therefore, we have

$$y''(x_i) = \frac{2}{\delta^2}[y(x_i - \delta) - y(x_i) + \delta D^+y(x_i)].$$

So from (13), we have

$$2\sigma(\rho)[y(x_i-\delta)-y(x_i)+\delta D^+y(x_i)]+\delta^2 p(x_i)D^0y(x_i)+\delta^2 q(x_i)y(x_i) = \delta^2 r(x_i)\,. \tag{14}$$

But from Taylor series expansion about the point $x = x_i$, we have

$$y(x_i - \delta) \approx y(x_i) - \delta y'(x_i) = y(x_i) - \delta D^- y(x_i).$$

Substituting the above in (14), we get a three term recurrence relation as follows:

$$E_i y_{i-1} + F_i y_i + G_i y_{i+1} = H_i\,, \quad 1 \le i \le N-1, \tag{15}$$

where

$$\begin{aligned} E_i &= \frac{2\sigma(\rho)}{h} - \frac{\delta p(x_i)}{2h}, \qquad F_i = \frac{-4\sigma(\rho)}{h} + \delta q(x_i), \\ G_i &= \frac{2\sigma(\rho)}{h} + \frac{\delta p(x_i)}{2h}, \qquad H_i = \delta r(x_i). \end{aligned}$$

Now (15) gives a system of $N-1$ equations with $N-1$ unknowns from y_1 to y_{N-1} where $y(x_i) = y_i$. To solve the tri-diagonal system, we use Thomas algorithm. A brief discussion on Thomas algorithm is as follows:

Thomas algorithm: A brief discussion on Thomas algorithm for solving the tri-diagonal system (15) is given below:

Consider the tri-diagonal system (15) with the boundary conditions. In Thomas algorithm, we set a recurrence relation

$$y_i = W_i\, y_{i+1} + T_i\,, \qquad \text{for } i = N-2, N-1, \cdots, 1, \tag{16}$$

where $W_i = W(x_i)$ and $T_i = T(x_i)$ are to be determined. For $i = 0$, we get $y_0 = W_0 y_1 + T_0$, but from boundary conditions, we already know that $y_0 = \alpha$. So by comparing the coefficients, we get $W_0 = 0$ and $T_0 = \alpha$. Again from (16), we have

$$y_{i-1} = W_{i-1}\, y_i + T_{i-1}\,, \qquad \text{for } i = N-2, N-1, \cdots, 1. \tag{17}$$

Substituting (17) in (15), we get

$$E_i[W_{i-1}\, y_i + T_{i-1}] + F_i y_i + G_i y_{i+1} = H_i\,, \tag{18}$$

and on simplifying, we obtain

$$y_i = \frac{-G_i}{F_i + E_i W_{i-1}}\, y_{i+1} + \frac{H_i - E_i T_{i-1}}{F_i + E_i W_{i-1}}\,. \tag{19}$$

Comparing (16) and (19), we get

$$W_i = \frac{-G_i}{F_i + E_i W_{i-1}}, \qquad T_i = \frac{H_i - E_i T_{i-1}}{F_i + E_i W_{i-1}}.$$

with the initial conditions $W_0 = 0$ and $T_0 = \alpha$. Now, we can calculate W_i, T_i and hence using the value $y_n = \beta$, we can get the values of y_i for $i = N-2, N-1, \cdots, 1$.

3.1 Stability analysis

By stability, we mean that the error committed at one stage is not propagated into larger to the later stage. Suppose a small error e_{i-1} has been made in calculating W_{i-1} given above. Now $\overline{W}_{i-1} = W_{i-1} + e_{i-1}$ and we want to calculate $\overline{W}_{i-1}$. So

$$\begin{aligned} e_i &= \frac{G_i}{F_i - E_i(W_{i-1} + e_{i-1})} - \frac{G_i}{F_i - E_i W_{i-1}} \\ &= \frac{G_i E_i e_{i-1}}{(F_i - E_i(W_{i-1} + e_{i-1}))(F_i - E_i W_{i-1})} = \frac{W_i^2 E_i}{G_i} e_{i-1}. \end{aligned}$$

From the assumption made earlier that $a(x) > 0$, $b(x) \leq 0$, so $|E_i| \leq |G_i|$. Now by the condition $|W_i| < 1$, $i = 1, 2, \cdots, N-1$, it follows that $|e_i| = |W_i|^2 \dfrac{|E_i|}{|G_i|} |e_{i-1}| < |e_{i-1}|$. Hence, the stability is guaranteed.

Lemma 4. *(Discrete comparison principle): Assume that the mesh function $V(x_i)$ satisfies $V(x_0) \geq 0$ and $V(x_N) \geq 0$. If $L^N V(x_i) \leq 0$ for $1 \leq i \leq N-1$, then $V(x_i) \geq 0$ for $0 \leq i \leq N$.*

Proof. Let us choose k such that $V(x_k) = \min V(x_i), 1 \leq i \leq N-1$. If $V(x_k) \geq 0$, then there is nothing to prove. It is obvious that $V(x_{k+1}) - V(x_k) \geq 0$ and $V(x_k) - V(x_{k-1}) \leq 0$. Now from (10), we have

$$L^N V(x_k) = \varepsilon\sigma(\rho)\frac{V(x_{k+1}) - 2V(x_k) + V(x_{k-1})}{h^2}$$

$$+a(x_k)\frac{V(x_{k+1}) - V(x_{k-1})}{2h} + b(x_k)V(x_k) \geq 0,$$

which contradicts $L^N V(x_i) \leq 0$. Hence, our assumption is wrong. □

4 Convergence Analysis

The following theorem shows the ε-uniform convergence of the proposed scheme.

Theorem 1. *Let y and Y be respectively the exact solution of (1) and the discrete solution of (13) respectively. Then, for sufficiently large N, we have the following ε–uniform error estimate:*

$$\sup_{0<\varepsilon\leq 1} \|y - Y\| \leq CN^{-1}(\ln N)^2, \quad x \in \overline{\Omega} \tag{20}$$

Proof. First, let us decompose the solution $y(x)$ of (1) into regular and singular parts as follows: $y(x) = r(x) + s(x)$. Now for $0 \leq k \leq 3$, the regular component $r(x)$ satisfies

$$|r^k(x)| \leq C[1 + \varepsilon^{2-k}e(x,a)], \quad \forall x \in [0,1]. \tag{21}$$

and the singular component $s(x)$ satisfies

$$|s^k(x)| \leq C\varepsilon^{-k}e(x,a), \quad \forall x \in [0,1]. \tag{22}$$

where $e(x,a) = e_1(x,a) + e_2(x,a) = \exp(\frac{-a_0 x}{\varepsilon}) + \exp(\frac{-a_0(1-x)}{\varepsilon})$.(for details see [11]).

Similarly, decompose the discrete solution Y of the problem (13) into regular (R_ε) and singular (S_ε) components. Thus $Y(x) = R_\varepsilon(x) + S_\varepsilon(x)$ where R_ε and S_ε are respectively the solution of the following problems:

$$L^N R_\varepsilon = f(x), \quad R_\varepsilon(0) = r(0), \quad R_\varepsilon(1) = r(1),$$

and

$$L^N S_\varepsilon = 0, \quad S_\varepsilon(0) = s(0), \quad S_\varepsilon(1) = s(1),$$

Thus $y(x) - Y(x) = [r(x) - R_\varepsilon(x)] + [s(x) - S_\varepsilon(x)]$ and the error can be estimated as

$$\|y(x) - Y(x)\| \leq \|r(x) - R_\varepsilon(x)\| + \|s(x) - S_\varepsilon(x)\|.$$

Now we need to calculate the errors in the regular and singular components separately.

Let us first calculate the error in the regular component. Consider the local truncation error defined as follows:

$$\begin{aligned} L^N(R_\varepsilon(x) - r(x)) &= (L - L^N)r(x) = f(x) - L^N(r(x)) \\ &= \varepsilon(D^2 - \Delta^2)r(x) + a(x)(D - D^0)r(x). \end{aligned} \tag{23}$$

Using Taylor's series expansion and neglecting higher order terms from fourth order, we get the following expansions for $y(x_i + h)$ and $y(x_i - h)$:

$$y(x_i \pm h) = y(x_i) \pm hy'(x_i) + \frac{h^2}{2}y''(x_i) \pm \frac{h^3}{6}y'''(\xi_1^{(i)}),$$

where $(\xi_1^{(i)}), (\xi_1^{(i)}) \in (x_{i-1}, x_{i+1})$. Simplifying the above two expressions, we can easily show that

$$(\Delta^2 y)(x_i) = y''(x_i) - \frac{h}{6}\big[y'''(\xi_1^{(i)}) - y'''(\xi_2^{(i)})\big].$$

So, $\|(\Delta^2 - \frac{d^2}{dx^2})y(x_i)\| \leq C\|y'''\|$, where $\|y'''\| = \sup_{x_i \in (x_0, x_N)} |y'''(x_i)|$. Similarly by Taylor's series expansion up to the second order terms we get

$$\|(D^0 - \frac{d}{dx})y(x_i)\| \leq C\|y''\|.$$

Now using the bounds of $r^k(x), s^k(x)$ and the assumption $\varepsilon \leq CN^{-1}$, the equation (23) reduces to

$$\|L^N(R_\varepsilon - r)(x_i)\| \leq CN^{-1}. \tag{24}$$

Hence, using the discrete maximum principle (Lemma 4), we get

$$\|(R_\varepsilon - r)\| \leq CN^{-1}. \tag{25}$$

Now we need to find out the error in the singular component. The local truncation error in the singular component is bounded in the standard way as done for the regular part and is given by

$$\|L^N(S_\varepsilon - s)(x_i)\| \leq C\varepsilon^{-2}N^{-1}.$$

Choose a constant K such that $K\varepsilon \ln N \geq \frac{1}{4}$, *i.e.*, $\varepsilon^{-1} \leq 4K \ln N$. So from above inequality, we have

$$\|L^N(S_\varepsilon - s)(x_i)\| \leq CN^{-1}(\ln N)^2.$$

Now again using the discrete comparison principle, we reach at

$$\|(S_\varepsilon - s)\| \leq CN^{-1}(\ln N)^2. \tag{26}$$

Finally, combining (25) and (26), we get our desired result. □

5 Numerical Results and discussions

To demonstrate the applicability of the method, we have applied the proposed scheme on several singular perturbation problems with left boundary layers. These examples are widely discussed in the literature. The exact solutions or sometimes uniformly valid approximate solutions are used for comparison purpose.

Example 1. Consider the homogeneous problem

$$\begin{cases} \varepsilon y''(x) + y'(x) - y(x) = 0, & x \in (0,1), \\ y(0) = 1, \quad y(1) = 1. \end{cases}$$

The exact solution is given by

$$y(x) = \frac{(\exp(m_2) - 1)\exp(m_1 x) - (1 - \exp(m_1))\exp(m_2 x)}{\exp(m_2) - \exp(m_1)},$$

where $m_{1,2} = \dfrac{-1 \pm \sqrt{1+4\varepsilon}}{2\varepsilon}$. This BVP has a boundary layer in the left end at $x = 0$.

Example 2. Consider the non-homogeneous singular perturbation problem

$$\begin{cases} \varepsilon y''(x) + y'(x) = 1 + 2x, & x \in (0,1), \\ y(0) = 0, \quad y(1) = 0. \end{cases}$$

Table 1: Maximum point-wise errors E_ε^N and the rate of convergence r_ε^N for Example 1.

ε	Number of intervals N					
	16	32	64	128	256	512
$1e-2$	7.5846e-3 0.9764	3.8548e-3 1.0110	1.9127e-3 1.0461	9.2632e-4 1.0183	4.5732e-4 1.0637	2.1879e-4
$1e-4$	1.1136e-2 0.9829	5.6347e-3 0.9994	2.8189e-3 1.0144	1.3953e-3 1.0366	6.8264e-4 1.0743	3.2347e-4
$1e-8$	1.1173e-2 0.9783	5.6771e-3 0.9896	2.8563e-3 0.9959	1.4326e-3 0.9979	7.1795e-4 0.9939	3.6449e-4
E^N r^N	1.1173e-2 0.9783	5.6771e-3 0.9896	2.8563e-3 0.9959	1.4326e-3 0.9979	7.1795e-4 0.9939	

The exact solution $y(x)$ is of the form

$$y(x) = x(1 + x - 2\varepsilon) + \frac{(2\varepsilon - 1)(1 - \exp(-x/\varepsilon))}{1 - \exp(-1/\varepsilon)}.$$

The above problem has a boundary layer at the left side of the domain near $x = 0$.

For any value of N and ε, we calculate the exact maximum point-wise errors E_ε^N and the corresponding rates of convergence by $E_\varepsilon^N = \max_{0\le j\le N} |y(x_j) - Y_j^N|$ and $r_\varepsilon^N = \log_2 \left(\frac{E_\varepsilon^N}{E_\varepsilon^{2N}} \right)$, where u is the exact solution and U_j^N is the numerical solution obtained by using N mesh intervals in the domain $\overline{\Omega}^N$. Now we would like to see uniform error and rate of convergence as $E^N = \max_{0\le\varepsilon\le1} E_\varepsilon^N$ and $r^N = \log_2 \left(\frac{E^N}{E^{2N}} \right)$.

5.1 Right end boundary layer problem

Finally, we consider the following singularly perturbed boundary value problem with right end boundary layer:

$$\begin{cases} Ly(x) \equiv -\varepsilon y''(x) + a(x)y'(x) + b(x)y(x) = f(x), \ x \in \Omega = (0, 1), \\ B_0 \equiv y(0) = \alpha, \quad B_1 \equiv y(1) = \beta, \end{cases} \tag{27}$$

where $0 < \varepsilon \ll 1$ is a small singular perturbation parameter, the functions $a(x), b(x), f(x)$ are sufficiently smooth and α, β are given constants.

Table 2: Maximum point-wise errors E_ε^N and the rate of convergence r_ε^N for Example 2.

ε	Number of intervals N					
	16	32	64	128	256	512
$1e-2$	4.3216e-2 0.8628	2.3764e-2 0.9728	1.2108e-3 1.0053	6.0325e-4 1.0130	2.9892e-4 1.0566	1.4375e-4
$1e-4$	5.8414e-2 0.9574	3.0087e-2 0.9862	1.5169e-3 1.0074	7.5535e-3 1.0326	3.6920e-3 1.0773	1.7547e-3
$1e-8$	5.8591e-2 0.9527	3.0274e-2 0.9768	1.5389e-2 0.9886	7.7512e-3 0.9942	3.8923e-3 0.9974	1.94967e-3
E^N r^N	5.8591e-2 0.9527	3.0274e-2 0.9768	1.5389e-2 0.9886	7.7512e-3 0.9942	3.8923e-3 0.9974	

Further, we assume that $a(x) \geq 2M > 0$ and $b(x) \geq 0$. Under these assumptions, the above problem (27) has a unique solution which exhibits a boundary layer at $x = 1$.

Using Taylor series expansion for $a(x)$ near the point $x = 1$, we get

$$y(x) = y_0(x) + (\beta - y_0(1)) \exp\left(- \frac{a(1)(1-x)}{\varepsilon} \right), \tag{28}$$

where $y_0(x)$ is the solution of the reduced problem of (27) which is given by $a(x)y_0'(x) + b(x)y_0(x) = f(x)$ with $y_0(0) = \alpha$. As $h \to 0$, we have the following limit $\lim_{h\to 0} y(ih) = y_0(0) + (\beta - y_0(1)) \exp\left(- \frac{a(1)(1-ih)}{\varepsilon} \right)$, which becomes

$$\lim_{h\to 0} y(ih) = y_0(0) + (\beta - y_0(1)) \exp(a(1)(1/\varepsilon - i\rho)), \tag{29}$$

where $\rho = \frac{h}{\varepsilon}$. Introducing an exponentially fitting factor $\sigma(\rho)$ in (27), we get

$$-\varepsilon\sigma(\rho)y''(x) + a(x)y'(x) + b(x)y(x) = f(x), \tag{30}$$

with boundary conditions $B_0 \equiv y(0) = \alpha$, and $B_1 \equiv y(1) = \beta$. On simplifying, we get

$$\sigma(\rho) = \frac{\sigma a(1)}{2} \coth\left[\frac{\sigma a(1)}{2}\right].$$

Now we can use the finite difference scheme and the techniques discussed for the left end boundary layer problem. Finally, we will reach at a three term recurrence relation as follows:

$$\widehat{E}_i y_{i-1} + \widehat{F}_i y_i + \widehat{G}_i y_{i+1} = \widehat{H}_i\,, \quad 1 \le i \le N-1, \tag{31}$$

where

$$\begin{aligned} \widehat{E}_i &= -\frac{2\sigma(\rho)}{h} - \frac{\delta p(x_i)}{2h}, \qquad \widehat{F}_i = \frac{4\sigma(\rho)}{h} + \delta q(x_i), \\ \widehat{G}_i &= -\frac{2\sigma(\rho)}{h} + \frac{\delta p(x_i)}{2h}, \qquad \widehat{H}_i = \delta r(x_i). \end{aligned}$$

Now (31) gives a system of $N-1$ equations with $N-1$ unknowns from y_1 to y_{N-1} where $y(x_i) = y_i$. Hence, we can use Thomas algorithm to solve the tri-diagonal system.

Example 3. Consider the following singular perturbation problem:

$$\begin{cases} -\varepsilon y''(x) + y'(x) + (1+\varepsilon)y(x) = 0, \quad x \in (0,1), \\ y(0) = 1 + \exp(-\frac{1+\varepsilon}{\varepsilon}), \quad y(1) = 1 + 1/e. \end{cases}$$

Here, $y(x)$ is of the form $y(x) = e^{(1+\varepsilon)(\frac{x-1}{\varepsilon})} + e^x$. and has a boundary layer at the right side of the domain near $x = 1$. The numerical results are shown in Table 3.

Table 3: Maximum point-wise errors E_ε^N and the rate of convergence r_ε^N for Example 3.

ε	Number of intervals N					
	16	32	64	128	256	512
$1e-4$	1.1143e-2 0.9836	5.6345e-3 0.9998	2.8197e-3 1.016	1.3958e-3 1.0308	6.8346e-4 1.0631	3.2758e-4
$1e-8$	1.1141e-2 0.9835	5.6343e-3 0.9989	2.8192e-3 1.014	1.3955e-3 1.0303	6.8342e-4 1.0625	3.2754e-4
E^N r^N	1.1141e-2 0.9835	5.6343e-3 0.9989	2.8192e-3 1.014	1.3955e-3 1.0303	6.8342e-4 1.0625	

6 Conclusion

An efficient exponentially fitted finite difference scheme for a class of singularly perturbed BVPs of the form (1) with left (or right) boundary layers is presented in this paper. A comparatively simple fitting factor is introduced and the solution thus obtained through a tri-diagonal system. We carried out the error analysis and numerical results obtained for some examples show that the proposed scheme is of almost first-order accurate up to an logarithm factor. Hence, the key result established here is that the solution thus obtained is uniformly convergent with respect to the perturbation parameter.

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