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Dynamical behavior and synchronization of hyperchaotic complex T-system

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Abstract. In this paper, we introduce a new hyperchaotic complex T-system. This system has complex nonlinear behavior which we study its dynamical properties including invariance, equilibria and their stability, Lyapunov exponents, bifurcation, chaotic behavior and chaotic attractors as well as necessary conditions for this system to generate chaos. We discuss the synchronization with certain and uncertain parameters via adaptive control. For synchronization, we use less controllers than the dimension of the proposed system. Also, we prove that the error system is asymptotically stable by using a Lyapunov function. Numerical simulations are computed to check the analytical expressions.

Keywords: Lyapunov stability, synchronization, chaos, adaptive control.
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1 Introduction

Chaos, which is an interesting phenomenon in nonlinear dynamical systems, has been studied over the last four decades [1, 9, 16, 25, 26, 35, 36]. Chaotic and hyperchaotic systems are nonlinear deterministic systems that displays complex and unpredictable behavior, and the sensitive dependence on initial conditions and on the systems parameters variation is a prominent

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characteristic of chaotic behavior. The chaotic and hyperchaotic systems have many important fields in applied nonlinear sciences, e.g., laser physics, secure communications, nonlinear circuits, synchronization, control, neural networks, and active wave propagation [2,3,5,7,9,11,22,27,33]. Also, there are many interesting cases involving complex variables which have not been actively explored, for example the complex Lorenz equations which are used to describe and simulate the physics of a detuned laser and thermal convection of liquid flows [21,23,28]. The electric field amplitude and the atomic polarization amplitude are both complex, for details see [28] and references therein. Complex Chen and Lü chaotic systems have also been introduced and studied recently in [18].

Chaos synchronization of chaotic systems with real variables has received a significant attention in the last few years, see the Pecora and Carroll results in 1990 [26]. Chaos synchronization, as an important topic in nonlinear science, has been widely investigated in many fields, such as physics, chemistry and ecological science [4, 8] and secure communications [35]. Recently synchronization of chaotic complex systems studied in [18]. The complete synchronization of two identical chaotic and hyperchaotic complex systems with certain and uncertain parameters was studied in [19]. The antisynchronization and adaptive antisynchronization of two different chaotic complex systems were investigated in [14, 15]. Phase and antiphase synchronization of two identical hyperchaotic complex nonlinear systems are studied in [20]. Projective synchronization and modified projective synchronization were performed on the chaotic and complex nonlinear systems in [10, 17].

In 2005, Tigan [30] introduced a new real chaotic nonlinear system of the form:

$$\begin{cases} \dot{x} = a(y - x), \\ \dot{y} = (c - a)x - axz, \\ \dot{z} = xy - bz, \end{cases} \tag{1}$$

with a, b and c positive real parameters and called it the T-system. Some results regarding the T-system were already presented in [13, 31, 32].

In this paper, we wish to study the dynamical properties and the phenomenon of chaos synchronization of a new chaotic complex T-system expressed by:

$$\begin{cases} \dot{x} = a(y - x), \\ \dot{y} = (c - a)x - axz, \\ \dot{z} = \frac{1}{2}(\bar{x}y + x\bar{y}) - bz, \end{cases} \tag{2}$$

where $x = u_1 + iu_2$ and $y = u_3 + iu_4$ are complex variables, $i = \sqrt{-1}$ and $z = u_5$ is a real variable; Dots represent derivatives with respect to time and the overbar $\bar{x}$ and $\bar{y}$ denotes the complex conjugate of x and y, respectively.

This paper is organized as follows: In Section 2, the dynamical properties of system (2) including invariance, dissipativity, equilibria and their stability will be discussed. The necessary conditions such as chaotic attractor, Lyapunov exponents and bifurcations for system (2) to generate chaos are given in section 3. Section 4 contains the study of chaos synchronization of (2) for some values of a, b and c using the method of adaptive control. Also, we calculate expressions for the control functions which are used to achieve chaos synchronization. These expressions are tested numerically and excellent agreement is found. A Lyapunov function is derived to prove that the error system is asymptotically stable. Some figures are presented to show pictorially the effect of chaos synchronization and the rapid decay of errors in time. Our concluding remarks are presented in Section 5.

2 Dynamical behaviors of chaotic complex T-system

In this section we present the basic dynamical analysis of our new system (2), which is a five-dimensional chaotic. The real version of (2) is:

$$\begin{cases} \dot{u_1} = a(u_3 - u_1), \\ \dot{u_2} = a(u_4 - u_2), \\ \dot{u_3} = (c-a)u_1 - au_1u_5, \\ \dot{u_4} = (c-a)u_2 - au_2u_5, \\ \dot{u_5} = u_1u_3 + u_2u_4 - bu_5. \end{cases} \tag{3}$$

System (3) has the following properties:

2.1 Symmetry and invariance:

1. Symmetry about the u_5-axis, due to invariance of the equations under the transformation symmetry

$$(u_1, u_2, u_3, u_4, u_5) \to (-u_1, -u_2, -u_3, -u_4, u_5).$$

Therefore, if $(u_1, u_2, u_3, u_4, u_5)$ is a solution of (3), then

$$(-u_1, -u_2, -u_3, -u_4, u_5),$$

is also a solution of the same system.

2. Symmetry about the u_1, u_3, u_5-axis, since

$$(u_1, u_2, u_3, u_4, u_5) \to (u_1, -u_2, u_3, -u_4, u_5),$$

does not change the equations. Therefore, if $(u_1, u_2, u_3, u_4, u_5)$ is a solution of (3), then $(u_1, -u_2, u_3, -u_4, u_5)$ is also a solution of the same system.

3. Symmetry about the u_2, u_4, u_5-axis, since

$$(u_1, u_2, u_3, u_4, u_5) \to (-u_1, u_2, -u_3, u_4, u_5),$$

does not change the equations. Therefore, if $(u_1, u_2, u_3, u_4, u_5)$ is a solution of (3), then$(u_1, -u_2, u_3, -u_4, u_5)$ is also a solution of the same system.

2.2 Dissipation:

The divergence of (3) is:

$$\nabla \cdot F = \sum_{i=1}^{5} \frac{\partial \dot{u}_i}{\partial u_i} = -2a - b.$$

Therefore, the system (3) is dissipative for the case:

$$\nabla \cdot F = -(2a + b) < 0.$$

2.3 Equilibria and their stability:

The equilibrium points of system (3) can be found by solving the following equations:

$$\begin{cases} a(u_3 - u_1) = 0, \\ a(u_4 - u_2) = 0, \\ (c - a)u_1 - au_1u_5 = 0, \\ (c - a)u_2 - au_2u_5 = 0, \\ u_1u_3 + u_2u_4 - bu_5 = 0. \end{cases} \tag{4}$$

Obviously, $E_0 = (0, 0, 0, 0, 0)$ is a trivial equilibria point of (3), and from (4), we have:

$$u_1^* = u_3^*, u_2^* = u_4^* \Rightarrow u_5^* = \frac{c - a}{a}.$$

If $b(c - a)/a > 0$, then system (4) has a whole circle of equilibrium points given by the expression:

$$(u_1^*)^2 + (u_2^*)^2 = (u_3^*)^2 + (u_4^*)^2 = b(\frac{c - a}{a}) = r^2, \tag{5}$$

so that

$$u_1^* = u_3^* = \pm k_1 \sqrt{b(\frac{c-a}{a})} = \pm k_1 \sqrt{b u_5^*},$$

$$u_2^* = u_4^* = \pm k_2 \sqrt{b(\frac{c-a}{a})} = \pm k_2 \sqrt{b u_5^*},$$

such that $r^2 = b(c-a)/a > 0$ and

$$(k_1, k_2) = (\sin(\theta), \cos(\theta)),$$

or

$$(k_1, k_2) = (\cos(\theta), \sin(\theta)),$$

for $\theta \in (0, 2\pi]$, and the non-isolated nontrivial equilibrium points are:

$$E_\theta = (\pm r\sin(\theta), \pm r\cos(\theta), \pm r\sin(\theta), \pm r\cos(\theta), \frac{c-a}{a}),$$

or

$$E_\theta = (\pm r\cos(\theta), \pm r\sin(\theta), \pm r\cos(\theta), \pm r\sin(\theta), \frac{c-a}{a}).$$

The Jacobian matrix of system (3) is:

$$J = \begin{bmatrix} -a & 0 & a & 0 & 0 \\ 0 & -a & 0 & a & 0 \\ c-a-a\,u_5 & 0 & 0 & 0 & -a\,u_1 \\ 0 & c-a-a\,u_5 & 0 & 0 & -a\,u_2 \\ u_3 & u_4 & u_1 & u_2 & -b \end{bmatrix}. \qquad (6)$$

The real part eigenvalues of J at E_0 and E_θ for $a = 2.1$, $b = 0.6$, and $0 < c < 40$ are shown in Figure 1 (let for example $\theta = 1.2$). So, E_0 and E_θ are unstable equilibrium points [6, 12, 24].

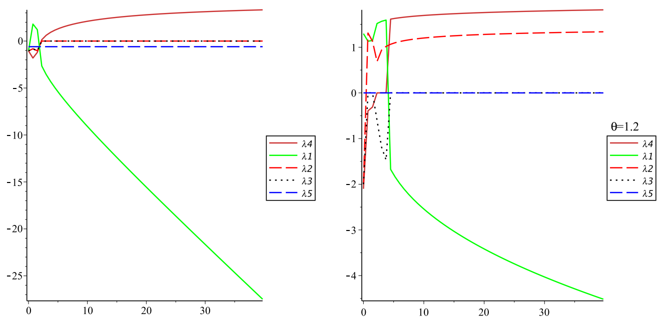
3 Lyapunov exponents, attractors and bifurcations

In this section, we discuss and calculate the Lyapunov exponents, attractors and bifurcations of complex T-system described in (3).

3.1 Lyapunov exponents and bifurcations:

System (3) in vector notation can be written as:

$$\dot{U}(t) = H(U(t); \eta), \qquad (7)$$



(a) (b)

Figure 1: Real part of eigenvalues of J for $a = 2.1$, $b = 0.6$ and $0 < c < 40$: (a) at E_0, (b) at E_θ for $\theta = 1.2$.

where $U(t) = [u_1(t), u_2(t), u_3(t), u_4(t), u_5(t)]^t$ is the state space vector, $H = [h_1, h_2, h_3, h_4, h_5]$, η is a set of parameters and $[\cdots]^t$ denotes transpose. The equations for small deviations δU from the trajectory $U(t)$ are:

$$\delta\dot{U}(t) = L_{i,j}(U(t);\eta)\delta U, \qquad i,j = 1,2,3,4,5, \tag{8}$$

where $L_{i,j} = \frac{\partial h_i}{\partial u_j} = J_{i,j}$ is the Jacobian matrix of system (3). The Lyapunov exponents are defined by [34]:

$$Ł_i = \lim_{t\to\infty} \frac{1}{t} \log \frac{\|\delta u_i(t)\|}{\|\delta u_i(0)\|}. \tag{9}$$

To find $Ł_i$, we numerically solve Equations (7) and (8) simultaneously by a simple Runge-Kutta method of order four.

Consider continuous-time system (7) with a parameter $\eta \in R$. As η changes, the limit sets of the system also change. Typically, a small change in η produces small quantitative changes in a limit set. For instance, perturbing η could change the position of a limit set slightly and, if the limit set is not an equilibrium point, its shape or size could also change. There is also the possibility that a small change in η can cause a limit set to undergo a qualitative change. Such a qualitative change is called a bifurcation and the value of η at which a bifurcation occurs is called a bifurcation value. The behavior of the critical points can be summarized on a bifurcation diagram.

For the choice $a = 2.1$, $b = 0.6$ and $0 < c < 40$, and the initial conditions $u_1(0) = 1$, $u_2(0) = -1$, $u_3(0) = 2$, $u_4(0) = -2$, and $u_5(0) = 0$,

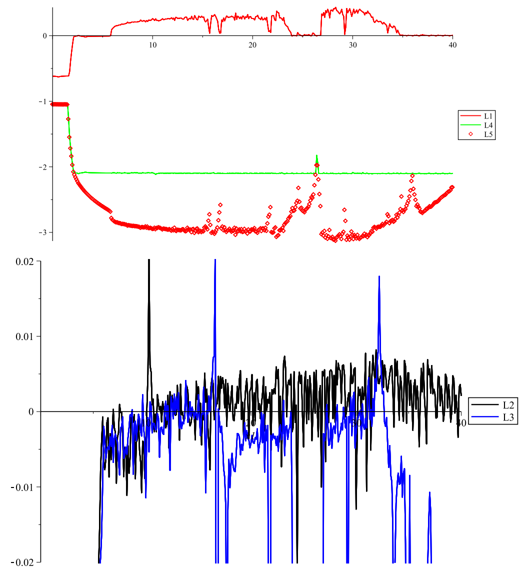


Figure 2: Lyapunov exponents of (3) for $a = 2.1$, $b = 0.6$ and $0 < c < 40$.

the Lyapunov exponents in Figure 2 show that the system (3) is a chaotic system. This means that system (3) (or (2)) for the some values of c is a hyperchaotic system, since at least two Lyapunov exponents, is positive, and it is a dissipative system, since sum of its Lyapunov exponents is negative. The bifurcation diagrams of system 3 are given in Figure 3. These diagrams such as Lyapunov exponents show that system (3) is hyperchaotic.

3.2 Attractors

The chaotic behavior for this system can also be shown by plotting the separation of two nearly initial conditions trajectories. Figure 4 shows two computed time series solutions of (3) with nearly initial conditions. Sensitivity for initial conditions are evident. Some attractors diagrams in Figure 5 show chaotic behaviors of system(3).

4 Chaos synchronization of chaotic complex T-system

In this section, we study chaos synchronization of system (3) for parameter values $a = 2.1$, $b = 0.6$ and $c = 30$ which generates hyperchaotic behavior.

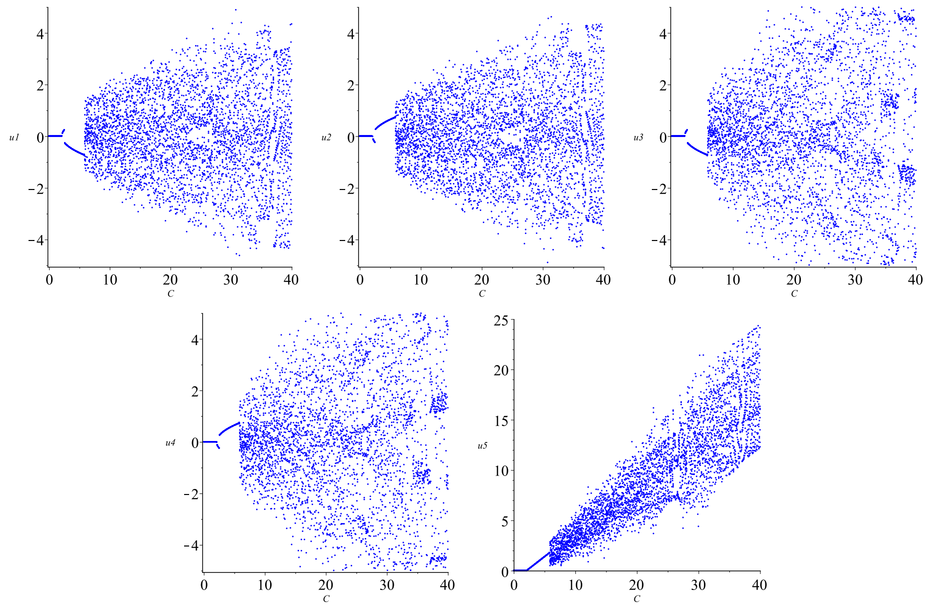


Figure 3: Bifurcation diagrams of (3) for $a = 2.1$, $b = 0.6$ and $0 < c < 40$ with initial conditions $t_0 = 0$, $u_1(0) = 1$, $u_2(0) = -1$, $u_3(0) = 2$,$u_4(0) = -2$, and $u_5(0) = 0$.

We assume that we have two identical complex hyperchaotic T-systems and the drive system with the subscript d is to control the response system with subscript r. We use the idea of adaptive control technique for synchronization [12, 29] and complete synchronization [19] of two identical complex hyperchaotic T-systems and for stability using Lyapunov stability method [12, 19, 29].

Our aim is to design a controller and make the response system follow the drive system, until they ultimately become the same. In spite of most papers that use n controller for n dimensions systems, we use three controller for five dimensional. The drive system for (3) is define as:

$$\begin{cases} \dot{u}_{1d} = a(u_{3d} - u_{1d}), \\ \dot{u}_{2d} = a(u_{4d} - u_{2d}), \\ \dot{u}_{3d} = (c - a)u_{1d} - au_{1d}u_{5d}, \\ \dot{u}_{4d} = (c - a)u_{2d} - au_{2d}u_{5d}, \\ \dot{u}_{5d} = u_{1d}u_{3d} + u_{2d}u_{4d} - bu_{5d}. \end{cases} \tag{10}$$

Now we discuss the synchronization with adaptive control in two cases:

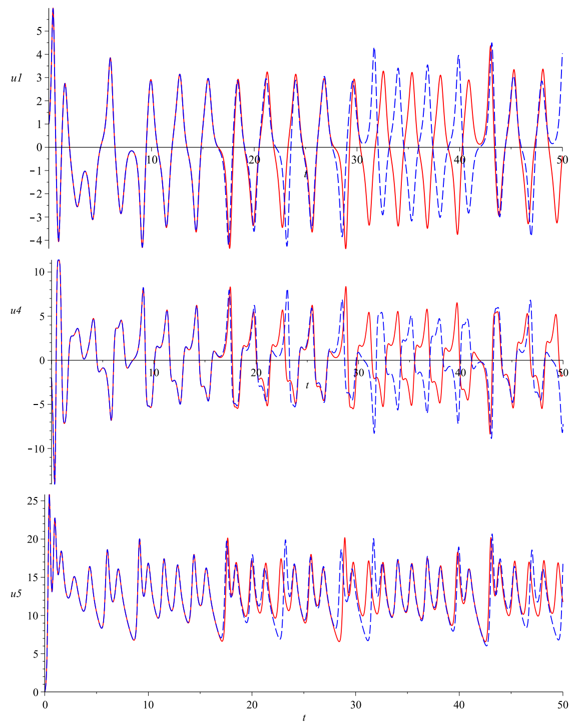


Figure 4: Sensitivity for initial condition of (3) for $a = 2.1$, $b = 0.6$ and $c = 30$ with initial conditions $(u_1(0) = 1, u_2(0) = -1, u_3(0) = 2, u_4(0) = -2, u_5(0) = 0)$(dashed line) and $(u_1(0) = 1.01, u_2(0) = -0.99, u_3(0) = 1.98, u_4(0) = -2, u_5(0) = 0)$ (solid line).

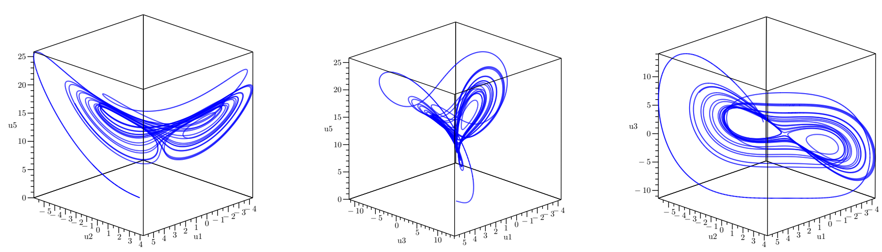


Figure 5: Attractors of (3) for $a = 2.1$, $b = 0.6$ and $c = 30$ with initial conditions: $u_1(0) = 1, u_2(0) = -1, u_3(0) = 2, u_4(0) = -2, u_5(0) = 0$.

Case 1. *Parameters of response system is certain:* The response system with known parameters system for (3) is define as:

$$\begin{cases} \dot{u}_{1r} = a(u_{3r} - u_{1r}), \\ \dot{u_{2r}} = a(u_{4r} - u_{2r}), \\ \dot{u_{3r}} = (c-a)u_{1r} - au_{1r}u_{5r} + v_1, \\ \dot{u_{4r}} = (c-a)u_{2r} - au_{2r}u_{5r} + v_2, \\ \dot{u_{5r}} = u_{1r}u_{3r} + u_{2r}u_{4r} - bu_{5r} + v_3. \end{cases} \tag{11}$$

The dynamical error system of (10) and (11) in real form is:

$$\begin{cases} \dot{e}_{u_1} = a(e_{u_3} - e_{u_1}), \\ \dot{e}_{u_2} = a(e_{u_4} - e_{u_2}), \\ \dot{e}_{u_3} = (c-a)e_{u_1} - a(u_{1r}u_{5r} - u_{1d}u_{5d}) + v_1, \\ \dot{e}_{u_4} = (c-a)e_{u_2} - a(u_{2r}u_{5r} - u_{2d}u_{5d}) + v_2, \\ \dot{e}_{u_5} = u_{1r}u_{3r} + u_{2r}u_{4r} - u_{1d}u_{3d} - u_{2d}u_{4d} - be_{u_5} + v_3. \end{cases} \tag{12}$$

The systems (11) and (10) can be effectively synchronized in the situation of uncertain parameters for controller functions. The drive system (10) and the response system (11) can be synchronized globally and asymptotically for any different initial condition with following adaptive controller laws:

$$\begin{cases} v_1 = -\hat{c}e_{u_1} + \hat{a}(u_{1r}u_{5r} - u_{1d}u_{5d}) - e_{u_3}, \\ v_2 = -\hat{c}e_{u_2} + \hat{a}(u_{2r}u_{5r} - u_{2d}u_{5d}) - e_{u_3}, \\ v_3 = -u_{1r}u_{3r} - u_{2r}u_{4r} + u_{1d}u_{3d} + u_{2d}u_{4d} + \hat{b}e_{u_5} - e_{u_5}, \end{cases} \tag{13}$$

where $\hat{a}$, $\hat{b}$ and $\hat{c}$ are the estimates of a, b and c respectively, and the adaptive parameter update laws are chosen as:

$$\begin{cases} \dot{\tilde{a}} = \dot{\hat{a}} = -e_{u_1}^2 - e_{u_2}^2 - e_{u_3}[u_{1r}u_{5r} - u_{1d}u_{5d}] - e_{u_4}[u_{2r}u_{5r} - u_{2d}u_{5d}] - \tilde{a}, \\ \dot{\tilde{b}} = \dot{\hat{b}} = -e_{u_5}^2 - \tilde{b}, \\ \dot{\tilde{c}} = \dot{\hat{c}} = e_{u_3}e_{u_1} + e_{u_4}e_{u_2} - \tilde{c}, \end{cases} \tag{14}$$

where $\tilde{a} = \hat{a} - a$, $\tilde{b} = \hat{b} - b$ and $\tilde{c} = \hat{c} - c$. Define the Lyapunov as below:

$$V(t) = \tfrac{1}{2}\sum_{i=1}^{5} e_{u_i}^2 + \frac{1}{2}(\tilde{a}^2 + \tilde{b}^2 + \tilde{c}^2). \tag{15}$$

Note that $\hat{a} - \tilde{a} = a > 0$, then with the above mentioned conditions we have:

$$\dot{V(t)} = -\left[(\hat{a} - \tilde{a})(e_{u_1}^2 + e_{u_2}^2) + e_{u_3}^2 + e_{u_4}^2 + e_{u_5}^2 + \tilde{a}^2 + \tilde{b}^2 + \tilde{c}^2\right] < 0.$$

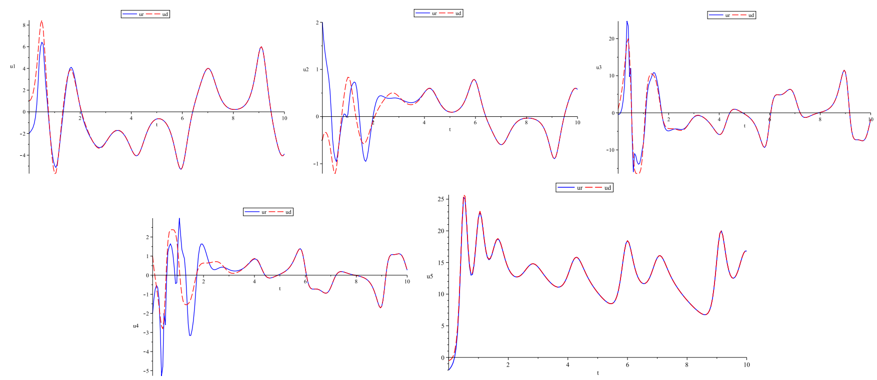


Figure 6: Synchronization of (10) and (11) for $u_{1d}(0) = 1$, $u_{2d}(0) = -0.5$, $u_{3d}(0) = 1$, $u_{4d}(0) = 1$, $u_{5d}(0) = -0.5$, $u_{1r}(0) = -2$, $u_{2r}(0) = 2$, $u_{3r}(0) = -0.4$, $u_{4r}(0) = -2$, $u_{5r}(0) = -2$.

To demonstrate and verify the validity of the proposed scheme, we discuss and illustrate the numerical simulations results for chaotic complex T-system (3). Systems (10), (11) and (14) with controllers (13) are solved numerically for $a = 2.1$, $b = 0.6$ and $c = 30$, where chaotic attractor exists (see Section 2) and with different initial conditions $u_{1d}(0) = 1$, $u_{2d}(0) = -0.5$, $u_{3d}(0) = 1$, $u_{4d}(0) = 1$, $u_{5d}(0) = -0.5$, $u_{1r}(0) = -2$, $u_{2r}(0) = 2$, $u_{3r}(0) = -0.4$, $u_{4r}(0) = -2$, $u_{5r}(0) = -2$ and the initial values of the parameters estimation laws are $\hat{a}(0) = 0.2$, $\hat{b}(0) = -1$ and $\hat{c}(0) = 20$. The results of chaotic synchronization of two identical chaotic complex T-systems via adaptive control is shown in Figure 6 and we plotted $u_{id}(t)$ and $u_{ir}(t)$ versus t and $i = 1, 2, 3, 4, 5$, respectively. Figure 6 shows the synchronization of (10) and (11) is achieved after small time interval and suitable for our scheme. The errors due of synchronization are plotted in Figure 7 which are the solutions of system (12). As expected from the above analytical considerations the synchronization errors e_{ui} converge to zero as $t \longrightarrow \infty$. Figure 8, shows the estimates $\hat{a}(t)$, $\hat{b}(t)$, $\hat{c}(t)$ of the unknown parameters of control functions, converge to $a = 2.1$, $b = 0.6$ and $c = 30$, respectively, as $t \longrightarrow \infty$.

Case 2. *Parameters of response system are uncertain:* In this case our main purpose is to investigate complete synchronization [19] hyperchaotic complex T-system with uncertain parameters. Let the response system

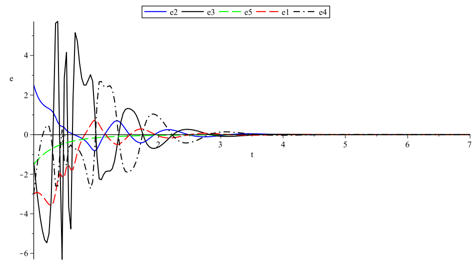


Figure 7: The errors due of synchronization (10) and (11) for $a = 2.1$, $b = 0.6$ and $c = 30$ with initial conditions $u_{1d}(0) = 1$, $u_{2d}(0) = -0.5$, $u_{3d}(0) = 1$, $u_{4d}(0) = 1$, $u_{5d}(0) = -0.5$, $u_{1r}(0) = -2$, $u_{2r}(0) = 2$, $u_{3r}(0) = -0.4$, $u_{4r}(0) = -2$, $u_{5r}(0) = -2$.

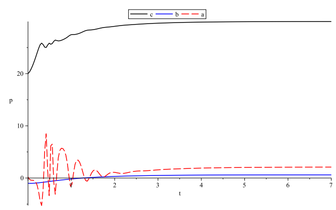


Figure 8: Estimation of parameter for adaptive control synchronization of (10) and (11) for $a = 2.1$, $b = 0.6$ and $c = 30$ with initial conditions $\hat{a}(0) = 0.2$, $\hat{b}(0) = -1$ and $\hat{c}(0) = 20$.

rewritten as the form:

$$\begin{cases} \dot{u}_{1r} = \hat{a}(u_{3r} - u_{1r}), \\ \dot{u_{2r}} = \hat{a}(u_{4r} - u_{2r}), \\ \dot{u_{3r}} = (\hat{c} - \hat{a})u_{1r} - \hat{a}u_{1r}u_{5r} + v_1, \\ \dot{u_{4r}} = (\hat{c} - \hat{a})u_{2r} - \hat{a}u_{2r}u_{5r} + v_2, \\ \dot{u_{5r}} = u_{1r}u_{3r} + u_{2r}u_{4r} - \hat{b}u_{5r} + v_3, \end{cases} \tag{16}$$

where $\hat{a}$, $\hat{b}$ and $\hat{c}$ are uncertain parameters, which need to be estimated in the response system. In order to obtain the complex error dynamical system, we subtracting (10) from (16), then the dynamical error is:

$$\begin{cases} \dot{e}_{u_1} = \hat{a}[e_{u3} - e_{u1}] + \tilde{a}[u_{3d} - u_{1d}], \\ \dot{e}_{u_2} = \hat{a}[e_{u4} - e_{u2}] + \tilde{a}[u_{4d} - u_{2d}], \\ \dot{e}_{u_3} = (\hat{c} - \hat{a})e_{u_1} - \hat{a}(u_{1r}u_{5r} - u_{1d}u_{5d}) + (\tilde{c} - \tilde{a})u_{1d} - \tilde{a}u_{1d}u_{5d} + v_1, \\ \dot{e}_{u_4} = (\hat{c} - \hat{a})e_{u_2} - \hat{a}(u_{2r}u_{5r} - u_{2d}u_{5d}) + (\tilde{c} - \tilde{a})u_{2d} - \tilde{a}u_{2d}u_{5d} + v_2, \\ \dot{e}_{u_5} = u_{1r}u_{3r} + u_{2r}u_{4r} - u_{1d}u_{3d} - u_{2d}u_{4d} - \hat{b}e_{u_5} - \tilde{b}u_{5d} + v_3, \end{cases} \tag{17}$$

where $\tilde{a} = \hat{a} - a$, $\tilde{b} = \hat{b} - b$ and $\tilde{c} = \hat{c} - c$. We define a Lyapunov function as (15). The derivative of $V(t)$ along the solution of system (17) is:

$$\begin{aligned} \dot{V(t)} &= \textstyle\sum_1^5 e_{u_i}\dot{e_{u_i}} + \tilde{a}\dot{\tilde{a}} + \tilde{b}\dot{\tilde{b}} + \tilde{c}\dot{\tilde{c}}, \\ &= e_{u1}[\hat{a}e_{u3} - \hat{a}e_{u1} + \tilde{a}(u_{3d} - u_{1d})] + e_{u2}[\hat{a}e_{u4} - \hat{a}e_{u2} + \tilde{a}(u_{4d} - u_{2d})] \\ &\quad + e_{u3}[(\hat{c} - \hat{a})e_{u_1} - \hat{a}(u_{1r}u_{5r} - u_{1d}u_{5d}) + (\tilde{c} - \tilde{a})u_{1d} - \tilde{a}u_{1d}u_{5d} + v_1] \\ &\quad + e_{u4}[(\hat{c} - \hat{a})e_{u_2} - \hat{a}(u_{2r}u_{5r} - u_{2d}u_{5d}) + (\tilde{c} - \tilde{a})u_{2d} - \tilde{a}u_{2d}u_{5d} + v_2] \\ &\quad + e_{u5}[u_{1r}u_{3r} + u_{2r}u_{4r} - u_{1d}u_{3d} - u_{2d}u_{4d} - \hat{b}e_{u_5} - \tilde{b}u_{5d} + v_3] \\ &\quad + \tilde{a}\dot{\tilde{a}} + \tilde{b}\dot{\tilde{b}} + \tilde{c}\dot{\tilde{c}}. \end{aligned} \tag{18}$$

Following conditions guarantees that $\dot{V(t)} < 0$,

$$\begin{cases} v_1 = -\hat{c}e_{u1} + \hat{a}(u_{1r}u_{5r} - u_{1d}u_{5d}) - \eta_1 e_{u3}, \\ v_2 = -\hat{c}e_{u2} + \hat{a}(u_{2r}u_{5r} - u_{2d}u_{5d}) - \eta_2 e_{u4}, \\ v_3 = -u_{1r}u_{3r} - u_{2r}u_{4r} + u_{1d}u_{3d} + u_{2d}u_{4d} + \hat{b}e_{u5} - \eta_3 e_{u5}, \end{cases} \tag{19}$$

and

$$\begin{cases} \dot{\hat{a}} = \dot{\tilde{a}} = (e_{u1}^2 + e_{u2}^2) + e_{u1}(u_{1d} - u_{3d}) + e_{u2}(u_{2d} - u_{4d}) \\ \qquad\qquad +(e_{u3}u_{1d} + e_{u4}u_{2d})(1 + u_{5d}), \\ \dot{\hat{b}} = \dot{\tilde{b}} = e_{u5}u_{5d}, \\ \dot{\hat{c}} = \dot{\tilde{c}} = -e_{u3}u_{1d} - e_{u4}u_{2d}, \end{cases} \tag{20}$$

where $\eta_i, (i = 1, 2, 3)$ are positive constants. Then

$$\dot{V(t)} = -a(e_{u1}^2 + e_{u2}^2) - (\eta_1 e_{u3}^2 + \eta_2 e_{u4}^2 + \eta_3 e_{u5}^2) < 0.$$

Therefore, Lyapunovs direct method implies that the equilibrium points of systems (17) and (20) are asymptotically stable.

To demonstrate and verify the validity of the proposed scheme, we discuss and illustrate the numerical simulations results for chaotic complex T-system (3). Systems (10), (16) and (20) with controllers (19) were solved numerically for $a = 2.1$, $b = 0.6$ and $c = 30$ with different initial conditions, $u_{1d}(0) = 1, u_{2d}(0) = -1, u_{3d}(0) = 2, u_{4d}(0) = -2, u_{5d}(0) = 0)$ and

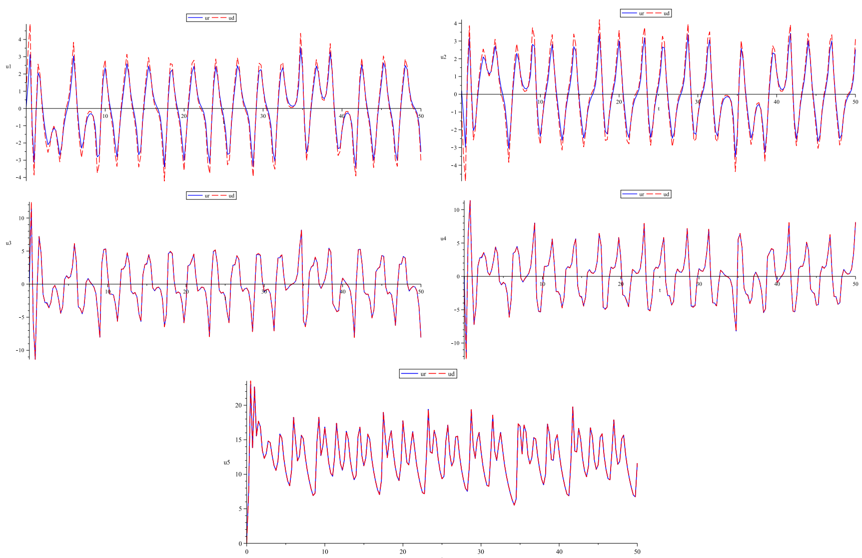


Figure 9: Complete synchronization of (10) and (16) for $a = 2.1$, $b = 0.6$ and $c = 30$ with initial conditions $u_{1d}(0) = 1$, $u_{2d}(0) = -1$, $u_{3d}(0) = 2$, $u_{4d}(0) = -2$, $u_{5d}(0) = 0)$, $u_{1r}(0) = 0.1$, $u_{2r}(0) = 0.2$, $u_{3r}(0) = 0.3$, $u_{4r}(0) = 0$ and $u_{5r}(0) = 0.5$.

$u_{1r}(0) = 0.1$, $u_{2r}(0) = 0.2$, $u_{3r}(0) = 0.3$, $u_{4r}(0) = 0$ and $u_{5r}(0) = 0.5$. We choose $\eta_1 = 12$, $\eta_2 = 13$, $\eta_3 = 9$ and the initial values of the parameters estimation laws are $\hat{a}(0) = 0.2$, $\hat{b}(0) = -1$, $\hat{c}(0) = 20$. The results of chaotic synchronization of two identical chaotic complex T-systems via adaptive control is shown in Figure 9 and we plotted $u_{id}(t)$ and $u_{ir}(t)$ $(i = 1, 2, 3, 4, 5)$ versus t respectively. Figure 9 shows that the complete synchronization of (10) and (16) is achieved after small time interval and suitable for our scheme. The synchronization errors are plotted in Figure 10 which are the solutions of system (17). As expected from the above analytical considerations the synchronization errors e_{ui} converge to zero as $t \longrightarrow \infty$. Figure 11 show the estimates $\hat{a}(t)$, $\hat{b}(t)$, $\hat{c}(t)$ of the unknown parameters of control functions, converge to $a = 2.1$, $b = 0.6$ and $c = 30$ respectively, as $t \longrightarrow \infty$.

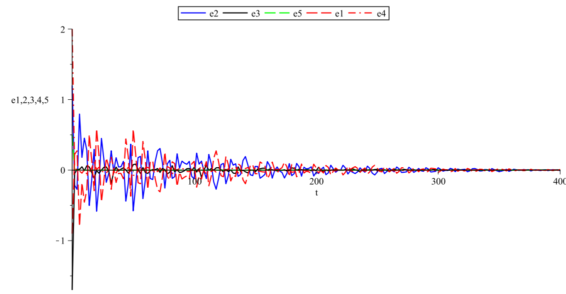


Figure 10: The errors due of synchronization (10) and (16) for $a = 2.1$, $b = 0.6$ and $c = 30$ with initial conditions $u_{1d}(0) = 1$, $u_{2d}(0) = -1$, $u_{3d}(0) = 2$, $u_{4d}(0) = -2$, $u_{5d}(0) = 0$), $u_{1r}(0) = 0.1$, $u_{2r}(0) = 0.2$, $u_{3r}(0) = 0.3$, $u_{4r}(0) = 0$ and $u_{5r}(0) = 0.5$.

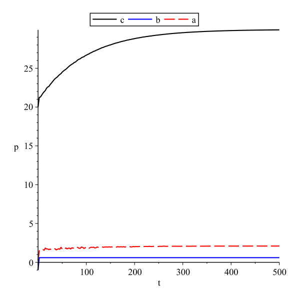


Figure 11: Estimation of parameter for adaptive control complete synchronization of (10) and (16) for $a = 2.1$, $b = 0.6$ and $c = 30$ with the initial conditions $\hat{a}(0) = 0.2$, $\hat{b}(0) = -1$ and $\hat{c}(0) = 20$.

5 conclusion

In this paper, we have studied the dynamics of a new hyperchaotic complex nonlinear system. This system has been introduced and studied with real variables in the recent literature. Our complexified version can be applied in engineering, for example, in secure communications. The basic properties of proposed system including invariance, dissipativity, equilibria and their stability, Lyapunov exponents, chaotic behavior and chaotic attractors were investigated and necessary conditions for proposed system to generate chaos was discussed. For synchronization, we used less controllers than the dimension of the dynamic system. The results of chaos synchronization and the exponential decay of errors were shown pictorially to agree very well with analytical predictions. A Lyapunov function were derived to prove that the associated error systems and parameter estimation rule are asymptotically stable.

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