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ON ZERO-DIVISOR GRAPHS OF QUOTIENT RINGS AND COMPLEMENTED ZERO-DIVISOR GRAPHS

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Abstract. For an arbitrary ring R, the zero-divisor graph of R, denoted by $\Gamma(R)$, is an undirected simple graph that its vertices are all nonzero zero-divisors of R in which any two vertices x and y are adjacent if and only if either $xy = 0$ or $yx = 0$. It is well-known that for any commutative ring R, $\Gamma(R) \cong \Gamma(T(R))$ where $T(R)$ is the (total) quotient ring of R. In this paper we extend this fact for certain noncommutative rings, for example, reduced rings, right (left) self-injective rings and one-sided Artinian rings. The necessary and sufficient conditions for two reduced right Goldie rings to have isomorphic zero-divisor graphs is given. Also, we extend some known results about the zero-divisor graphs from the commutative to noncommutative setting: in particular, complemented and uniquely complemented graphs.

1. Introduction

Throughout the paper, R denotes a ring with identity element (not necessarily commutative) and a zero-divisor in R is an element of R which is either a left or a right zero-divisor. We denote the set of all zero-divisors of R and the set of all regular elements of R by $Z(R)$ and C_R, respectively. Also, the set of all minimal prime ideals of R is denoted by minSpec(R). The zero-divisor graph of R, denoted by $\Gamma(R)$, is an undirected simple graph with the vertex set $Z(R)^* = Z(R)\setminus\{0\}$ in which any two distinct vertices x and y are adjacent if and only if either $xy = 0$ or $yx = 0$. The notion of zero-divisor graph of a commutative

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ring with identity element was first introduced by I. Beck in [6] and has been studied by many authors (see for example [1, 2, 3, 5, 11]). In [10], Redmond has extended this notion to noncommutative rings and showed that for any ring R, the graph $\Gamma(R)$ is connected and its diameter is at most 3. Moreover if $\Gamma(R)$ contains a cycle, then the girth of $\Gamma(R)$ is at most 4.

A multiplicative set $S \subseteq R$ is called a *right Ore set* if for any $a \in R$ and $s \in S$, $aS \cap sR \neq \emptyset$. We say that R is a *right Ore ring* if C_R is a right Ore set. For a right Ore ring R, we define a relation "$\sim$" on $R \times C_R$ as follow:
$(a_1, s_1) \sim (a_2, s_2)$ if and only if there exist $b_1, b_2 \in R$ such that $s_1b_1 = s_2b_2 \in C_R$ and $a_1b_1 = a_2b_2 \in R$. It can be seen that the relation "$\sim$" is an equivalence relation and so we write a/s or as^{-1} for the equivalence class (a, s). The set of all equivalence classes is denoted by RC_R^{-1}. For any $a_1/s_1, a_2/s_2 \in RC_R^{-1}$, there exist $s, s' \in C_R$ and $r, r' \in R$ such that $s_1s = s_2r \in C_R$ and $a_2s' = s_1r'$. Thus we define $a_1/s_1 + a_2/s_2 = (a_1s + a_2r)/t$, where $t = s_1s = s_2r$ and $(a_1/s_1)(a_2/s_2) = a_1r'/s_2s'$. It is well-known that the addition and the multiplication defined on RC_R^{-1} are binary operations and under these operations RC_R^{-1} becomes a ring (for more details see [8, p. 301-302]). The ring RC_R^{-1} is usually called the (*classical*) *right quotient ring* of R.

In Section 1, we prove that for any reduced right Ore ring R, $\Gamma(R)$ and $\Gamma(RC_R^{-1})$ are isomorphic (Theorem 2.2). Also it is shown that if R is a von Neumann regular ring, a right (left) self-injective ring or a right (left) Artinian ring, then $\Gamma(R)$ and $\Gamma(RC_R^{-1})$ are isomorphic. We show that if R is a reduced right Goldie ring, then $\Gamma(R) \cong \Gamma(D_1 \times D_2 \times \cdots \times D_n)$ for suitable division rings $D_1, D_2, \ldots, D_n$ and integer number n (Proposition 2.4). In Section 2, first complemented and uniquely complemented are introduced and then we give some results about them. For example, it is shown that for any reduced ring R, $\Gamma(R)$ is complemented if and only if $\Gamma(R)$ is uniquely complemented (Proposition 3.3). Also we prove that for any reduced right Ore ring R, if RC_R^{-1} is von Neumann regular, then $\Gamma(R)$ is uniquely complemented and while $\Gamma(R)$ is complemented, then every prime ideal of RC_R^{-1} is maximal (Proposition 3.6). Next we show that for an Artinian ring R with $\mathrm{Nil}_*(R)$ nonzero:
(1) If $\Gamma(R)$ is complemented, then either $|R| = 8$, $|R| = 9$, or $|R| > 9$ and $\mathrm{Nil}_*(R) = \{0, x\}$, for some $0 \neq x \in R$.
(2) If $\Gamma(R)$ is uniquely complemented and $|R| > 9$, then any complement of the nonzero nilpotent element of R is an end (Theorem 3.7).

2. Zero-divisor graphs of quotient rings

Remark 2.1. For a reduced right Ore ring R, the right quotient ring RC_R^{-1} is also reduced. To see this, suppose that $xy = 0$, where $x, y \in R$. First we show that $xs^{-1}y = 0$, for any $s \in C_R$. Since R is a right Ore ring, there exist $r_1 \in R$ and $s_1 \in C_R$ such that $sr_1 = ys_1$ and so $y = sr_1{s_1}^{-1}$. Thus $0 = xy = xsr_1{s_1}^{-1}$ and hence $xsr_1 = 0$. Since R is reduced, we have $r_1xs = 0$. It follows that $r_1x = 0$, and so $xr_1 = 0$. Thus $0 = xr_1{s_1}^{-1} = xs^{-1}y$. Now suppose that $xs^{-1}yt^{-1} = 0$, where $x, y \in R$ and $s, t \in C_R$. Then $xs^{-1}y = 0$ and so $xr_1{s_1}^{-1} = 0$ (note that $s^{-1}y = r_1{s_1}^{-1}$). Thus $xr_1 = 0 = r_1x$ because R is reduced. By the first part of the proof, $0 = r_1(ts_1)^{-1}x = r_1{s_1}^{-1}t^{-1}x$. Therefore $s^{-1}yt^{-1}x = 0$ and so $yt^{-1}xs^{-1} = 0$. Thus RC_R^{-1} is reduced.

Let G be an undirected simple graph. As in [9], for every two vertices a and b of G, we define $a \leq b$ if a and b are not adjacent and each vertex of G adjacent to b is also adjacent to a. We write $a \sim b$ if both $a \leq b$ and $b \leq a$. It is easy to see that $\sim$ is an equivalence relation on G. We denote the equivalence class of a vertex x of G by $[x]$. Note that for any ring R with $a, b \in Z(R)^*$, we have $a \sim b$ in $\Gamma(R)$ if and only if $(\mathrm{ann}_l(a) \cup \mathrm{ann}_r(a)) \setminus \{a\} = (\mathrm{ann}_l(b) \cup \mathrm{ann}_r(b)) \setminus \{b\}$. If R is a right Ore ring and $A \subseteq R$, then the set $\{a/s \mid a \in A, s \in C_R\}$ is denoted by A_{C_R}.
In [4], the authors proved that for any commutative ring R, $\Gamma(R) \cong \Gamma(T(R))$ where $T(R)$ is the quotient ring of R. Here, by the same method as [4], we extend this fact to the reduced right Ore rings.

Theorem 2.2. *Let R be a reduced right Ore ring with right quotient ring RC_R^{-1}. Then the graphs $\Gamma(R)$ and $\Gamma(RC_R^{-1})$ are isomorphic.*

Proof. Let $S = C_R$ and $T = RS^{-1}$. Denote the equivalence relations defined above on $Z(R)^*$ and $Z(T)^*$ by $\sim_R$ and $\sim_T$, respectively, and denote their respective equivalence classes by $[a]_R$ and $[a]_T$. Since R and RS^{-1} are reduced, we note that $\mathrm{ann}_T(x/s) = \mathrm{ann}_R(x)_S$ and $\mathrm{ann}_T(x/s) \cap R = \mathrm{ann}_R(x)$; thus $x/s \sim_T x/t$, $x \sim_R y \Leftrightarrow x/s \sim_T y/s$, $([x]_R)_S = [x/1]_T$ and $[x/s]_T \cap R = [x]_R$ for all $x, y \in Z(R)^*$ and $s, t \in S$. Since $Z(T) = Z(R)_S$, by the above comments, we have $Z(R)^* = \bigcup_{\alpha \in A} [a_\alpha]_R$ and $Z(T)^* = \bigcup_{\alpha \in A} [a_\alpha/1]_T$ (both disjoint unions) for some $\{a_\alpha\}_{\alpha \in A} \subseteq R$.
We next show that $|[a]|_R = |[a/1]|_T$ for each $a \in Z(R)^*$. First assume that $[a]_R$ is finite. Then it is clear $[a]_R \subseteq [a/1]_T$. For the inverse inclusion, let $x \in [a/1]_T$. Then $x = b/s$ with $b \in [a]_R$ and $s \in S$. Since $\{bs^n \mid n \geq 1\} \subseteq [a]_R$ is finite, $b = bs^i$ for some

integer $i > 1$, and hence $b/s = bs^i/s = bs^{i-1} \in [a]_R$. Now suppose that $[a]_R$ is infinite. Clearly $|[a]_R| \leq |[a/1]_T|$. Define an equivalence relation $\approx$ on S by $s \approx t$ if and only if $sa = ta$. Then $s \approx t$ if and only if $sb = tb$ for all $b \in [a]_R$. It is easily verified that the map $[a]_R \times S/_{\approx} \longrightarrow [a/1]_T$, given by $(b, [s]) \to b/s$, is well-defined and surjective. Thus $|[a/1]_T| \leq |[a]_R||S/_{\approx}|$. Also the map $S/_{\approx} \longrightarrow [a]_R$, given by $[s] \to sa$, is clearly well-defined and injective. Hence $|S/_{\approx}| \leq |[a]_R|$, and so $|[a/1]_T| \leq |[a]_R|^2 = |[a]_R|$ since $|[a]_R|$ is infinite. Thus $|[a]_R| = |[a/1]_T|$. Therefore there is a bijection $\phi_\alpha : [a_\alpha] \longrightarrow [a_\alpha/1]$ for each $\alpha \in A$. Define $\phi : Z(R)^* \longrightarrow Z(T)^*$ by $\phi(x) = \phi_\alpha(x)$ if $x \in [a_\alpha]$. Thus we need only show that x and y are adjacent in $\Gamma(R)$ if and only if $\phi(x)$ and $\phi(y)$ are adjacent in $\Gamma(T)$; i.e., $xy = 0$ if and only if $\phi(x)\phi(y) = 0$. Let $x \in [a]_R, y \in [b]_R, w \in [a/1]_T$ and $z \in [b/1]_T$. It is sufficient to show that $xy = 0$ if and only if $zw = 0$. Note that $\mathrm{ann}_T(x) = \mathrm{ann}_T(a) = \mathrm{ann}_T(w)$ and $\mathrm{ann}_T(y) = \mathrm{ann}_T(b) = \mathrm{ann}_T(z)$. Thus $xy = 0 \Leftrightarrow y \in \mathrm{ann}_T(x) = \mathrm{ann}_T(w) \Leftrightarrow yw = 0 \Leftrightarrow w \in \mathrm{ann}_T(y) = \mathrm{ann}_T(z) \Leftrightarrow wz = 0$. Hence $\Gamma(R)$ and $\Gamma(T(R))$ are isomorphic as graphs. $\square$

Let R be a ring. We denote the group of unit elements of R by $U(R)$. By [8, Proposition 11.4], the right quotient ring of R exists and $R \cong RC_R^{-1}$ if and only if $C_R = U(R)$. In this case, we say that R is a *classical ring* and it is clear that $\Gamma(R) \cong \Gamma(RC_R^{-1})$. Recall that R is *von Neumann regular* if for each $x \in R$, there exists $y \in R$ such that $x = xyx$. In the following we give some examples of noncommutative rings R for which $\Gamma(R) \cong \Gamma(RC_R^{-1})$.

Example 2.3. (a) For any von Neumann regular ring R, we have $\Gamma(R) \cong \Gamma(RC_R^{-1})$. To see this, let $q \in C_R$. Then there exists $q' \in R$ such that $q = qq'q$. So $q(1 - q'q) = 0 = (1 - qq')q$. Since q is regular, $qq' = q'q = 1$ and hence $q \in U(R)$. Thus $C_R = U(R)$ and this implies that R is a classical ring. Therefore $\Gamma(R) \cong \Gamma(RC_R^{-1})$.

(b) Let R be a ring in which for any $q \in R$, the chain $qR \supseteq q^2R \supseteq \dots$ stabilizes. Then R is a classical ring. Indeed, if $q \in C_R$, then by hypothesis, there exists $n \geq 1$ such that $q^nR = q^{n+1}R$. Thus $q^n = q^{n+1}q'$, for some $q' \in R$. Since q is regular, $qq' = 1$. Also $q(1 - q'q) = 0$ and hence $q'q = 1$. Thus $C_R = U(R)$ and we conclude that $\Gamma(R) \cong \Gamma(RC_R^{-1})$. In particular if R is a right (left) Artinian ring, then $\Gamma(R) \cong \Gamma(RC_R^{-1})$.

(c) Let V be a vector space over a division ring K. Then $R = End(V_R)$ is a classical ring. To see this, we note that V is a semisimple K-module and by [7, Proposition 4.27], R is a von Neumann regular ring. Now the assertion is obtained from (a).

(d) Every left (right) self-injective ring is a classical ring. Suppose that R is a left self-injective ring and a is a regular element in R. We show that $a \in U(R)$. Define R-monomorphism $f : R \to R$ by $f(r) = ra$. Since R is self-injective, there exists R-homomorphism $g : R \to R$ such that $gf = 1$. Now $a = 1(a) = gf(a) = g(a^2) = a^2 g(1)$. Since a is a regular element, we have $1 = ag(1)$. Thus $a = ag(1)a$ and hence $1 = g(1)a$ because again a is regular. Therefore $a \in U(R)$ and so R is a classical ring. This implies that $\Gamma(R) \cong \Gamma(RC_R^{-1})$.

(e) Let R be a right Ore ring such that RC_R^{-1} is a Noetherian right R-module. Then $\Gamma(R) \cong \Gamma(RC_R^{-1})$. Clearly, the natural homomorphism $\phi : R \longrightarrow RC_R^{-1}$, given by $\phi(r) = r/1$, is injective. We show that ϕ is an isomorphism. Let $rs^{-1} \in RC_R^{-1}$. Then the chain $s^{-1}R \subseteq s^{-2}R \subseteq \ldots$ stabilizes because RC_R^{-1} is Noetherian as right R-module. Thus there exists $n \geq 1$ such that $s^{-n}R = s^{-n-1}R$, and so $s^{-n-1} = s^{-n}r_1$ for some $r_1 \in R$. Hence $s^{-1} = r_1 \in R$ and so $\phi(rr_1) = rr_1 = rs^{-1}$. This implies that ϕ is epimorphism; thus $\Gamma(R) \cong \Gamma(RC_R^{-1})$.

Proposition 2.4. *Let R be a reduced right Goldie ring. Then $\Gamma(R) \cong \Gamma(D_1 \times D_2 \times \cdots \times D_n)$ for suitable division rings $D_1, D_2, \ldots, D_n$ and integer number n.*

Proof. By Goldie's Theorem [8, Theorem 11.13], RC_R^{-1} is a semisimple ring. Also by Remark 2.1, RC_R^{-1} is reduced. Using the Weddernborn-Artin Theorem, we conclude that $RC_R^{-1} \cong D_1 \times D_2 \times \cdots \times D_n$ for suitable division rings $D_1, D_2, \ldots, D_n$ and integer number n. Now by Theorem 2.2, $\Gamma(R) \cong \Gamma(D_1 \times D_2 \times \cdots \times D_n)$. □

Let x be a vertex of a graph G. We say that x is a *primitive vertex*, if it is a minimal element in the ordering $\leq$.

Theorem 2.5. *Let $\{A_i\}_{i\in I}$ and $\{B_j\}_{j\in J}$ be two families of domains and let $A = \prod_{i\in I} A_i$ and $B = \prod_{j\in J} B_j$. Then $\Gamma(A) \cong \Gamma(B)$ if and only if there exists a bijection $\psi : I \to J$ such that $|A_i| = |B_{\psi(i)}|$ for each $i \in I$.*

Proof. One direction of the proof is clear. For the other direction, suppose that $\phi : \Gamma(A) \to \Gamma(B)$ is an isomorphism. We note that each primitive vertex in $\Gamma(A)$ has exactly one nonzero component. Let $x = (x_i)_{i\in I}$ be a primitive vertex in $\Gamma(A)$. Then there is $i_0 \in I$ such that $x_{i_0} \neq 0$ and $x_i = 0$ for each $i_0 \neq i \in I$. Thus the set $\{[z] \mid z$ is a primitive vertex of $\Gamma(A)\}$ has cardinality $|I|$. Similarly, the set $\{[z] \mid z$ is a primitive vertex of $\Gamma(B)\}$ has cardinality $|J|$. One can easily see that z is a primitive vertex of $\Gamma(A)$ if and only if $\phi(z)$ is a primitive vertex of $\Gamma(B)$. Also $[z]=[z']$ if and only if $[\phi(z)]=[\phi(z')]$. Thus we have $|I|=|J|$. On the other hand, $z \in [x]$ if and only if $\phi(z) \in [\phi(x)]$ and hence $|[x]|=|[\phi(x)]|$. Moreover $|[x]|=|A_{i_0}|$ and $|[\phi(x)]|=|B_j|$ for some $j \in J$. Clearly ϕ induces the required bijection ψ. □

Corollary 2.6. *Let A and B be two reduced right Goldie rings which are not domains. Then $\Gamma(A) \cong \Gamma(B)$ if and only if there exists a bijection $\phi : \mathrm{minSpec}(A) \to \mathrm{minSpec}(B)$ such that $|A/P| = |B/\phi(P)|$ for each $P \in \mathrm{minSpec}(A)$.*

Proof. Set $T(A)=AC_A^{-1}$ and $T(B)=BC_B^{-1}$. Since A and B are reduced right Goldie rings, by [8, Proposition 11.22], we may assume that $\mathrm{minSpec}(A)=\{P_1,P_2,\ldots,P_m\}$, $\mathrm{minSpec}(B)=\{Q_1,Q_2,\ldots,Q_n\}$ and $T(A)\cong K_1\times\ldots\times K_m$, $T(B)\cong L_1\times\ldots\times L_n$, where division rings K_i and L_j are the quotient rings of A/P_i and B/Q_j, respectively for $1\le i\le m$ and $1\le j\le n$. By Theorem 2.2, $\Gamma(A)\cong\Gamma(K_1\times K_2\times\cdots\times K_m)$ and $\Gamma(B)\cong\Gamma(L_1\times L_2\times\cdots\times L_n)$. Now suppose that $\Gamma(A)\cong\Gamma(B)$. By Theorem 2.5, we conclude that $m=n$ and there exists a permutation ρ of $\{1,\ldots,n\}$ such that $|A/P_i|=|K_i|=|L_{\rho(i)}|=|B/Q_{\rho(i)}|$ for $1\le i\le n$. It is clear that ρ induces the required bijection ϕ. Conversely, if there exists such a bijection ϕ, then by Theorem 2.5, $\Gamma(K_1\times\ldots\times K_m)\cong\Gamma(L_1\times\ldots\times L_n)$ and hence $\Gamma(A)\cong\Gamma(B)$. □

3. Complemented zero-divisor graphs

Let G be an undirected simple graph. As in [9], for distinct vertices a and b of G, we say that a and b are *orthogonal*, written by $a\perp b$, if a and b are adjacent and there is no vertex c of G which is adjacent to both a and b, i.e., the edge $a-b$ is not part of any triangle of G. Thus for $a,b\in Z(R)^*$, we have $a\perp b$ in $\Gamma(R)$ if and only if $ab=0$ or $ba=0$ and

$$(\mathrm{ann}_l(a)\cup\mathrm{ann}_r(a))\cap(\mathrm{ann}_l(b)\cup\mathrm{ann}_r(b))\subseteq\{0,a,b\}.$$

Finally, we say that G is a *complemented graph* if for each vertex a of G, there exists a vertex b of G (called a *complement* of a) such that $a\perp b$, and that G is *uniquely complemented* if it is complemented and whenever $a\perp b$ and $a\perp c$, then $b\sim c$.

In this section, we first show that for any reduced ring R, $\Gamma(R)$ is complemented if and only if $\Gamma(R)$ is uniquely complemented. Next we prove that if R is a reduced and von Neumann regular ring, then $\Gamma(R)$ is complemented. In the end of this section, we show that if R is not reduced, then under certain conditions, $\Gamma(R)$ is complemented or $\Gamma(R)$ is uniquely complemented. In order to show these results, we need the following two lemmas which translate the above graph-theoretic concepts into ring-theoretic terms.

Lemma 3.1. *Consider the following statements for a ring R and $a, b \in Z(R)^*$.*
(1) $a \sim b$.
(2) $aR = bR$.
(3) $\mathrm{ann}_l(a) = \mathrm{ann}_l(b)$.
(a) *If R is reduced, then* (1) *and* (3) *are equivalent.*
(b) *If R is a reduced von Neumann regular ring, then all three statements are equivalent.*

Proof. (a). If R is reduced, then $\mathrm{ann}_l(x) = \mathrm{ann}_r(x)$ for each $x \in Z(R)^*$. Thus we have $a \sim b$ if and only if $\mathrm{ann}_l(a) = \mathrm{ann}_l(b)$.
(b). Since R is a reduced ring, it is enough to show that (2) and (3) are equivalent. (2) $\Rightarrow$ (3) is clear. To show (3) $\Rightarrow$ (2), let $a = aca$ for some $c \in R$. Thus $a(1 - ca) = 0$ and so $1 - ca \in \mathrm{ann}_r(a) = \mathrm{ann}_l(a)$. Since $\mathrm{ann}_l(a) = \mathrm{ann}_l(b)$, we have $(1 - ca)b = 0$ and hence $b(1 - ca) = 0$. Therefore $b = bca \in Ra$. This implies that $Rb \subseteq Ra$. Similarly, $Ra \subseteq Rb$ and so $Ra = Rb$. $\square$

Lemma 3.2. *Let R be a reduced ring and $a, b \in Z(R)^*$. Then the following statements are equivalent.*
(1) $a \perp b$.
(2) $ab = 0$ *and* $a + b$ *is a regular element of* R.

Proof. (1) $\Rightarrow$ (2). Since $a \perp b$ and R is reduced, we have $ab = 0$. Suppose that $(a + b)c = 0$ for some $c \in Z(R)^*$. Let $y = ac = -bc$. Then $by = ay = 0$. Since $a \perp b$, we conclude that $y \in \{0, a, b\}$. If $y = a$, then $a^2 = ay = 0$ which a contradiction. Similarly, $y = b$ implies that $b^2 = 0$, again a contradiction. Hence $y = 0$ and so $ac = bc = 0$. It follows that $c \in \{0, a, b\}$ because $a \perp b$. If $c = a$, then $a^2 = 0$, a contradiction. Similarly, $c \neq b$ and hence $a + b$ is regular.
(2) $\Rightarrow$ (1). Suppose that $ca = cb = 0$ for some $c \in Z(R)^*$. Then $c(a + b) = 0$, a contradiction because $a + b$ is regular. Since $ab = 0$, we have $a \perp b$. $\square$

Proposition 3.3. *Let R be a reduced ring and $a, b, c \in Z(R)^*$. If $a \perp b$ and $a \perp c$, then $b \sim c$. Consequently, $\Gamma(R)$ is uniquely complemented if and only if $\Gamma(R)$ is complemented.*

Proof. Since $a \perp b$ and $a \perp c$, we have $ab = ac = 0$. We first show that $bc \neq 0$. If $bc = 0$, then $c \in \{0, a, b\}$ because $ac = 0$ and $a \perp b$.

By our assumption, $c = a$ or $c = b$. If $c = a$, then $ac = a^2 = 0$ and hence $a = 0$, a contradiction. Similarly $c \neq b$. Thus $bc \neq 0$. Now suppose that $db = 0$ for some $d \in Z(R)^*$. Then $0 = (ac)d = a(cd)$ and $0 = (db)c = c(db) = (cd)b$. It follows that $cd \in \{0, a, b\}$ because $a \perp b$. If $cd \neq 0$, then $cd = a$ or $cd = b$ and hence $a^2 = 0$ or $b^2 = 0$, which a contradiction. Therefore $cd = 0$ and so $c \leq b$. Similarly $b \leq c$, and thus $b \sim c$. □

Remark 3.4. Let R be a reduced von Neumann regular ring. Then for any $a \in Z(R)^*$, we have $a = ue$ where $u \in U(R)$ and $e \in R$ is idempotent. To see this, let $a \in R$. Since R is von Neumann regular, there exists $b \in R$ such that $a = aba$. Then $a(1 - ba) = 0$ and hence $(1 - ba)a = 0$ because R is reduced. Thus $a = ba^2$. Similarly, $a = a^2b$. We set $x = b^2a$, $e = ax$ and $u = (1 - e + a)$. Then $e^2 = axax = ab^2aab^2a = ab^2a^2b^2a = ab^2a^2bba = ab^2aba = ab^2a = e$. Also, since $a = a^2b^2a$, and $0 = ab^2 - a^2b^2ab^2 = a(b^2 - ab^2ab^2)$, we have $(b^2 - ab^2ab^2)a = 0$ and so $b^2a = ab^2ab^2a$. This implies that $u(1 - e + x) = (1 - e + a)(1 - e + x) = 1$. On the other hand, $ab^2a^2 = a$ and $a^2b^2 - ab = 0$. Hence $a(ab^2 - b) = 0 = (ab^2 - b)a = 0$ and so $ab^2a = ba = b^2a^2$. Also $ab^2 - abab^2 = 0$ implies that $a(b^2 - bab^2) = 0$ and hence $(b^2 - bab^2)a = 0$. Thus $b^2a = bab^2a = b^2a^2b^2a$. Now, we conclude that $(1 - e + x)u = (1 - e + x)(1 - e + a) = 1$.

Corollary 3.5. *If R is a reduced von Neumann regular ring, then $\Gamma(R)$ is a uniquely complemented graph.*

Proof. By Remark 3.4, for any $a \in Z(R)^*$, there exist $u \in U(R)$ and idempotent $e \in R$ such that $a = ue$. Clearly, $a(1 - e) = 0$. Suppose that $ax = 0$ and $(1 - e)x = 0$, for some $x \in R$. Then $x = ex$ and hence $ux = uex = 0$. Since $u \in U(R)$, we conclude that $x = 0$. Thus $a \perp (1 - e)$. □

Proposition 3.6. *Let R be a reduced right Ore ring. Then:*
(a) *If RC_R^{-1} is von Neumann regular, then $\Gamma(R)$ is uniquely complemented.*
(b) *If $\Gamma(R)$ is complemented, then every prime ideal of RC_R^{-1} is maximal.*

Proof. (a). Since R is reduced, by Theorem 2.2, $\Gamma(R) \cong \Gamma(RC_R^{-1})$. Also by Corollary 3.5, $\Gamma(RC_R^{-1})$ is uniquely complemented. Thus $\Gamma(R)$ is uniquely complemented.
(b). Let P and Q be two prime ideals of RC_R^{-1} such that $P \subsetneqq Q$. Thus there exists $xs^{-1} \in Q$ such that $xs^{-1} \notin P$. Then $x \in Z(R)^*$ because $P \neq R$. Since $\Gamma(R)$ is complemented, there exists $y \in Z(R)^*$ such that

$x \perp y$. Now by Lemma 3.2, $xy = 0$ and $x+y$ is a regular element. Also since R is reduced, we have $xRC_R^{-1}y = 0$ and so $xRC_R^{-1}y \subseteq P$. This implies that $y \in P$ because P is prime and $x \notin P$. Thus $x+y \in Q$ and hence $Q = RC_R^{-1}$, a contradiction. It follows that every prime ideal of RC_R^{-1} is maximal. □

Recall that a vertex of a graph is called an end if there is only one other vertex adjacent to it. We say that a ring R is an *Artinian ring* if R is both a left and a right Artinian ring. Let R be a ring. The prime radical of R, denoted by $\mathrm{Nil}_*(R)$, is the intersection of all prime ideals in R and the Jacobson radical of R, denoted by $\mathrm{Rad}(R)$, is the intersection of all maximal right ideals of R. We conclude the paper with the following theorem which gives the necessary conditions for an Artinian ring R with $\mathrm{Nil}_*(R) \neq 0$, such that $\Gamma(R)$ is a complemented or uniquely complemented graph.

Theorem 3.7. *Let R be an Artinian ring with $\mathrm{Nil}_*(R)$ nonzero.*
(a) *If $\Gamma(R)$ is complemented, then either $|R| = 8$, $|R| = 9$, or $|R| > 9$ and $\mathrm{Nil}_*(R) = \{0, x\}$ for some $0 \neq x \in R$.*
(b) *If $\Gamma(R)$ is uniquely complemented and $|R| > 9$, then any complement of the nonzero nilpotent element of R is an end.*

Proof. (a). Suppose that $\Gamma(R)$ is complemented and let $a \in \mathrm{Nil}_*(R)$ have index of nilpotence $n \geq 3$. Let $y \in Z(R)^*$ be a complement of a. Then $a^{n-1}y = 0 = a^{n-1}a$; so $y = a^{n-1}$, because $a \perp y$. Thus $a \perp a^{n-1}$ and this implies that $\mathrm{ann}_l(a) \cup \mathrm{ann}_r(a) = \{0, a^{n-1}\}$. Similarly, $a^i \perp a^{n-1}$ for each $1 \leq i \leq n-2$. Suppose that $n > 3$. Then $a^{n-2}+a^{n-1}$ kills both a^{n-2} and a^{n-1}, a contradiction, because $a^{n-2} \perp a^{n-1}$ and $a^{n-2} + a^{n-1} \notin \{0, a^{n-2}, a^{n-1}\}$. Thus if R has a nilpotent element with index $n \geq 3$, then $n = 3$. In this case, $Ra^2 = \{0, a^2\}$ because any $z \in Ra^2$ kills both a and a^2 and $a \perp a^2$. Also if $za^2 = 0$, then $za \in \mathrm{ann}_l(a) = \{0, a^2\}$ and so either $za = 0$ or $za = a^2$. If $za = 0$, then $z = 0$ or $z = a^2$ while if $za = a^2$, then $(z - a)a = 0$ and hence $z = a$ or $z = a + a^2$. Therefore $\mathrm{ann}_l(a^2) = \{0, a, a^2, a + a^2\}$. Thus the R-epimorphism $r \longrightarrow ra^2$, from R onto Ra^2 implies that R is a local ring with $|R| = 8$, $\mathrm{Nil}_*(R) = \mathrm{ann}_l(a^2)$ its maximal ideal and $\Gamma(R)$ is a star graph with center a^2 and two edges.

Now suppose that each nonzero nilpotent element of R has index of nilpotence 2. Let $y \in \mathrm{Nil}_*(R)$ have complement $z \in Z(R)^*$ and assume that $2y \neq 0$. Without loss of generality, we can assume that $yz = 0$. Note that $(ry)y = 0 = (ry)z$ for all $r \in R$. Thus $Ry \subseteq \{0, y, z\}$. Then necessarily $2y = z$ since $2y \in Ry \subseteq \{0, y, z\}$. Also $\mathrm{ann}_r(y) = \{0, y, 2y\}$ since $y \perp 2y$. Thus $Ry = \{0, y, 2y\}$; so we have $|R| = 9$. In this case,

R is local with maximal ideal $\mathrm{Nil}_*(R) = \mathrm{ann}_r(y)$ and $\Gamma(R)$ is a star graph with one edge.

Next suppose that each nonzero nilpotent element of R has index nilpotence 2 and $|R| \neq 9$. By above, we must have $2y = 0$. We show that $\mathrm{Nil}_*(R) = \{0, y\}$. Suppose that z is another nonzero nilpotent element of R; so $z^2 = 0$. Then $y + z \in \mathrm{Nil}_*(R)$ and hence $(y + z)^2 = 0$. Suppose that y' and z' are complements of y and z, respectively. Then we have $yy' = 0$ or $y'y = 0$ and $zz' = 0$ or $z'z = 0$. We proceed by cases.

Case 1. $yy' = 0$ and $zz' = 0$. Since $y \perp y'$ and $z \perp z'$, $Ry \subseteq \{0, y, y'\}$ and $Rz \subseteq \{0, z, z'\}$. We claim that $yz = zy = 0$. Note that $yz \in Rz \subseteq \{0, z, z'\}$. If $yz \neq 0$, then either $yz = z$ or $yz = z'$. If $yz = z$, then $0 = y(yz) = yz$, a contradiction. Thus $yz = z'$. It follows that $z' \in \mathrm{Nil}_*(R)$ and so $z'^2 = 0$. Since $z \perp z'$ and $z(z + z') = (z + z')z' = 0$, we conclude that $z + z' = 0$; so $z' = -z = z$, a contradiction (by the definition of complement). Thus $yz = 0$. Similarly, $zy = 0$. Let w be a complement of $y + z$. Then $w(y + z) = 0$ or $(y + z)w = 0$. We note that $w \neq y$. For if $w = y$, then $(y + z)z = 0$ and $wz = 0$ and hence $z \in \{0, w, y + z\} = \{0, y, y + z\}$, a contradiction. Similarly, $w \neq z$. We claim that $(y + z)w = 0$. Otherwise $w(y + z) = 0$. Then $wy = wz \in Ry \cap Rz$. Thus $wy = wz = 0$ or $wy = wz = y' = z'$. If $wy = 0$, then since $(y + z)y = 0$ and $(y + z) \perp w$, we conclude that $y \in \{0, y + z, w\}$, a contradiction. If $wy = wz = y' = z'$, then $y' \in \mathrm{Nil}_*(R)$ which again is a contradiction (similar to what was described above for $z' \in \mathrm{Nil}_*(R)$). Thus $(y + z)w = 0$. On the other hand, $z'y \in Ry \subseteq \{0, y, y'\}$. If $z'y = 0$, then since $yz = 0$ and $z \perp z'$, we have $y \in \{0, z, z'\}$ and hence $y = z'$. Thus $z' \in \mathrm{Nil}_*(R)$, a contradiction. If $z'y = y'$, then $y' \in \mathrm{Nil}_*(R)$ which again is a contradiction. Thus $z'y = y$. Similarly, $y'z = z$. Also $y'y \in Ry \subseteq \{0, y, y'\}$. We claim that $y'y = y$. If $y'y = y'$, then $y' \in \mathrm{Nil}_*(R)$, a contradiction. If $y'y = 0$, then $y'(y + z) = y'z = z \in \{0, y + z, w\}$ which is a contradiction (because $zw = 0$ and $(y + z)z = 0$). Thus $y'y = y$ and so $y'^2 \neq 0$. Since R is an Artinian ring, every right zero-divisor of R is a left zero-divisor. Thus $y't = 0$ for some nonzero $t \in R$. Now $ty \in Ry$ and so $ty = 0$, $ty = y$ or $ty = y'$. If $ty = 0$, then since $y't = 0$ and $y \perp y'$, we have $t \in \{0, y, y'\}$. Then $0 = y't = y'^2$ or $0 = y't = y'y = y$, which is a contradiction. Thus $ty = y$ or $ty = y'$ and we conclude that $0 = y'ty = y'y = y$ or $0 = y'ty = y'^2$, a contradiction.

Case 2. $y'y = 0$ and $zz' = 0$. Then $yR \subseteq \{0, y, y'\}$ and $Rz \subseteq \{0, z, z'\}$ and so $yy' \in \{0, y, y'\}$. If $yy' = y'$, then $y' \in \mathrm{Nil}_*(R)$, a contradiction. If $yy' = 0$, then by Case 1, we are done. Thus we have $yy' = y$. Now since

R is an Artinian ring and $y'y = 0$, $ty' = 0$ for some nonzero $t \in R$. On the other hand, $yt \in \{0, y, y'\}$. If $yt = 0$, then $t \in \{0, y, y'\}$ (because $ty' = 0$ and $y \perp y'$). Therefore either $t = y$ or $t = y'$. This implies that $0 = ty' = yy' = y$ or $0 = ty' = {y'}^2$, again a contradiction. Thus we have $yt = y$ or $yt = y'$. Then $0 = yty' = yy' = y$ or $0 = y'yt = {y'}^2$, a contradiction.
Case 3. $y'y = 0$ and $z'z = 0$. It is similar to Case 1.
Case 4. $yy' = 0$ and $z'z = 0$. It is similar to Case 2.
(b). Suppose that $\Gamma(R)$ is uniquely complemented and $|R| > 9$. Let $0 \neq x \in \mathrm{Nil}_*(R)$. By part (a), $\mathrm{Nil}_*(R) = \{0, x\}$. Let y be a complement of x. Then $xy = 0$ or $yx = 0$. Without loss of generality, we may assume that $xy = 0$. Clearly $x(x+y) = 0$, since $x^2 = 0$. We claim that $x \perp (x+y)$. Suppose $w \in Z(R)^*$ such that $x - w$ and $(x+y) - w$ are two edges of $\Gamma(R)$. Now we proceed by cases.
Case 1. $xw = 0$ and $(x+y)w = 0$. Then $yw = 0$ and since $xw = 0$ and $x \perp y$, we conclude that $w \in \{0, x, y\}$. If $w = y$, then $y^2 = 0$ and hence $x(x+y) = 0$ and $(x+y)y = 0$. This contradicts that $x \perp y$. Thus $w = x$ and we are done.
Case 2. $wx = 0$ and $(x+y)w = 0$. Then $xw = yw$ (note that $x = -x$) and since $xw \in \{0, x\}$, either $xw = 0$ or $xw = x$. If $xw = 0$, then $xw = yw = 0$ and similar to Case 1, $w = x$. Thus suppose that $xw = yw = x$. Since $xy = 0$ and R is an Artinian ring, $yt = 0$ for some $t \in Z(R)^*$. Now if $tx = 0$, then $t \in \{0, x, y\}$ (note that $x \perp y$) and we deduce $t = x$ (since $y^2 \neq 0$). If $tx = x$, then $0 = ytx = yx$. Thus in any case, we have $yx = 0$. On the other hand, $(wy)x = 0$ and $y(wy) = 0$ and hence $wy \in \{0, x, y\}$. Now we continue the proof by subcases;

Subcase 1. $wy = 0$. Then since $wx = 0$ and $x \perp y$ and $w \in \{0, x, y\}$; so $w = x$ and we are done.

Subcases 2. $wy = y$. Then $(w-1)y = 0$ and also $x(w-1) = 0$. Thus $(w-1) \in \{0, x, y\}$. Clearly $w \neq 1$. If $w - 1 = x$, then $w = 1 + x$ is invertible (note that $x \in \mathrm{Rad}(R) = \mathrm{Nil}_*(R)$), a contradiction. Therefore $w - 1 = y$ and this implies that $0 = wx = x + yx = x$, which again is a contradiction.

Subcases 3. $wy = x$. Then $0 = wx = w(yw) = (wy)w = xw = x$, a contradiction.
Case 3. $xw = 0$ and $w(x+y) = 0$. It is similar to Case 2.
Case 4. $wx = 0$ and $w(x+y) = 0$. It is similar to Case 1.
Thus in any case, we conclude that $x \perp (x+y)$. Since $\Gamma(R)$ is uniquely complemented and $x \perp y$, we have that $(x+y) \sim y$. Suppose that there exists $z \in Z(R)^* \setminus \{x\}$ such that $zy = 0$ or $yz = 0$. Without loss of generality, we can assume that $zy = 0$. Then since $(x+y) \sim y$, we

have $z(x+y)=0$ or $(x+y)z=0$. If $z(x+y)=0$, Then $zx=0$, a contradiction. Thus $(x+y)z=0$. Then $xz+yz=0$ and since $xz\neq 0$, we have $x+yz=0$. Now $zx=zyz=0$, which again is a contradiction. Thus no such z can exist; so y is an end. □

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References

1. M. Afkhami and K. Khashyarmanesh, *On the cozero-divisor graphs of commutative rings*, App. Mathematics, **4** (2013), 979-985.
2. S. Akbari and A. Mohammadian, *On the zero-divisor graph of a commutative ring*, J. Algebra, **274** (2004), 847-855.
3. S. Akbari and A. Mohammadian, *Zero-divisor graphs of noncommutative rings*, J. Algebra, **296** (2006), 462-479.
4. D. F. Anderson, R. Levy and J. Shapiro, *Zero-divisor graphs, von Neumann regular rings, and Boolean algebras*, J. Pure Appl. Algebra, **180** (2003), 221-241.
5. D. F. Anderson and P. S. Livingston, *The zero-divisor graph of a commutative ring*, J. Algebra, **217** (1999), 434-447.
6. I. Beck, *Coloring of commutative rings*, J. Algebra, **116** (1988), 208-226.
7. T. Y. Lam, *A first course in noncommutative rings*, New York: Springer-Verlag, 1991.
8. T. Y. Lam, *Lectures on modules and rings*, Graduate Texts in Math, New York-Heidelberg- Berlin: Springer-Verlag, 1999.
9. R. Levy and J. Shapiro, *The zero-divisor graph of von Neumann regular rings*, Comm. Algebra, **30** (2002), 745-750.
10. S. P. Redmond, *The zero-divisor graph of a noncommutative ring*, Internat J. Commutative Rings, (4) **1** (2002), 203-211.
11. T. Tamizh Chelvam and S. Nithya, *A note on the zero-divisor graph of a lattice*, Transactions on Combinatorics, (3) **3** (2014), 51-59.

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