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ON COMPONENT EXTENSIONS OF LOCALLY COMPACT ABELIAN GROUPS

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Abstract. Let $\pounds$ be the category of locally compact abelian groups and $A, C \in \pounds$. In this paper, we define component extensions of A by C and show that the set of all component extensions of A by C forms a subgroup of $Ext(C, A)$ whenever A is a connected group. We establish conditions under which the component extensions split and determine LCA groups which are component projective. We also gives a necessary condition for an LCA group to be component injective in $\pounds$.

1. Introduction

Let $\pounds$ denote the category of locally compact abelian (LCA) groups (will be written additively) with continuous homomorphisms as morphisms. The identity component of a group $G \in \pounds$ is denoted by G_0. A morphism is called proper if it is open onto its image and a short exact sequence $0 \to A \xrightarrow{\phi} B \xrightarrow{\psi} C \to 0$ in $\pounds$ is said to be proper exact if ϕ and ψ are proper morphisms. In this case the sequence is called an extension of A by C (in $\pounds$). Following [4], we let $Ext(C, A)$ denote the (discrete) group of extensions of A by C.The group operation on $Ext(C, A)$ is as in Theorem 2.19. The splitting problem in LCA groups is finding conditions on A and C under which $Ext(C, A) = 0$. In [2, 3, 5, 7, 8, 9, 12, 13] the splitting problem is studied. Sometimes, the splitting problem is limited to a subgroup or a subset of $Ext(C, A)$. Some subgroups of $Ext(C, A)$ such as

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$Pext(C, A)$,$^*Pext(C, A)$,$Tpext(C, A)$ and $Apext(C, A)$ have been studied in [2, 7, 8, 9]. In [13], we define s-pure extensions and obtained some results. In [8] the question investigated is connected with the search for condition under which the group of pure extensions , $Pext(C, A)$, is null. In this paper, we introduce a new subgroup of $Ext(C, A)$, namely $Ext(C, A)_0$. We study the vanishing problem for this subgroup and will find a classification for the group C. An extension $0 \to A \xrightarrow{\phi} B \xrightarrow{\psi} C \to 0$ is called a component extension if $0 \to A_0 \xrightarrow{\phi} B_0 \xrightarrow{\psi} C_0 \to 0$ is an extension. Let $Ext(C, A)_0$ denote the set of all component extensions of A by C. In Section 2, we show that $Ext(C, A)_0$ is a subgroup of $Ext(C, A)$ whenever A is a connected group (Theorem 2.19). In Section 3, we introduce component injective and component projective in $\pounds$. An LCA group G is a component projective group in $\pounds$ if and only if $G \cong \mathbb{R}^n \bigoplus C \bigoplus A$ where C is a compact connected group having a cotorsion dual and A a discrete free group (Theorem 3.6). If G is a component injective group in $\pounds$, then $G \cong \mathbb{R}^n \bigoplus (\mathbb{R}/\mathbb{Z})^\sigma \bigoplus H$ where n is a nonnegative integer, σ a cardinal number and H a totally disconnected, LCA group (Theorem 3.3).

The additive topological group of real numbers is denoted by $\mathbb{R}$, $\mathbb{Q}$ is the group of rationals with the discrete topology and $\mathbb{Z}$ is the group of integers with the discrete topology. The Pontrjagin dual of a group G is denoted by $\hat{G}$. For more on locally compact abelian groups, see [6].

2. Component extensions

Let $A, C \in \pounds$. In this section, we will define component extensions and will show that the set of all component extensions of A by C is a subgroup of $Ext(C, A)$ whenever A is a connected group.

Definition 2.1. An extension $0 \to A \xrightarrow{\phi} B \xrightarrow{\psi} C \to 0$ is called a component extension if $0 \to A_0 \xrightarrow{\phi} B_0 \xrightarrow{\psi} C_0 \to 0$ is an extension.

Lemma 2.2. *An extension $E : 0 \to A \xrightarrow{\phi} B \xrightarrow{\psi} C \to 0$ is a component extension if and only if $0 \to A_0 \xrightarrow{\phi} B_0 \xrightarrow{\psi} C_0 \to 0$ is an exact sequence.*

Proof. Let $0 \to A_0 \xrightarrow{\phi} B_0 \xrightarrow{\psi} C_0 \to 0$ be an exact sequence. By [6, Theorem 5.29], $\phi : A_0 \to B_0$ and $\psi : B_0 \to C_0$ are proper morphisms. Hence E is a component extension. □

Remark 2.3. Let $G \in \pounds$ and H be a connected subgroup of G. We know that $(G/H)_0$ is the intersection of all open subgroups of G/H. But an open subgroup of G/H has the form K/H where K is an open subgroup of G containing H. Since H is connected, then by [6, Theorem 7.8], $H \subseteq K$ for every open subgroup K of G. Hence, $(G/H)_0 = G_0/H$.

Lemma 2.4. *Every extension of a connected LCA group by a LCA group is a component extension.*

Proof. Let $E : 0 \to A \xrightarrow{\phi} B \to C \to 0$ be an extension such that A is connected. Since $\phi(A)$ is a connected subgroup of B, so by Remark 2.3, $C_0 \cong B_0/\phi(A)$. Hence $0 \to A \xrightarrow{\phi} B_0 \to C_0 \to 0$ is a component extension. □

Lemma 2.5. *Every extension of a totally disconnected group by a totally disconnected group is a component extension.*

Proof. Let $E : 0 \to A \xrightarrow{\phi} B \xrightarrow{\psi} C \to 0$ be an extension such that A and C are totally disconnected. We claim that B is totally disconnected. Since $\psi(B_0) \subseteq C_0$ and C is totally disconnected, it follows that $\psi(B_0) = 0$. Hence, $B_0 \subseteq Im\phi$. But, $Im\phi$ is a totally disconnected group. Therefore, $B_0 = 0$ and B is a totally disconnected group. □

The extension $0 \to A \to A \bigoplus C \to C \to 0$ is called the trivial extension.

Lemma 2.6. *The trivial extension of A by C is a component extension.*

Proof. It is clear. □

Recall that two extensions $0 \to A \xrightarrow{\phi_1} B \xrightarrow{\psi_1} C \to 0$ and $0 \to A \xrightarrow{\phi_2} X \xrightarrow{\psi_2} C \to 0$ are said to be equivalent if there is a topological isomorphism $\beta : B \to X$ such that the following diagram

$$\begin{tikzcd}
0 \arrow[r] & A \arrow[r, "\phi_1"] \arrow[d, "1_A"] & B \arrow[r, "\psi_1"] \arrow[d, "\beta"] & C \arrow[r] \arrow[d, "1_C"] & 0 \\
0 \arrow[r] & A \arrow[r, "\phi_2"] & X \arrow[r, "\psi_2"] & C \arrow[r] & 0
\end{tikzcd}$$

is commutative.

Lemma 2.7. *An extension equivalent to a component extension is a component extension.*

Proof. Let

$$E_1 : 0 \to A \xrightarrow{\phi_1} B \xrightarrow{\psi_1} C \to 0$$

and

$$E_2 : 0 \to A \xrightarrow{\phi_2} X \xrightarrow{\psi_2} C \to 0$$

be two equivalent extensions such that E_1 is a component extension. Then, there is a topological isomorphism $\beta : B \to X$ such that $\beta\phi_1 =$

ϕ_2 and $\psi_2\beta = \psi_1$. Let $c_0 \in C_0$. Since E_1 is a component extension, so $\psi_1(b_0) = c_0$ for some $b_0 \in B_0$. Hence, $\psi_2(\beta(b_0)) = \psi_1(b_0) = c_0$. So, $\psi_2 : X_0 \to C_0$ is surjective. Now, let $\psi_2(x_0) = 0$ for some $x_0 \in X_0$. Since $\beta(B_0) = X_0$, so there exists $b_0 \in B_0$ such that $\beta(b_0) = c_0$. Hence, $\psi_1(b_0) = \psi_2(\beta(b_0)) = 0$. Since E_1 is a component extension,then $\phi_1(a_0) = b_0$ for some $a_0 \in A_0$. Consequently, $\phi_2(a_0) = \beta(\phi_1(a_0)) = x_0$. $\square$

Definition 2.8. Let $E : 0 \to A \xrightarrow{\phi} B \xrightarrow{\psi} C \to 0$ be an extension and $\alpha : A \to A'$ be a proper morphism. We define the sequence αE as follows:

$$\alpha E : 0 \to A' \xrightarrow{\phi'} X \xrightarrow{\psi'} C \to 0$$

where

$$X = (A' \bigoplus B)/H$$

$$H = \{(-\alpha(a), \phi(a)); a \in A\}$$

$$\phi'(a') = (a', 0) + H$$

$$\psi'((a', b) + H) = \psi(b)$$

Then, αE is an extension which is called the standard pushout of E (See [4, Proposition 2.3]).

Let $\gamma : C' \to C$ be a proper morphism. We define the sequence $E\gamma$ as follows:

$$E\gamma : 0 \to A \xrightarrow{\phi'} X \xrightarrow{\psi'} C' \to 0$$

where

$$X = \{(b, c'); b \in B, c' \in C', \psi(b) = \gamma(c')\}$$

$$\phi'(a) = (\phi(a), 0)$$

$$\psi'(b, c') = c'$$

Then, $E\gamma$ is an extension which is called the standard pullback of E (See [4, Proposition 2.3]).

Lemma 2.9. *A pullback of a component extension is a component extension.*

Proof. Suppose $0 \to A \xrightarrow{\phi} B \xrightarrow{\psi} C \to 0$ is a component extension and

$$\begin{tikzcd} 0 \arrow[r] & A \arrow[r, "\phi'"] \arrow[d, "1_A"] & B' \arrow[r, "\psi'"] \arrow[d] & C' \arrow[r] \arrow[d, "\gamma"] & 0 \\ 0 \arrow[r] & A \arrow[r, "\phi"] & B \arrow[r, "\psi"] & C \arrow[r] & 0 \end{tikzcd}$$

is the standard pullback diagram. Then,

$$B' = \{(b, c'); \psi(b) = \gamma(c')\}$$

and

$$\phi' : a \longmapsto (\phi(a), 0),\ \ \psi' : (b, c') \longmapsto c'$$

We show that $0 \to A_0 \xrightarrow{\phi'} B'_0 \xrightarrow{\psi'} C'_0 \to 0$ is exact. Let $c'_0 \in C'_0$. Then, $\gamma(c'_0) \in C_0$. Since $\psi : B_0 \to C_0$ is surjective, so there exists $b_0 \in B_0$ such that $\psi(b_0) = \gamma(c'_0)$. Hence, $(b_0, c'_0) \in B'_0$ and $\psi'(b_0, c'_0) = c'_0$. So $\psi' : B'_0 \to C'_0$ is surjective. Now, suppose that $(b, c') \in B'_0$ and $\psi'(b, c') = 0$. Then, $c' = 0$ and $b \in B_0$. Since $\psi(b) = 0$, so there exists $a_0 \in A_0$ such that $\phi(a_0) = b$. Hence, $\phi'(a_0) = (b, 0) = (b, c')$. This shows that $Ker\psi' \mid_{B'_0} \subseteq Im\phi' \mid_{A_0}$. □

Remark 2.10. Let $f : A \to C$ be a proper morphism and $G \in \pounds$. Then

(1) $f_* : Ext(G, A) \to Ext(G, C)$ defined by $f_*([E]) = [fE]$
and
(2) $f^* : Ext(C, G) \to Ext(A, G)$ defined by $f^*([E]) = [Ef]$
are group homomorphisms (See [10, Theorem 2.1]).

Recall that $Ext(C, A)_0$ denotes the set of all component extensions of A by C. We say that $Ext(C, A)_0 = 0$ if every component extension of A by C splits.

Lemma 2.11. *Let $f : A \to C$ be a proper morphism. Then, $f^*(Ext(C, G)_0) \subseteq Ext(A, G)_0$ for all $G \in \pounds$.*

Proof. Let $E \in Ext(C, G)_0$. Then, by definition, Ef is a pullback of E.By Remak 2.10 (2), $f^*(E) = [Ef]$. Now, by Lemma 2.9,$f^*(E) \in Ext(A, G)_0$. □

Theorem 2.12. *If C is a connected group and A a totally disconnedted group, then $Ext(C, A)_0 = 0$.*

Proof. Let $E : 0 \to A \overset{\phi}{\to} B \overset{\psi}{\to} C \to 0$ be a component extension. Then, $0 \to A_0 \overset{\phi}{\to} B_0 \overset{\psi}{\to} C_0 \to 0$ is an extension. Since A is totally disconnected and C connected, it follows that $\psi : B_0 \to C$ is a topological isomorphism. We claim that $B = B_0 + \phi(A)$. Let $b \in B$. Then $\psi(b) \in C$.

So $\psi(b) = \psi(b_0)$ for some $b_0 \in B_0$. Hence $b - b_0 \in ker\psi = im\phi$. So $b = b_0 + \phi(a)$ for some $a \in A$. Now, we show that $\phi(A) \bigcap B_0 = 0$. Let $b \in \phi(A) \bigcap B_0$. Then, $\psi(b) = 0$. Since $\psi : B_0 \to C$ is injective, so $b = 0$. Hence, $\phi(A) \bigcap B_0 = 0$. Since B_0 is $\sigma-$compact, it follows from [4, Corollary 3.2] that $B \cong \phi(A) \bigoplus B_0$. This shows that E splits. □

Lemma 2.13. *Let $G \in \pounds$ be a non discrete group which contains a compact open subgroup K. Then $Ext(G, \mathbb{Z}) \neq 0$.*

Proof. Consider the exact sequence $0 \to K \to G \to G/K \to 0$. By [4, Corollary 2.10], we have the exact sequence

$$\to Ext(G/K, \mathbb{Z}) \to Ext(G, \mathbb{Z}) \to Ext(K, \mathbb{Z}) \to 0$$

If $Ext(G, \mathbb{Z}) = 0$, then $Ext(K, \mathbb{Z}) = 0$. By [4, Theorem 2.12], $Ext(\hat{\mathbb{Z}}, \hat{K}) \cong Ext(K, \mathbb{Z}) = 0$. It follows from [4, Proposition 2.17] that $\hat{K} = 0$ which is a contradiction because G is not a discrete group. □

Lemma 2.14. *Let $G \in \pounds$ be a non connected and non totally disconnected group such that G_0 is not an open subgroup of G. Then $Ext(G, \mathbb{Z})_0 \neq 0$.*

Proof. Consider the extension $0 \to G_0 \xrightarrow{i} G \longrightarrow G/G_0 \to 0$. By [5, Corollary 2.10], we have the exact sequence

$$0 \to Ext(G/G_0, \mathbb{Z}) \xrightarrow{\pi^*} Ext(G, \mathbb{Z}) \xrightarrow{i^*} Ext(G_0, \mathbb{Z}) \to 0$$

Now, suppose that $Ext(G, \mathbb{Z})_0 = 0$. We have $\pi^*(Ext(G/G_0, \mathbb{Z})_0) \subseteq Ext(G, \mathbb{Z})_0$, so $\pi^*(Ext(G/G_0, \mathbb{Z})_0 = 0$. Since π^* is injective, then $Ext(G/G_0, \mathbb{Z})_0 = 0$. By Lemma 2.5, $Ext(G/G_0, \mathbb{Z}) = 0$ which is a contradiction because G/G_0 is a totally disconnected group which contains a compact open subgroup. On the other hand, G/G_0 is not a discrete group. So, by Lemma 2.13, $Ext(G/G_0, \mathbb{Z}) \neq 0$. □

Lemma 2.15. *Let A be a discrete divisible group and G a LCA group. Then $Ext(G, A)_0 = 0$.*

Proof. Consider the extension $0 \to G_0 \xrightarrow{i} G \longrightarrow G/G_0 \to 0$. By [5, Corollary 2.10], we have the exact sequence

$$0 \to Ext(G/G_0, A) \to Ext(G, A) \xrightarrow{i^*} Ext(G_0, A) \to 0$$

Since, G/G_0 is a totally disconnected group, so by [4, Theorem 3.4] $Ext(G/G_0, A) = 0$. Hence, i^* is injective. By Lemma 2.11,$i^*(Ext(G, A)_0) \subseteq Ext(G_0, A)_0$. By Theorem 2.12, $Ext(G_0, A)_0 = 0$. Hence, $i^*(Ext(G, A)_0) = 0$. Now $Ext(G, A)_0 = 0$ because i^* is injective. □

The following remark shows that the pushout of a component extension need not be a component extension.

Remark 2.16. By Lemma 2.14, if $G = \hat{\mathbb{Q}} \bigoplus \widehat{(\mathbb{Q}/\mathbb{Z})}$, then $Ext(G, \mathbb{Z})_0 \neq 0$. So there exists a non splitting component extension $E : 0 \to \mathbb{Z} \xrightarrow{\phi} X \to G \to 0$. Consider the standard pushout iE:

$$iE : 0 \to \mathbb{Q} \to (\mathbb{Q} \bigoplus X)/H \to G \to 0$$

where $H = \{(-n, \phi(n)); n \in \mathbb{Z}\}$. Let iE be a component extension. Then by Lemma 2.15, it splits. So $(\mathbb{Q} \bigoplus X)/H \cong \mathbb{Q} \bigoplus G$. Since $\mathbb{Q} \bigoplus G$ is torsion-free, so H is a pure subgroup. On the other hand, $H \subseteq \mathbb{Q} \bigoplus nX$. So H is a divisible group. Hence, for a positive integer $m \neq 1$, there exists $(-n, f(n)) \in H$ such that $m(-n, f(n)) = (-1, f(1))$. It follows that $mn = 1$ which is a contradiction. So iE is not a component extension.

Lemma 2.17. *Let A be a connected group. Then a pushout of a component extension of A by C is a component extension.*

Proof. Suppose $0 \to A \xrightarrow{\phi} B \xrightarrow{\psi} C \to 0$ is a component extension and

$$\begin{tikzcd}
0 \arrow[r] & A \arrow[r, "\phi"] \arrow[d, "\mu"] & B \arrow[r, "\psi"] \arrow[d] & C \arrow[r] \arrow[d, "1_C"] & 0 \\
0 \arrow[r] & A' \arrow[r, "\phi'"] & (A' \bigoplus B)/H \arrow[r, "\psi'"] & C \arrow[r] & 0
\end{tikzcd}$$

is a standard pushout diagram. Then

$$H = \{(\mu(a), -\phi(a)), a \in A\}$$

and

$$\phi' : a' \longmapsto (a', 0) + H, \ \psi' : (a', b) + H \longmapsto \psi(b)$$

We show that $0 \to A'_0 \xrightarrow{\phi'} ((A' \bigoplus B)/H)_0 \xrightarrow{\psi'} C_0 \to 0$ is exact. Let $c_0 \in C_0$. Since $\psi : B_0 \to C_0$ is surjective, so there exists $b_0 \in B_0$ such that $\psi(b_0) = c_0$. Hence $(0, b_0) + H \in ((A' \bigoplus B)/H)_0$. Since $\psi'((0, b_0) + H) = c_0$, so $\psi' : ((A' \bigoplus B)/H)_0 \to C_0$ is surjective. Let $(a', b) + H \in ((A' \bigoplus B)/H)_0$ and $\psi'((a', b) + H) = 0$. So $\psi(b) = 0$. Hence, there is $a \in A$ such that $\phi(a) = -b$. Since A is connected, then $\mu(A) \subseteq A'_0$. $\square$

Let $E_1 : 0 \to A \xrightarrow{\phi} B \xrightarrow{\psi} C \to 0$ and $E_2 : 0 \to A \xrightarrow{\phi'} B' \xrightarrow{\psi'} C \to 0$ be two extensions of A by C. Then the direct sum of E_1 and E_2 is denoted by $E_1 \bigoplus E_2$ and defined as follows:

$$E_1 \bigoplus E_2 : 0 \to A \bigoplus A \xrightarrow{\phi \bigoplus \phi'} B \bigoplus B' \xrightarrow{\psi \bigoplus \psi'} C \bigoplus C \to 0$$

Lemma 2.18. *The direct sum of two component extensions is a component extension.*

Proof. It is clear. □

Theorem 2.19. *Let $A, C \in \pounds$ such that A is a connected group. Then, $Ext(C, A)_0$ is an subgroup of $Ext(C, A)$ with respect to the operation defined by*

$$[E_1] + [E_2] = [\nabla_A(E_1 \bigoplus E_2)\triangle_C]$$

where E_1 and E_2 are component extensions of A by C and ∇_A and $\triangle_C$ are the diagonal and codiagonal homomorphism.

Proof. Clearly, $0 \to A \to A \bigoplus C \to C \to 0$ is a component extension. Let $[E] \in Ext(C, A)_0$. The inverse of $[E]$ is $[-1_A E]$ which belongs to $Ext(C, A)_0$ (Lemma 2.17). By Lemma 2.9 and Lemma 2.17, $[E_1] + [E_2] \in Ext(C, A)_0$ for two component extensions E_1 and E_2 of A by C. Therefore, $Ext(C, A)_0$ is a subgroup of $Ext(C, A)$. □

3. COMPONENT INJECTIVE AND PROJECTIVE IN $\pounds$

In this section, we define the concept of component injective and component projective in $\pounds$ and classify them.

Definition 3.1. Let $G \in \pounds$. We call G a component injective group in $\pounds$ if for every component extension

$$0 \to A \xrightarrow{\phi} B \to C \to 0$$

and a morphism $f : A \to G$, there is a morphism $\bar{f} : B \longrightarrow G$ such that $\bar{f}\phi = f$.

We call G a component projective group in $\pounds$ if for every component extension

$$0 \to A \to B \xrightarrow{\psi} C \to 0$$

and a morphism $f : G \to C$, there is a morphism $\bar{f} : G \to B$ such that $\psi\bar{f} = f$.

Lemma 3.2. $\mathbf{Q}$ *is not a component injective in $\pounds$.*

Proof. By Lemma 2.14, there is a non splitting component extension $E : 0 \to \mathbf{Z} \xrightarrow{\phi} X \xrightarrow{\psi} G \to 0$. Let $\mathbf{Q}$ be a component injective. Then, there is a morphism $f : X \to \mathbf{Q}$ such that $f\phi = i$ where $i : \mathbf{Z} \hookrightarrow \mathbf{Q}$ is an inclusion. Since E is a component extension, so $\psi(X_0) = G_0$. Hence,

$X = X_0 + \phi(\mathbf{Z})$. Since f is continuous and $\mathbf{Q}$ totally disconnected,then $f(X_0) = 0$. We claim that $f(x) = n$ for some $n \in \mathbf{Z}$. Let $x \in X$. Then $x = x_0 + \phi(n)$ for some $n \in \mathbf{Z}$. So $f(x) = f(\phi(n)) = n$. An easy calculation shows that $f^{-1}(0) = X_0$. So X_0 is open in X. Hence $\psi(X_0) = G_0$ is open in G which is a contradiction. □

Theorem 3.3. *Let $G \in \pounds$. Consider the following conditions for G:*

(1) *G is component injective in $\pounds$.*
(2) *$Ext(X, G)_0 = 0$ for all $X \in \pounds$.*
(3) *$G \cong \mathbb{R}^n \bigoplus (\mathbb{R}/\mathbb{Z})^\sigma \bigoplus H$ where n is a nonnegative integer, σ a cardinal number and H is a totally disconnected, LCA group.*

Then: (1) $\Rightarrow$ (2) $\Rightarrow$ (3).

Proof. (1) $\Rightarrow$ (2): It is clear.

(2) $\Rightarrow$ (3): Let $G \in \pounds$ and $Ext(X, G)_0 = 0$ for all $X \in \pounds$. Let C be a connected group. Consider the following exact sequence

$$0 \to Ext(C, G_0) \xrightarrow{i_*} Ext(C, G) \xrightarrow{\pi_*} Ext(C, G/G_0) \to 0$$

Since $Ext(C, G)_0 = 0$ and $i_*(Ext(C, G_0)_0) \subseteq Ext(C, G)_0 = 0$, so $Ext(C, G_0)_0 = 0$. By Lemma 2.4 (2), $Ext(C, G_0) = 0$. So By [5, Theorem 3.3], $G_0 \cong \mathbb{R}^n \bigoplus (\mathbb{R}/\mathbb{Z})^\sigma$. Hence $G \cong G_0 \bigoplus G/G_0$. Set $H = G/G_0$. So $G \cong \mathbb{R}^n \bigoplus (\mathbb{R}/\mathbb{Z})^\sigma \bigoplus H$. □

Remark 3.4. In Theorem 3.3, (3) may not imply (1).For example, take $G = \mathbf{Q}$. Then, by Lemma 3.2, G is not a component injective group. Also, if $X = \hat{\mathbb{Q}} \bigoplus \widehat{(\mathbb{Q}/\mathbb{Z})}$ then by Lemma 2.14 $Ext(X, \mathbb{Z})_0 \neq 0$. Hence, (3) may not imply (2) as well.

Lemma 3.5. *Let $G \in \pounds$ and $\{H_i; i \in I\}$ be a collection in $\pounds$ such that $\bigoplus_{i\in I} H_i \in \pounds$. If $Ext(H_i, G)_0 = 0$ for every i, then $Ext(\bigoplus_{i\in I} H_i, G)_0 =$* 0.

Proof. By [4, Theorem 2.13], $\sigma : Ext(\bigoplus_{i\in I} H_i, G) \to \prod_{i\in I} Ext(H_i, G)$ defined by $\sigma([E]) = ([El_i])_{i\in I}$ is an isomorphism where $l_i : H_i \to \bigoplus_{i\in I} H_i$ is an injection. By Lemma 2.9, $[El_i] \in Ext(H_i, G)_0$ for every i. So $\sigma(Ext(\bigoplus_{i\in I} H_i, G)_0) \subseteq \prod_{i\in I} Ext(H_i, G)_0$. Now, suppose that $Ext(H_i, G)_0 = 0$ for every i. Then, $\sigma(Ext(\bigoplus_{i\in I} H_i, G)_0) = 0$. Since σ is injective, then $Ext(\bigoplus_{i\in I} H_i, G)_0 = 0$. □

Theorem 3.6. *Let $G \in \pounds$. The following statements are equivalent:*

(1) *G is component projective in $\pounds$.*
(2) *$Ext(G, X)_0 = 0$ for all $X \in \pounds$.*
(3) *$G \cong \mathbb{R}^n \bigoplus C \bigoplus A$ where C is a compact connected group having a cotorsion dual and A a discrete free group.*

Proof. (1) $\Rightarrow$ (2): It is clear. (2) $\Rightarrow$ (3): Let $G \in \pounds$ and $Ext(G, X)_0 = 0$ for all $X \in \pounds$. Let X be a totally disconnected group. Consider the following exact sequence

$$0 \to Ext(G/G_0, X) \xrightarrow{\pi^*} Ext(G, X) \xrightarrow{i^*} Ext(G_0, X) \to 0$$

Since $Ext(G, X)_0 = 0$, so $Ext(G/G_0, X)_0 = 0$. Hence $Ext(G/G_0, X) = 0$ for all totally disconnected groups X. By [4, Theorem 4.1], G/G_0 is a free group. So $G \cong G_0 \bigoplus G/G_0$. By Lemma 2.4, $Ext(G, C) = Ext(G, C)_0 = 0$ for all connected groups $C \in \pounds$. So $Ext(G_0, C) = 0$ for all connected groups $C \in \pounds$. by [5, Theorem 3.6], $G_0 \cong \mathbb{R}^n \bigoplus C$ where C is a compact group having a cotorsion dual. (3) $\Rightarrow$ (2): Let $G \cong \mathbb{R}^n \bigoplus C \bigoplus A$ where C is a compact connected group having a cotorsion dual and A a discrete free group. By [11, Theorem 3.3], $Ext(\mathbb{R}^n \bigoplus A, X) = 0$ for all $X \in \pounds$. Let $X \in \pounds$. Now, we show that $Ext(C, X)_0 = 0$. Consider the following exact sequence

$$0 \to Ext(C, X_0) \to Ext(C, X) \xrightarrow{\pi_*} Ext(C, X/X_0) \to 0$$

Since $\hat{C}$ is a cotorsion group, so $Ext(C, X_0) \cong Ext(\widehat{(X_0)}, \hat{C}) = 0$. By Theorem 2.12, $Ext(C, X/X_0)_0 = 0$. Hence $\pi_*(Ext(C, X)_0) = 0$. So $Ext(C, X)_0 = 0$. Hence, by Lemma 3.5, $Ext(\mathbb{R}^n \bigoplus C \bigoplus A, X)_0 = 0$. (2) $\Rightarrow$ (1): Let $0 \to A \xrightarrow{\phi} B \xrightarrow{\psi} C \to 0$ be a component extension and $f : G \to C$ a morphism. Consider the pullback diagram

$$\begin{tikzcd}
0 \arrow[r] & A \arrow[r, "\phi'"] \arrow[d, "1_A"] & B' \arrow[r, "\psi'"] \arrow[d, "\beta"] & G \arrow[r] \arrow[d, "f"] & 0 \\
0 \arrow[r] & A \arrow[r, "\phi"] & B \arrow[r, "\psi"] & C \arrow[r] & 0
\end{tikzcd}$$

By Lemma 2.9, Ef is a component extension. So by assumption, Ef splits. Hence, there is a morphism $h : C \to B'$ such that $\psi' h = f$. Now, βh is a morphism of G to B and $\psi \beta h = f$. So G is component projective in $\pounds$.

$\square$

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