A CLASS OF \( J \)-QUASIPOLAR RINGS

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Abstract. In this paper, we introduce a class of \( J \)-quasipolar rings. Let \( R \) be a ring with identity. An element \( a \) of a ring \( R \) is called weakly \( J \)-quasipolar if there exists \( p^2 = p \in \text{comm}^2(a) \) such that \( a + p \) or \( a - p \) are contained in \( J(R) \) and the ring \( R \) is called weakly \( J \)-quasipolar if every element of \( R \) is weakly \( J \)-quasipolar. We give many characterizations and investigate general properties of weakly \( J \)-quasipolar rings. If \( R \) is a weakly \( J \)-quasipolar ring, then we show that (1) \( R/J(R) \) is weakly \( J \)-quasipolar, (2) \( R/J(R) \) is commutative, (3) \( R/J(R) \) is reduced. We use weakly \( J \)-quasipolar rings to obtain more results for \( J \)-quasipolar rings. We prove that the class of weakly \( J \)-quasipolar rings lies between the class of \( J \)-quasipolar rings and the class of quasipolar rings. Among others it is shown that a ring \( R \) is abelian weakly \( J \)-quasipolar if and only if \( R \) is uniquely clean.

1. Introduction

Throughout this paper all rings are associative with identity unless otherwise stated. Given a ring \( R \), the symbol \( U(R) \) and \( J(R) \) stand for the group of units and the Jacobson radical of \( R \), respectively.

Let \( R \) be a ring and \( a \in R \). We adopt the notations \( \text{comm}(a) = \{ b \in R \mid ab = ba \} \) while the second commutant \( \text{comm}^2(a) = \{ b \in R \mid bc = cb \text{ for all } c \in \text{comm}(a) \} \) and \( R^{\text{nil}} = \{ a \in R \mid 1 + ax \text{ is invertible for each } x \in \text{comm}(a) \} \). An element \( a \) of a ring \( R \) is called quasipolar (see [8]) if there exists \( p^2 = p \in R \) such that \( p \in \text{comm}^2(a), a + p \in U(R) \) and


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Any idempotent $p$ satisfying the above conditions is called a *spectral idempotent* of $a$, and this term is borrowed from spectral theory in Banach algebra and it is unique for $a$. Quasipolar rings have been studied by many ring theorists (see [5], [7], [8] and [12]). Recently, $J$-quasipolar rings are introduced in [6]. For an element $a$ of a ring $R$, if there exists $p^2 = p \in \text{comm}^2(a)$ such that $a + p \in J(R)$, then $a$ is called a *$J$-quasipolar* and a ring $R$ is called $J$-quasipolar, if every element of $R$ is $J$-quasipolar. It is proved that every $J$-quasipolar ring is quasipolar.

Motivated by these classes of polarity versions of rings, we introduce weakly $J$-quasipolar rings, generalizing $J$-quasipolar rings. Throughout this paper, some basic properties of weakly $J$-quasipolar ring are studied, also examples and counter examples are given. We show that the class of weakly $J$-quasipolar rings lies properly between the class of $J$-quasipolar rings and the class of quasipolar rings. It is proved that $R$ is $J$-quasipolar if and only if $R$ is weakly $J$-quasipolar and $2 \in J(R)$. Then some of the main results of $J$-quasipolar rings are special cases of our results for this general setting. Given a ring $R$, if $M_n(R)$ and $T_n(R)$ denote the ring of all $n \times n$ matrices and triangular matrices over $R$, then we investigate necessary and sufficient conditions as to weakly $J$-quasipolarity of $T_2(R)$ over a commutative local ring $R$. Further, it is proven that $M_n(R)$ is not weakly $J$-quasipolar for $n \geq 2$. Finally, we determine under what conditions a $2 \times 2$ matrix over a commutative local ring is weakly $J$-quasipolar.

In what follows, $\mathbb{N}$ and $\mathbb{Z}$ denote the set of natural numbers, the ring of integers and for a positive integer $n$, $\mathbb{Z}_n$ is the ring of integers modulo $n$. The notations $\det A$ and $\text{tr} A$ denote the determinant and the trace of a square matrix $A$ over a commutative ring and $I_n$ denotes the $n \times n$ identity matrix.

## 2. Weakly $J$-Quasipolar Rings

In this section, we introduce a class of quasipolar rings which is a generalization of $J$-quasipolar rings. By using weakly $J$-quasipolar rings, we obtain more results for $J$-quasipolar rings. It is clear that every $J$-quasipolar ring is weakly $J$-quasipolar and we supply an example to show that the converse does not hold in general (see Example 2.9). Moreover, it is shown that the class of weakly $J$-quasipolar rings lies strictly between the class of $J$-quasipolar rings and the class of quasipolar rings (see Example 2.9, Corollary 2.11 and Example 2.12). We investigate general properties of weakly $J$-quasipolar rings.
**Definition 2.1.** Let $R$ be a ring and $a \in R$. The element $a$ is called weakly $J$-quasipolar if there exists $p^2 = p \in \text{comm}^2(a)$ such that $a + p \in J(R)$ or $a - p \in J(R)$. The idempotent which satisfies the above condition is called a weakly $J$-spectral idempotent and $R$ is called weakly $J$-quasipolar if every element of $R$ is weakly $J$-quasipolar.

Lemma 2.2 shows that weakly $J$-quasipolar elements and rings are abundant.

**Lemma 2.2.** Let $R$ be a ring. Then we have the followings.

1. Every idempotent in $R$ is weakly $J$-quasipolar.
2. An element $a \in R$ is weakly $J$-quasipolar if and only if $-a \in R$ is weakly $J$-quasipolar.
3. Every element in $J(R)$ is weakly $J$-quasipolar.
4. Boolean rings are weakly $J$-quasipolar.
5. $J$-quasipolar rings are weakly $J$-quasipolar.

In the sequel, we state elementary properties of weakly $J$-quasipolar elements and weakly $J$-quasipolar rings.

**Lemma 2.3.** Let $R$ be a ring. If $u \in U(R)$ is weakly $J$-quasipolar, then $1$ is the weakly $J$-spectral idempotent of $u$.

**Proof.** Let $u \in U(R)$ be weakly $J$-quasipolar, so $u + p \in J(R)$ or $u - p \in J(R)$ such that $p^2 = p \in \text{comm}^2(u)$. If $u - p \in J(R)$, then $u^{-1}u - u^{-1}p = 1 - u^{-1}p \in J(R)$. Hence, $u^{-1}p \in U(R)$ and so $p \in U(R)$. Thus, we have $p = 1$. In case $u + p \in J(R)$, the proof is similar. □

By using the concept of $J$-quasipolarity, we obtain a characterization for local rings.

**Proposition 2.4.** Let $R$ be a weakly $J$-quasipolar ring. Then $R$ is a local ring if and only if $R$ has only trivial idempotents.

**Proof.** Assume that $R$ is a weakly $J$-quasipolar ring and has only trivial idempotents. Let $a \in R$, so $a + 1 \in J(R)$ or $a - 1 \in J(R)$ or $a \in J(R)$. If $a + 1 \in J(R)$ or $a - 1 \in J(R)$, then $a \in U(R)$. In the last condition, $a \in J(R)$. Consequently, $R$ is a local ring. The converse statement is clear. □

**Lemma 2.5.** Let $R$ be a ring. If $a \in R$ and $u \in U(R)$, then $a$ is weakly $J$-quasipolar if and only if $u^{-1}au$ is weakly $J$-quasipolar.

**Proof.** Assume that $a$ is weakly $J$-quasipolar. Then there exists $p^2 = p \in \text{comm}^2(a)$ such that $a - p \in J(R)$. If $q$ is taken as $q = u^{-1}pu$, then $q^2 = q \in R$ and $u^{-1}au - u^{-1}pu = u^{-1}(a - p)u \in J(R)$. Let $b \in \text{comm}(u^{-1}au)$, then $(u^{-1}au)b = b(u^{-1}au)$ and so $a(ubu^{-1}) = (ubu^{-1})a$. 

Thus \( ubu^{-1} \in \text{comm}(a) \). Since \( p \in \text{comm}^2(a) \), \( (ubu^{-1})p = p(ubu^{-1}) \). Hence \( b(ubu^{-1})p = (ubu^{-1})p = p(ubu^{-1}) \). Consequently, \( u^{-1}pu \in \text{comm}^2(u^{-1}au) \) and so \( u^{-1}au \) is weakly \( J \)-quasipolar. Conversely, assume that \( u^{-1}au - q \in J(R) \), so \( a - uqu^{-1} \in J(R) \). Also \( (uqu^{-1})^2 = uqu^{-1} \in \text{comm}^2(a) \). If \( a + p \in J(R) \), then proof is similar. □

The proof of Lemma 2.5 reveals that \( p \) is weakly \( J \)-spectral idempotent of \( a \) if and only if \( u^{-1}pu \) is the weakly \( J \)-spectral idempotent of \( u^{-1}au \). We need the following lemma in order to prove Theorem 2.7.

**Lemma 2.6.** Let \( R \) be a ring. If \( a = j_1 - p \in J(R) \) or \( a = j_2 + p \in J(R) \) is weakly \( J \)-quasipolar decomposition of \( a \) in \( R \), then \( \text{ann}_l(a) \subseteq \text{ann}_l(p) \)and \( \text{ann}_r(a) \subseteq \text{ann}_r(p) \).

**Proof.** If \( r \in \text{ann}_l(a) \), then \( ra = 0 \). Assume that \( a + p = j_1 \in J(R) \) such that \( p^2 = p \in \text{comm}^2(a) \). Then \( rp = r(j_1 - a) = rj_1 \) and so \( rp = rj_1p = rpj_1 \). Hence \( rp(1 - j_1) = rp - rpj_1 = 0 \). Since \( 1 - j_1 \in U(R) \), \( r \in \text{ann}_l(p) \). If \( r \in \text{ann}_r(a) \), then \( ar = 0 \). Thus \( pr = (j_1 - a)r = j_1r \) and so \( pr = pj_1r \). Since \( a \in \text{comm}(a) \) and \( p \in \text{comm}^2(a) \), \( ap = pa \). Hence \( (j_1 - p)p = p(j_1 - p) \) and so \( j_1p = pj_1 \). Therefore \( pr = pj_1r = j_1pr \). Also \( (1 - j_1)pr = pr - j_1pr = 0 \). Because of \( 1 - j_1 \in U(R) \), \( r \in \text{ann}_r(p) \). If \( a - p = j_2 \in J(R) \) such that \( p^2 = p \in \text{comm}^2(a) \), then the proof is similar to above. □

**Theorem 2.7.** If \( R \) is weakly \( J \)-quasipolar, then so is \( fRf \) for all \( f^2 = f \in R \).

**Proof.** For every \( a \in fRf \) there exists \( p \in \text{comm}^2(a) \) such that \( a - p \in J(R) \) or \( a + p \in J(R) \). Let \( a + p = j_1 \in J(R) \) or \( a - p = j_2 \in J(R) \). By Lemma 2.6, we have \( 1 - f \in \text{ann}_l(a) \) and \( \text{ann}_r(a) \subseteq \text{ann}_l(p) \in \text{ann}_r(p) \). If \( (1 - j_1)p = p(1 - j_1) \), then \( pj_1f = f(j_1p) \) and so \( a = fj_1f - fpf \). Then \( fp = ffp \) and \( f(j_1f) = f(j_1f) \). Lastly, let \( xa = ax \) and \( x \in fRf \), so \( x(fpf) = (fpf)x \). If \( a - p = j_2 \in J(R) \), then proof is similar. Consequently, \( a \) is weakly \( J \)-quasipolar in \( fRf \). □

By the definition of weakly \( J \)-quasipolar rings, it is clear that every \( J \)-quasipolar ring is weakly \( J \)-quasipolar. We now investigate under what condition a weakly \( J \)-quasipolar ring is \( J \)-quasipolar.

**Proposition 2.8.** A ring \( R \) is \( J \)-quasipolar if and only if \( R \) is weakly \( J \)-quasipolar and \( 2 \in J(R) \).

**Proof.** Let \( R \) be a weakly \( J \)-quasipolar ring and \( 2 \in J(R) \). If \( a + p \in J(R) \) such that \( p^2 = p \in \text{comm}^2(a) \), then it is clear. Let \( a - p \in J(R) \) and \( p^2 = p \in \text{comm}^2(a) \). Since \( 2 \in J(R) \), \( a - p + 2p \in J(R) \) and so \( a \) is \( J \)-quasipolar. The converse is clear. □
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The next example illustrates that there are weakly J-quasipolar rings but not J-quasipolar.

Example 2.9. The ring $\mathbb{Z}_6$ is weakly J-quasipolar but not J-quasipolar.

Proof. It is obvious that $\mathbb{Z}_6$ is weakly J-quasipolar. Since $1 + 1 \notin J(\mathbb{Z}_6) = 0$, by Proposition 2.8, $\mathbb{Z}_6$ is not J-quasipolar. □

In [6], it is shown that every J-quasipolar element is quasipolar. We obtain the following result for this general setting.

Proposition 2.10. Every weakly J-quasipolar element in a ring $R$ is quasipolar.

Proof. Let $a \in R$ be weakly J-quasipolar. Then there exists $p^2 = p \in \text{comm}^2(a)$ such that $a + p \in J(R)$ or $a - p \in J(R)$. If $a + p \in J(R)$, then $a$ is quasipolar from [6, Proposition 2.4]. If $a - p \in J(R)$ such that $p^2 = p \in \text{comm}^2(a)$, then $a + (1 - p) \in U(R)$ and also $(a - p)(1 - p) = a(1 - p) \in J(R) \subseteq R^{\text{qnil}}$. Therefore $a$ is a quasipolar element. □

Corollary 2.11. If $R$ is weakly J-quasipolar, then it is quasipolar.

The converse statement of Corollary 2.11 is not true in general, i.e., there are quasipolar rings but not weakly J-quasipolar.

Example 2.12. Let $R = \mathbb{Z}_{(5)}$ be the localization ring of $\mathbb{Z}$ at the prime 5. Then $R$ is a local ring and thus quasipolar by [12, Corollary 3.3]. Since $\frac{1}{3} \in \mathbb{Z}_{(5)}$ is not weakly J-quasipolar, $\mathbb{Z}_{(5)}$ is not weakly J-quasipolar.

By Example 2.9, Corollary 2.11 and Example 2.12, it is clear that the class of weakly J-quasipolar rings lies strictly between the class of J-quasipolar rings and the class of quasipolar rings.

Proposition 2.13. Any weakly J-quasipolar element $a \in R$ has a unique weakly J-spectral idempotent.

Proof. Assume that $p, q$ are weakly J-spectral idempotents of $a \in R$. Case 1: If $a + p \in J(R)$ and $a + q \in J(R)$, then $1 - p$ and $1 - q$ are spectral idempotents of $-a$ by the proof of Proposition 2.10. By [6], the spectral idempotent of $a$ and $-a$ is equal. Also by [8, Proposition 2.3], the spectral idempotent of $a$ is unique, so we obtain that $1 - p = 1 - q$. Then $p = q$.

Case 2: Assume that $a + p \in J(R)$ and $a - q \in J(R)$. Then $1 - p$ is spectral idempotent of $-a$ and $1 - q$ is spectral idempotent of $a$. The remaining proof is similar to Case 1.

Case 3: Assume that $a - p \in J(R)$ and $a + q \in J(R)$, then similarly $p = q$. 
Case 4: Assume that \( a - p \in J(R) \) and \( a - q \in J(R) \), then similarly \( p = q \).

In [2], an element of a ring is called strongly \( J \)-clean provided that it can be written as the sum of an idempotent and an element in its Jacobson radical that commute. A ring is strongly \( J \)-clean in case each of its elements is strongly \( J \)-clean. From the definition of a strongly \( J \)-clean ring, one may suspects that every weakly \( J \)-quasipolar ring is strongly \( J \)-clean. But the following example erases possibility.

Example 2.14. It is clear that the ring \( \mathbb{Z}_3 \) is weakly \( J \)-quasipolar. Since \( 2 \notin J(\mathbb{Z}_3) \), it is not strongly \( J \)-clean by [2, Proposition 3.1].

Recall that, a ring \( R \) is called periodic if for each \( x \in R \), there exists distinct positive integers \( m, n \) depending on \( x \), for which \( x^n = x^m \). For an easy reference, we mention Lemma 2.15 which is one of Jacobson’s theorem given in [9] relating to periodicity and commutativity of the rings.

Lemma 2.15. Let \( R \) be a ring in which for every \( a \in R \) there exists an integer \( n(a) > 1 \), depending on \( a \) such that \( a^{n(a)} = a \), then \( R \) is commutative.

We now give a useful result to determine whether \( R \) is weakly \( J \)-quasipolar.

Theorem 2.16. If a ring \( R \) is weakly \( J \)-quasipolar, then \( R/J(R) \) is a periodic ring which has three period and \( R/J(R) \) is commutative.

Proof. Let \( R \) be weakly \( J \)-quasipolar and \( r \in R \). So \( r + p \in J(R) \) or \( r - p \in J(R) \) such that \( p^2 = p \in \text{comm}^2(a) \). Clearly, \( \overline{r} = \overline{p} \) or \( \overline{r} = -\overline{p} \) and \( \overline{p^2} = \overline{p} \). If \( \overline{r} = \overline{p} \), then \( \overline{r^2} = \overline{r} \) and so \( \overline{r^3} = \overline{r} \). If \( \overline{r} = -\overline{p} \), then it is clear that \( \overline{r^3} = \overline{r} \). Hence \( R/J(R) \) is a periodic ring which has period three. By Lemma 2.15, \( R/J(R) \) is commutative.

The following example shows that the converse statement of Theorem 2.16 is not true in general.

Example 2.17. It is clear that the ring \( \mathbb{Z} \) is commutative, \( J(\mathbb{Z}) = 0 \) and \( \mathbb{Z}/J(\mathbb{Z}) \cong \mathbb{Z} \). But \( \mathbb{Z} \) is not weakly \( J \)-quasipolar.

By Theorem 2.16, we obtain the following important result for weakly \( J \)-quasipolar rings.

Corollary 2.18. If \( R \) is weakly \( J \)-quasipolar, then \( R/J(R) \) is weakly \( J \)-quasipolar.

Proof. Proof is clear from Lemma 2.2 (1) and (2).
Recall that a ring \( R \) is said to be clean if for each \( a \in R \) there exists \( e^2 = e \in R \) such that \( a - e \in U(R) \). According to Nicholson and Zhou [11], a ring \( R \) is said to be uniquely clean if for each \( a \in R \) there exists unique idempotent \( e \in R \) such that \( a - e \in U(R) \). In [6], it is proved that a ring \( R \) is uniquely clean if and only if \( R \) is abelian (i.e., each idempotent of \( R \) is central) \( J \)-quasipolar. In this direction we generalize this result for weakly \( J \)-quasipolar rings.

**Theorem 2.19.** A ring \( R \) is abelian weakly \( J \)-quasipolar if and only if \( R \) is uniquely clean.

**Proof.** Given \( a \in R \), then \( -a \in R \). Hence \( -a + p \in J(R) \) or \( -a - p \in J(R) \) such that \( p^2 = p \in R \). If \( -a + p \in J(R) \), so \( a \) is uniquely clean. If \( -a - p \in J(R) \), then \( a - (1 - p) \in U(R) \). Uniqueness of the idempotent \( p \) follows from Proposition 2.13. Therefore \( R \) is a uniquely clean ring. The converse is clear by [6, Theorem 2.7]. \( \square \)

The next example illustrates that “abelian” condition is not superfluous in Theorem 2.19.

**Example 2.20.** The matrix ring \( T_2(\mathbb{Z}_2) \) is weakly \( J \)-quasipolar, but not abelian. By [11, Lemma 4], \( T_2(\mathbb{Z}_2) \) is not a uniquely clean ring.

In [1], Ungor et al. introduced and studied a new class of reduced rings (i.e., it has no nonzero nilpotent elements). A ring \( R \) is called feckly reduced if \( R/J(R) \) is a reduced ring. In this direction we show that every weakly \( J \)-quasipolar ring is feckly reduced.

**Theorem 2.21.** If \( R \) is a weakly \( J \)-quasipolar ring, then it is feckly reduced.

**Proof.** Let \( R \) be weakly \( J \)-quasipolar and \( r^2 = 0 \). Therefore there exists \( p^2 = p \in \text{comm}^2(r) \) such that \( r + p \in J(R) \) or \( r - p \in J(R) \). If \( r - p \in J(R) \), then \( r(r - p) = r^2 - rp \in J(R) \). Since \( r^2 = 0 \in J(R) \), \( rp \in J(R) \). Also \( (r - p)p = rp - p \in J(R) \). Hence \( p \in J(R) \) and so \( p = 0 \). Thus \( r \in J(R) \) and \( R/J(R) \) is reduced. If \( r + p \in J(R) \), then similarly \( r \in J(R) \) and \( R/J(R) \) is reduced. Consequently, \( R \) is a feckly reduced ring. \( \square \)

Let \( J^t(R) \) denote the subset \( \{ x \in R \mid \exists n \in \mathbb{N} \text{ such that } x^n \in J(R) \} \) of \( R \). It is obvious that \( J(R) \subseteq J^t(R) \). Weakly \( J \)-quasipolar rings play an important role for the reverse inclusion.

**Corollary 2.22.** If \( R \) is a weakly \( J \)-quasipolar ring, then \( J(R) = J^t(R) \)

**Proof.** Let \( R \) be a weakly \( J \)-quasipolar ring. By Theorem 2.21, \( R \) is feckly reduced and so \( J(R) = J^t(R) \) from [1, Proposition 2.6]. \( \square \)
The following result follows from Corollary 2.22.

**Corollary 2.23.** If \( R \) is a \( J \)-quasipolar ring, then \( J(R) = J^\#(R) \).

Corollary 2.22 is helpful to show that a ring is not weakly \( J \)-quasipolar.

**Example 2.24.** Let \( R \) denote the ring \( M_2(\mathbb{Z}_2) \). Then

\[
J^\#(R) = \left\{ \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} \right\}
\]

and \( J(R) = \left\{ \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \right\} \). By Corollary 2.22, \( R \) is not weakly \( J \)-quasipolar.

Let \( R \) be a ring and \( a, b \in R \). Then \( R \) is called *directly finite*, if \( ab = 1 \) then \( ba = 1 \). It is well known that \( R \) is directly finite if and only if \( R/J(R) \) is directly finite.

**Proposition 2.25.** If a ring \( R \) is weakly \( J \)-quasipolar, then \( R \) is directly finite.

**Proof.** The proof is clear from [1, Proposition 4.8]. \( \square \)

Since every \( J \)-quasipolar ring is weakly \( J \)-quasipolar, the following result follows from Proposition 2.25.

**Corollary 2.26.** If \( R \) is a \( J \)-quasipolar ring, then \( R \) is directly finite.

In [10], strongly clean rings are introduced and studied. A ring \( R \) is *strongly clean*, if for every \( a \in R \) there exists \( e^2 = e \in R \) such that \( a - e \in U(R) \) and \( ae = ea \). At the end of that paper, the authors ask some open questions. One of them is “Is every strongly clean ring directly finite?” By Proposition 2.25, weakly \( J \)-quasipolar rings are both strongly clean and directly finite.

### 3. Weakly \( J \)-Quasipolarity of Matrix rings

In this section we study weakly \( J \)-quasipolarity of some matrix rings. It is important to determine whether an individual matrix is weakly \( J \)-quasipolar. In particular, we investigate necessary and sufficient conditions weakly \( J \)-quasipolarity of the matrix ring \( T_2(R) \) over a commutative local ring \( R \). We determine under what conditions a single \( 2 \times 2 \) matrix over a commutative local ring is weakly \( J \)-quasipolar.

We start with the obvious proposition.

**Proposition 3.1.** (1) Let \( R \) be a commutative local ring. Then \( A \in M_2(R) \) is an idempotent if and only if either \( A = 0 \), or \( A = I_2 \), or \( A = \begin{bmatrix} a & b \\ c & 1 - a \end{bmatrix} \) where \( bc = a - a^2 \).
(2) Let \( R \) be a commutative local ring and \( P \in T_2(R) \). Then \( P \) is an idempotent if and only if \( P \) has a form \[
\begin{bmatrix}
1 & 0 \\
0 & 1 \\
0 & x
\end{bmatrix}
\] for some \( x \in R \).

Proof. (1) is clear from [3, Lemma 16.4.10] and (2) is straightforward.

\[\square\]

Proposition 3.2. Let \( R \) be a commutative local ring. \( A = \begin{bmatrix}
a_1 & a_2 \\
0 & a_3
\end{bmatrix} \) is weakly \( J \)-quasipolar in \( T_2(R) \) if and only if one of the following holds:

(1) \( A \in J(T_2(R)) \),

(2) \( A \in \pm 1 + J(T_2(R)) \),

(3) \( A + P \) or \( A - P \in J(T_2(R)) \) where \( P = \begin{bmatrix} 1 & x \\ 0 & 0 \end{bmatrix} \) such that \( x = (a_1 - a_3)^{-1}a_2 \),

(4) \( A - P \) or \( A + P \in J(T_2(R)) \) where \( P = \begin{bmatrix} 0 & x \\ 0 & 1 \end{bmatrix} \) such that \( x = (a_3 - a_1)^{-1}a_2 \).

Proof. Assume that \( A \) is weakly \( J \)-quasipolar.

Case 1: Let \( A + P \in J(T_2(R)) \) such that \( P^2 = P \in \text{comm}^2(A) \).

Since \( A + P = \begin{bmatrix} a_1 + p_1 & a_2 + p_2 \\ 0 & a_3 + p_3 \end{bmatrix} \in J(T_2(R)) \), \( a_1 + p_1 \in J(R) \) and \( a_3 + p_3 \in J(R) \). Besides assume that \( B \in \text{comm}(A) \) and take \( B = \begin{bmatrix} b_1 & b_2 \\ 0 & b_3 \end{bmatrix} \), so

\[
\begin{bmatrix}
b_1a_1 & b_1a_2 + b_2a_3 \\
0 & b_3a_3
\end{bmatrix} = \begin{bmatrix}
a_1b_1 & a_1b_2 + a_2b_3 \\
0 & a_3b_3
\end{bmatrix}.
\]

Therefore \( a_2(b_1 - b_3) = b_2(a_1 - a_3) \).

(i) If \( a_1, a_3 \in J(R) \), then \( p_1 = p_3 = 0 \). Hence \( p_2 = 0 \).

(ii) If \( a_1, a_3 \in U(R) \), then \( p_1 = p_3 = 1 \). Hence \( p_2 = 0 \).

(iii) If \( a_1 \in J(R), a_3 \in U(R) \), then \( p_1 = 0, p_3 = 1 \) and \( p_2 = x \in R \).

Since \( a_1 - a_3 \in U(R) \), \( b_2 = (a_1 - a_3)^{-1}a_2(b_1 - b_3) \). Providing \( x = (a_3 - a_1)^{-1}a_2 \), then \( P \in \text{comm}(B) \). Hence \( P \in \text{comm}^2(A) \).

(iv) If \( a_1 \in U(R), a_3 \in J(R) \), then \( p_1 = 1, p_3 = 0 \) and \( p_2 = x \in R \).

Because of \( a_1 - a_3 \in U(R) \), \( b_2 = (a_1 - a_3)^{-1}a_2(b_1 - b_3) \). Providing \( x = (a_1 - a_3)^{-1}a_2 \), then \( P \in \text{comm}(B) \). Therefore \( P \in \text{comm}^2(A) \).

Case 2: Let \( A - P \in J(T_2(R)) \) such that \( P^2 = P \in \text{comm}^2(A) \). Proof is similar to proof of Case 1.

The converse statement is clear. \[\square\]
The following result is a direct consequence of Proposition 3.2 for $J$-quasipolar rings.

**Corollary 3.3.** Let $R$ be a commutative local ring. $A = \begin{bmatrix} a_1 & a_2 \\ 0 & a_3 \end{bmatrix}$ is $J$-quasipolar in $T_2(R)$ if and only if one of the following holds:

1. $A \in J(T_2(R))$.
2. $A \in -1 + J(T_2(R))$.
3. $A + P \in J(T_2(R))$ where $P = \begin{bmatrix} 1 & x \\ 0 & 0 \end{bmatrix}$ such that $x = (a_1 - a_3)^{-1}a_2$ or $x = (a_3 - a_1)^{-1}a_2$.

**Corollary 3.4.** Let $R$ be a ring. If $T_n(R)$ with $n \geq 2$ is weakly $J$-quasipolar, then $R$ is weakly $J$-quasipolar.

*Proof.* Assume that $T_n(R)$ is weakly $J$-quasipolar. Let $f$ be the unit matrix with $(1,1)$ entry is 1 and the other entries are 0, then $fT_n(R)f \cong R$. By Theorem 2.7, $R$ is weakly $J$-quasipolar. □

The following example illustrates that the converse statement of Corollary 3.4 is not true in general.

**Example 3.5.** If $R = \mathbb{Z}_3$, then $R$ is weakly $J$-quasipolar. For $A = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \in U(T_2(R))$, $A + I_2 \notin J(T_2(R))$ and $A - I_2 \notin J(T_2(R))$. Therefore $T_2(R)$ is not weakly $J$-quasipolar.

Our next endeavor is to find conditions under which an individual matrix in $M_2(R)$ is weakly $J$-quasipolar.

**Lemma 3.6.** Let $R$ be a ring. Then $A \in U(M_2(R))$ and $A$ is weakly $J$-quasipolar if and only if $A - I_2 \in J(M_2(R))$ or $A + I_2 \in J(M_2(R))$.

*Proof.* Let $A$ be weakly $J$-quasipolar. Since $A \in U(M_2(R))$, weakly $J$-spectral idempotent of $A$ is $I_2$. Hence $A + I_2 \in J(M_2(R))$ or $A - I_2 \in J(M_2(R))$. Conversely, if $A - I_2 \in J(M_2(R))$, then $A \in I_2 + J(M_2(R)) \subseteq U(M_2(R))$. If $A + I_2 \in J(M_2(R))$, then it is clear from the proof of [6, Lemma 4.3] that $A \in U(M_2(R))$. □

The following lemma is important to study especially in a matrix ring.

**Lemma 3.7.** If $R$ is a weakly $J$-quasipolar ring, then $6 \in J(R)$.

*Proof.* Let $R$ be a weakly $J$-quasipolar ring, then there exists $p^2 = p \in \text{comm}^2(2)$ such that $2 - p \in J(R)$ or $2 + p \in J(R)$. Assume that $2 - p = j \in J(R)$, therefore $2 - j = p$ and $(2 - j)^2 = 2 - j$. Thus
2 = j(3 − j) ∈ J(R). As a consequence 6 ∈ J(R). If 2 + p = j_1 ∈ J(R), then (j_1 − 2)^2 = (j_1 − 2). So 6 = j_1(5 − j_1) ∈ J(R). \[\Box\]

Lemma 3.7 is helpful to show a ring is not weakly J-quasipolar.

**Example 3.8.** Since 6 \(\notin J(\mathbb{Z}_{15}) = 0\), by Lemma 3.7, \(\mathbb{Z}_{15}\) is not weakly J-quasipolar.

The converse statement of Lemma 3.7 is not true in general, i.e., for a ring \(R\), if 6 \(\in J(R)\), then \(R\) need not be weakly J-quasipolar.

**Example 3.9.** It is obvious that 6 \(\in J(T_2(\mathbb{Z}_3))\). By Example 3.5, the ring \(T_2(\mathbb{Z}_3)\) is not weakly J-quasipolar.

Proposition 2.8 shows that in case of 2 \(\in J(R)\), weakly J-quasipolar rings and J-quasipolar rings are the same. The following example indicates that it does not hold in case of 6 \(\in J(R)\).

**Example 3.10.** The ring \(\mathbb{Z}_9\) is weakly J-quasipolar and 6 \(\in J(\mathbb{Z}_9)\). Since there is not a J-spectral idempotent for 4 such that 4 + p \(\in J(\mathbb{Z}_9)\), it is not J-quasipolar.

**Lemma 3.11.** Let \(R\) be a ring with 6 \(\in J(R)\). If \(a \in R\) is weakly J-quasipolar, then \(a + 5\) or \(a − 5\) is weakly J-quasipolar.

**Proof.** Let \(a \in R\) be weakly J-quasipolar. Thus \(a + p \in J(R)\) or \(a − p \in J(R)\) such that \(p^2 = p \in \text{comm}^2(a)\). Assume that \(a + p \in J(R)\) and \(p^2 = p \in \text{comm}^2(a)\). Since 6 \(\in J(R)\), \(a − 6 + p = (a − 5) − (1 − p) \in J(R)\). So \(a − 5\) is weakly J-quasipolar. If \(a − p \in J(R)\) such that \(p^2 = p \in \text{comm}^2(a)\), \(a + 6 − p = (a + 5) + (1 − p) \in J(R)\). \[\Box\]

**Proposition 3.12.** Let \(R\) be a commutative ring with 6 \(\in J(R)\) and \(A \in M_2(R)\) such that \(A \notin J(M_2(R))\). If both \(\det A\) and \(\text{tr} A\) are in \(J(R)\), then \(A\) is not weakly J-quasipolar.

**Proof.** If \(A\) is weakly J-quasipolar, then \(A − 5\) or \(A + 5\) weakly J-quasipolar by Lemma 3.11. Note that \(\det(A − 5) = \det A − 5(\text{tr} A + 5) \in U(R)\). Hence weakly J-spectral idempotent of \(A − 5\) is \(I_2\) by Lemma 2.3. So \(A − 5 − I_2 \in J(M_2(R))\) or \(A − 5 + I_2 \in J(M_2(R))\). If \(A − 5 − I_2 \in J(M_2(R))\), then \(A\) is weakly J-quasipolar, which contradicts the assumption. In other condition, let \(A − 5 + I_2 \in J(M_2(R))\) and so \(A − 4 \in J(M_2(R))\). Therefore \(a_{11} − 4, a_{22} − 4 \in J(R)\), \(a_{11} + a_{22} − 8 = \text{tr} A − 8 \in J(R)\). Since \(\text{tr} A \in J(R)\), so \(8 \in J(R)\) and \(8 − 6 = 2 \in J(R)\). Thus \(A − 4 + 4 \in J(M_2(R))\) is a contradiction. As a consequence \(A\) is not weakly J-quasipolar. Also in case of \(A + 5 \in J(M_2(R))\), proof is similar. Finally \(A\) is not weakly J-quasipolar. \[\Box\]
Lemma 3.13. Let $R$ be a commutative local ring. Then $A = \begin{bmatrix} j & 0 \\ 0 & u \end{bmatrix}$ is weakly $J$-quasipolar in $M_2(R)$ if and only if one of the following holds.

1. $A \in J(M_2(R))$.
2. $A + I_2 \in J(M_2(R))$.
3. $A - I_2 \in J(M_2(R))$.
4. $u \in -1 + J(R)$ and $j \in J(R)$.
5. $u \in J(R)$ and $j \in -1 + J(R)$.
6. $u \in J(R)$ and $j \in 1 + J(R)$.
7. $u \in 1 + J(R)$ and $j \in J(R)$.

Proof. Let $A$ be weakly $J$-quasipolar. Then, there exists $P^2 = P \in \text{comm}^2(A)$ such that $A + P \in J(M_2(R))$ or $A - P \in J(M_2(R))$. If $A + P \in J(M_2(R))$, then (1), (2), (3), (4), (5) hold by [6, Lemma 4.7]. Assume that $A - P \in J(M_2(R))$. If $P = 0$ or $P = I_2$ it is clear. Let $P \neq 0$ and $P \neq I_2$. By Proposition 3.1, $P = \begin{bmatrix} a & b \\ c & 1 - a \end{bmatrix}$ where $bc = a - a^2$.

Since $F = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \in \text{comm}(A)$ and $P \in \text{comm}^2(A)$, $FP = PF$. Then, $b = c = 0$. Thus, $P = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ or $P = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$. Since $A - P \in J(M_2(R))$, $u \in J(R)$ and $j \in 1 + J(R)$ or $u \in J(R)$ and $j \in J(R)$.

Conversely, if $A \in J(M_2(R))$ or $A + I_2 \in J(M_2(R))$ or $A - I_2 \in J(M_2(R))$, then $A$ is weakly $J$-quasipolar. If $u \in -1 + J(R)$ and $j \in J(R)$ or $u \in J(R)$ and $j \in -1 + J(R)$, then it follows from [6, Lemma 4.7]. Suppose that $u \in J(R)$ and $j \in 1 + J(R)$. Let $P = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$. Then $P^2 = P$ and $A - P \in J(M_2(R))$. To show that $P^2 = P \in \text{comm}^2(A)$, let $B = \begin{bmatrix} x & y \\ z & t \end{bmatrix} \in \text{comm}(A)$. Hence $y = z = 0$ and so $PB = BP$. Thus $A$ is weakly $J$-quasipolar. If $u \in J(R)$ and $j \in 1 + J(R)$, similarly $A$ is weakly $J$-quasipolar.

Proposition 3.14. Let $R$ be a commutative local ring with $6 \in J(R)$ and let $A \in M_2(R)$ such that $A \notin J(M_2(R))$ and $\det A \in J(R)$. Then $A$ is weakly $J$-quasipolar if and only if $x^2 - (\text{tr}A)x + \det A = 0$ has a root in $J(R)$ and a root in $\pm 1 + J(R)$.

Proof. Let $A$ be weakly $J$-quasipolar, $A \notin J(M_2(R))$ and $\det A \in J(R)$. Then there exists $P^2 = P \in \text{comm}^2(A)$ such that $A - P \in J(M_2(R))$ or $A + P \in J(M_2(R))$. Let $A - P \in J(M_2(R))$. So $\text{tr}A \in U(R)$, by Proposition 3.12. If $x^2 - (\text{tr}A)x = -\det A$, then $x(x(\text{tr}A)^{-1} - 1) =$
$-\det A (\text{tr}A)^{-1}$. As $R$ is commutative local, $J(R)$ is a prime ideal in $R$. Hence $x \in J(R)$ or $x(\text{tr}A)^{-1} - 1 \in J(R)$. We discuss the following cases.

**Case 1:** If $x \in J(R)$, then $x(\text{tr}A)^{-1} - 1 \in -1 + J(R)$.

**Case 2:** If $x(\text{tr}A)^{-1} - 1 \in J(R)$, then $x \in 1 + J(R)$. In case of $A + P \in J(M_2(R))$, the proof is similar. Conversely, let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. Assume that $\gamma_1$ and $\gamma_2$ are roots of characteristic equation of $A$ such that $\gamma_1 \in J(R)$ and $\gamma_2 \in \mp 1 + J(R)$. It is clear that $\text{tr}A = \gamma_1 + \gamma_2 \in U(R)$. Without loss of generality, we may assume that $a \in U(R)$. Let $W = \begin{bmatrix} b & a - \gamma_1 \\ \gamma_1 - a & c \end{bmatrix} \in M_2(R)$. Then $\det W = bc - (a - \gamma_1)(\gamma_1 - a) \in U(R)$ and $W \in U(M_2(R))$. So $W^{-1}AW = \begin{bmatrix} \gamma_1 & 0 \\ 0 & \gamma_2 \end{bmatrix}$. By Lemma 3.13, $W^{-1}AW$ is weakly $J$-quasipolar. Therefore $A$ is weakly $J$-quasipolar by Lemma 2.5.

**Theorem 3.15.** Let $R$ be a commutative local ring with $6 \in J(R)$. The matrix $A \in M_2(R)$ is weakly $J$-quasipolar if and only if one of the following holds:

1. Either $A$ or $A - I_2$ or $A + I_2$ is in $J(M_2(R))$.
2. The equation $x^2 - (\text{tr}A)x + \det A = 0$ has a root in $J(R)$ and a root in $\mp 1 + J(R)$.

**Proof.** For the sufficiency, in the case (1) clearly $A$ is weakly $J$-quasipolar. Suppose that (2) holds. Then $A \notin J(M_2(R))$ and $\det A \in J(R)$, so $A$ is weakly $J$-quasipolar, by Proposition 3.14.

For the necessity, suppose that $A$, $A - I_2$ and $A + I_2$ are not contained in $J(M_2(R))$. Hence $\det A \in J(R)$ by Lemma 3.6. Therefore (2) is guaranteed by Proposition 3.14.

**Lemma 3.16.** [4, Lemma 1.5] Let $R$ be a commutative domain. Then $A \in M_2(R)$ is an idempotent if and only if either $A = 0$ or $A = I_2$ or $A = \begin{bmatrix} a & b \\ c & 1 - a \end{bmatrix}$ where $bc = a - a^2$.

**Proposition 3.17.** $A \in M_2(\mathbb{Z})$ is weakly $J$-quasipolar if and only if one of the following hold:

1. $A = \begin{bmatrix} -a & b \\ c & a - 1 \end{bmatrix}$ such that $bc = a - a^2$.
2. $A$ is idempotent.
3. $A = \begin{bmatrix} -a & -b \\ -c & a - 1 \end{bmatrix}$ such that $bc = a - a^2$. 
Proof. Assume that $A$ is weakly $J$-quasipolar. Since $J(M_2(\mathbb{Z})) = 0$, proof is clear. Conversely, If $A = \begin{bmatrix} a & b \\ c & 1-a \end{bmatrix}$ and $bc = a - a^2$, then $A$ is idempotent. So $A$ is weakly $J$-quasipolar. Let $A = \begin{bmatrix} -a & b \\ c & a-1 \end{bmatrix}$. If idempotent is chosen as $P = \begin{bmatrix} a & -b \\ -c & 1-a \end{bmatrix}$, then it is clear. Lately, let $A = \begin{bmatrix} -a & -b \\ -c & a-1 \end{bmatrix}$. The idempotent is chosen as $P = \begin{bmatrix} a & b \\ c & 1-a \end{bmatrix}$, it is clear. \hfill \Box

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