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ON THE FITTING IDEALS OF A COMULTIPLICATION MODULE

S. KARIMZADEH * AND S. HADJIREZAEI

Abstract. Let R be a commutative ring. In this paper we assert some properties of finitely generated comultiplication modules and Fitting ideals of them.

1. Introduction

Let R be a commutative ring with identity and M be a finitely generated R-module. For a set $\{x_1, \ldots, x_n\}$ of generators of M there is an exact sequence $0 \longrightarrow N \longrightarrow R^n \xrightarrow{\varphi} M \longrightarrow 0$ where R^n is a free R-module with the set $\{e_1, \ldots, e_n\}$ of basis, the R-homomorphism φ is defined by $\varphi(e_j) = x_j$ and N is the kernel of φ. Let N be generated by $u_\lambda = a_{1\lambda}e_1 + \ldots + a_{n\lambda}e_n$, with λ in some index set Λ. Let $\mathrm{Fitt}_i(M)$ be the ideal of R generated by the minors of size $n - i$ of the matrix

$$\begin{pmatrix} \cdots & a_{1\lambda} & \cdots \\ \vdots & \vdots & \vdots \\ \cdots & a_{n\lambda} & \cdots \end{pmatrix}.$$

For $i > n$, $\mathrm{Fitt}_i(M)$ is defined to be R, and for $i < 0$, $\mathrm{Fitt}_i(M)$ is defined to be the zero ideal. It is known that $\mathrm{Fitt}_i(M)$ is an invariant ideal determined by M, that is, it is determined uniquely by M and it does not depend on the choice of the set of generators of M [10]. The ideal $\mathrm{Fitt}_i(M)$ will be called the i-th Fitting ideal of the module M. It follows from the definition of $\mathrm{Fitt}_i(M)$ that $\mathrm{Fitt}_i(M) \subseteq \mathrm{Fitt}_{i+1}(M)$.

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∗Corresponding author.

Moreover, it is shown that $\mathrm{Fitt}_0(M) \subseteq \mathrm{Ann}_R(M)$ and $(\mathrm{Ann}_R(M))^n \subseteq \mathrm{Fitt}_0(M)$ (M is generated by n elements) and $\mathrm{Fitt}_i(M)_P = \mathrm{Fitt}_i(M_P)$, for every prime ideal P of R [9]. The most important Fitting ideal of M is the first of the $\mathrm{Fitt}_j(M)$ that is nonzero. We shall denote this Fitting ideal by $I(M)$. Note that if $I(M)$ contains a nonzerodivisor, then $I(M_P) = I(M)_P$ for every prime ideal P of R. Fitting ideals are strong tools to identify properties of modules and sometimes to characterize modules. For example Buchsbaum and Eisenbud have shown in [8] that for a finitely generated R-module M, $I(M) = R$ if and only if M is a projective of constant rank module. A lemma of Lipman asserts that if R is a local ring and $M = R^m/K$ and $I(M)$ is the $(m-q)$th Fitting ideal of M, then $I(M)$ is a regular principal ideal if and only if K is finitely generated free and $M/T(M)$ is free of rank $m - q$ ([14]). Finally it is shown in [11] that if M is a finitely generated module over a Noetherian local UFD (R, P), then $I(M) = P$ if and only if

1. M is isomorphic to $R^n / < (a_1, \dots, a_n)^t >$, where $P =< a_1, \dots, a_n >$ and n is a positive integer if M is torsionfree, and
2. M is isomorphic to $R^n \oplus R/P$, for some positive integer n if M is not torsionfree.

An R-module M is said to be a comultiplication module if for any submodule N of M there exists an ideal I of R such that $N = (0 :_M I)$ [5]. Ansari-Toroghy and Farshadifar have shown in [5] that an R-module M is a comultiplication module if and only if for each submodule N of M, $N = (0 :_M \mathrm{Ann}_R(N))$. An R-module M satisfies the double annihilator conditions (DAC for short) if for each ideal I of R, we have $I = \mathrm{Ann}_R(0 :_M I)$. M is said to be a strong comultiplication module if M is a comultiplication R-module which satisfies the double annihilator conditions [4].

1.1. **Comultiplication Module.**

Lemma 1.1. *Let R be an integral domain. If R is a comultiplication R-module, then R is a field.*

Proof. Let I be a nonzero ideal of R. Since R is a domain, $\mathrm{Ann}_R(I) = 0$. Since R is a comultiplication R-module, $I = (0 :_R \mathrm{Ann}_R(I)) = R$. So R is a field. □

Lemma 1.2. *Let M be a finitely generated comultiplication R-module. If there exists a submodule N of M such that $Ann_R(N) = Ann_R(M)$, then $N = M$.*

Proof. [3, Proposition 3.2] □

Theorem 1.3. *Let M be a finitely generated comultiplication R-module. If R is an integral domain, then $I(M) = Fitt_0(M)$ or $M \cong R$.*

Proof. Let M be generated by $\{x_1, \ldots, x_n\}$. Consider the exact sequence $0 \longrightarrow N \longrightarrow R^n \xrightarrow{\varphi} M \longrightarrow 0$, where $\varphi(e_j) = x_j$ and $N = Ker(\varphi)$. Let $r_i \in \text{Ann}_R(x_i)$ for $i = 1, ..., n$. Consider the matrix

$$\begin{bmatrix} r_1 & 0 & \ldots & 0 \\ 0 & r_2 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & \ldots & 0 & r_n \end{bmatrix}.$$

Since each column of this matrix belongs to N, so we have $\text{Ann}_R(x_1)...\text{Ann}_R(x_n) \subseteq \text{Fitt}_0(M)$. If $\text{Ann}_R(x_1)...\text{Ann}_R(x_n) \neq 0$, then $\text{Fitt}_0(M) = I(M)$. If there is an integer i, $1 \leq i \leq n$, such that $\text{Ann}_R(x_i) = 0$, then $\text{Ann}_R(M) = \cap_{i=1}^n \text{Ann}_R(x_i) = \text{Ann}_R(x_i) = 0$. Thus by Lemma 1.2, $M = Rx_i \cong R$. □

An R-module M is said to be a prime module if $\text{Ann}_R(N) = \text{Ann}_R(M)$, for every non-zero submodule N of M [17].

Proposition 1.4. *Let M be a finitely generated comultiplication R-module. If M is a prime module, then M is a simple R-module.*

Proof. [3, Proposition 3.18] □

Corollary 1.5. *Let M be a finitely generated faithful comultiplication R-module. If R is an integral domain, then R is a field and $M \cong R$.*

Proof. [3, Theorem 3.3] □

Proposition 1.6. *Let M be a finitely generated comultiplication module over an integral domain R. If $I(M)$ is a prime ideal of R, then M is a simple R-module.*

Proof. Let $M = \langle x_1, ..., x_n \rangle$. By Theorem 1.3, we have $M \cong R$ or $I(M) =\text{Fitt}_0(M)$. If $M \cong R$, then by Corollary 1.5, R is a field and hence M is simple. If $I(M) =\text{Fitt}_0(M)$, then as the proof of Theorem 1.3, we can conclude that $\text{Ann}_R(x_1)...\text{Ann}_R(x_n) \subseteq\text{Fitt}_0(M)$. So there exists some x_i, $1 \leq i \leq n$, such that $\text{Ann}_R(x_i) \subseteq \text{Fitt}_0(M)$. Since $\text{Ann}_R(x_i) \subseteq \text{Fitt}_0(M) \subseteq \text{Ann}_R(M) \subseteq \text{Ann}_R(x_i)$, $I(M) = \text{Ann}_R(M) = \text{Ann}_R(x_i)$. Thus by Lemma 1.2, $M = Rx_i$. Let N be a nonzero submodule of M and $0 \neq n \in N$. So there exists $a \in R$ such that $n = ax_i$. Suppose that $r \in \text{Ann}_R(N)$, $0 = rn = rax_i$. Thus $ra \in \text{Ann}_R(M)$. Since $\text{Ann}_R(M)$ is a prime ideal of R, $r \in \text{Ann}_R(M)$ or $a \in \text{Ann}_R(M)$. Since n is a nonzero element of N, $r \in \text{Ann}_R(M)$.

Hence $\mathrm{Ann}_R(N) = \mathrm{Ann}_R(M)$. So M is a prime module. Therefore by Proposition 1.4, M is simple. □

Proposition 1.7. *Every finitely generated comultiplication module over a valuation ring is cyclic.*

Proof. Let $M = \langle x_1, ..., x_n \rangle$. Since R is a valuation ring, there exists a positive integer i, $1 \leq i \leq n$, such that $\mathrm{Ann}_R(x_i) \subseteq \mathrm{Ann}_R(x_j)$ for all $1 \leq j \leq n$. Hence $\mathrm{Ann}_R(M) = \mathrm{Ann}_R(x_i)$. Thus by Lemma 1.2, $M = \langle x_i \rangle$. □

Lemma 1.8. *Let M be a finitely generated comultiplication R-module. If R is a Dedekind domain, then M is cyclic.*

Proof. Let $M = \langle x_1, ..., x_n \rangle$ and P be a maximal ideal of R. By [1, Corollary 2.6], M_P is a comultiplcation module. Since R_P is a valuation ring, by Proposition 1.7, M_P is a cyclic module. So by [6, Perposition 5], M is a multiplication module. By [1, Corollary 2.2], $R/\mathrm{Ann}_R(M)$ is a semi- local ring. Therefore by [6, Proposition 4], M is a cyclic. □

Lemma 1.9. *Let M be a finitely generated comultiplication R-module. If $M = \langle x_1, ..., x_n \rangle$ and $\cap_{i=1}^n Rx_i = 0$, then $Fitt_{n-1}(M) = R$.*

Proof. Consider the exact sequence $0 \longrightarrow N \longrightarrow R^n \xrightarrow{\varphi} M \longrightarrow 0$, where $\varphi(e_j) = x_j$ and $N = Ker(\varphi)$. Let $r_i \in \mathrm{Ann}_R(x_i)$ for $i = 1, ..., n$. Consider the matrix

$$\begin{bmatrix} r_1 & 0 & \dots & 0 \\ 0 & r_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & \dots & 0 & r_n \end{bmatrix}.$$

Since each column of this matrix belongs to N, so we have $\Sigma_{i=1}^n \mathrm{Ann}_R(x_i) \subseteq \mathrm{Fitt}_{n-1}(M)$. By [1, lemma 2.3], $\Sigma_{i=1}^n \mathrm{Ann}_R(x_i) = R$. Thus $\mathrm{Fitt}_{n-1}(M) = R$. □

Lemma 1.10. *Let M be a comultiplication R-module generated by n elements. If M is a decomposable R-module, then $Fitt_{n-1}(M) = R$.*

Proof. Let M be a decomposable R-module. Since M is a finitely generated decomposable module, there exist finitely generated submodules $N_1, ..., N_k$ of M such that $M = \oplus_{i=1}^k N_i$ and $N_i = \langle x_{i1}, ..., x_{in_i} \rangle$, for some elements $x_ij \in M$, $1 \leq j \leq i_{n_i}, 1 \leq i \leq k$. By [1, Lemma 2.3], $\Sigma_{i=1}^k \mathrm{Ann}_R(N_i) = R$. So $M = \langle x_{11}, ..., x_{1n_1}, x_{21}, ..., x_{kn_k} \rangle$. Put $n = n_1 + ... + n_k$. Therefore $\Sigma_{i,j} \mathrm{Ann}_R(x_{ij}) \subseteq \mathrm{Fitt}_{n-1}(M)$. Since $\Sigma_{i=1}^k \mathrm{Ann}_R(N_i) \subseteq \Sigma_{i,j} \mathrm{Ann}_R(x_{ij})$, $\mathrm{Fitt}_{n-1}(M) = R$. □

Proposition 1.11. *Let M be a decomposable comultiplication module. If M is generated by two elements, then $Fitt_0(M) = Ann_R(M)$.*

Proof. By Lemma 1.10, $\mathrm{Fitt}_1(M) = R$. By [9, Proposition 20.7], $\mathrm{Ann}_R(M)\mathrm{Fitt}_1(M) \subseteq \mathrm{Fitt}_0(M)$. So $\mathrm{Fitt}_0(M) = \mathrm{Ann}_R(M)$. □

Theorem 1.12. *Let M be a finitely generated R-module.*
(i) If $I(M)$ is a prime ideal of R, then $Ann_R(M) \subseteq I(M)$.
(ii) If $I(M) = Q_1 ... Q_n$ such that Q_i are distinct maximal ideals of R, then $Ann_R(M) \subseteq I(M)$.
(iii) If $Ann_R(M) = Q^n$, for some maximal ideal Q of R and positive integer n, then $I(M) = R$ or $I(M)$ is a Q-primary ideal of R.

Proof. Let M be generated by n elements.
(i) By [9, Proposition 20.7], $(\mathrm{Ann}_R(M))^n \subseteq \mathrm{Fitt}_0(M) \subseteq I(M)$. So $\mathrm{Ann}_R(M) \subseteq I(M)$.
(ii) We shall show that $\mathrm{Ann}_R(M) \subseteq Q_i$, for all $i = 1, ..., n$. Assume that $\mathrm{Ann}_R(M) \nsubseteq Q_i$, for some $i = 1, ..., n$. So $\mathrm{Ann}_R(M) + Q_i = R$. Hence there is a $q \in Q_i$ such that $1 - q \in \mathrm{Ann}_R(M)$. By [9, Proposition 20.7], $(\mathrm{Ann}_R(M))^n \subseteq \mathrm{Fitt}_0(M) \subseteq I(M)$. So $(1 - q)^n \in I(M) \subseteq Q_i$. Therefore $1 - q \in Q_i$, a contradiction. Thus $\mathrm{Ann}_R(M) \subseteq Q_1 \cap ... \cap Q_n = Q_1 ... Q_n = I(M)$.
(iii) By [9, Proposition 20.7], $(\mathrm{Ann}_R(M))^m = Q^{nm} \subseteq \mathrm{Fitt}_0(M) \subseteq I(M)$. So $Q \subseteq \sqrt{I(M)}$. Since Q is a maximal ideal of R, $\sqrt{I(M)} = Q$ or $\sqrt{I(M)} = R$. This implies that $I(M)$ is a primary ideal or $I(M) = R$. □

Proposition 1.13. *Let M be a comultiplication R-module. If M is a decomposable module and $M = \langle x_1, ..., x_n \rangle$, then $(Ann_R(M))^{n-1} \subseteq Fitt_0(M)$.*

Proof. By Lemma 1.10, $\mathrm{Fitt}_{n-1}(M) = R$. Hence by [9, Proposition 20.7], $(\mathrm{Ann}_R(M))^{n-1} \subseteq \mathrm{Fitt}_0(M)$. □

Theorem 1.14. *Let M be a finitely generated comultiplication module. If $Ann_R(M) = Q_1 ... Q_n$, where Q_i, $1 \le i \le n$, are distinct maximal ideals of R, then $M \cong R/Q_1 \oplus ... \oplus R/Q_n$.*

Proof. Assume that $(0 : Q_j) = 0$, for some $1 \le j \le n$. By [1, Lemma 2.1], $Q_j M = M$. So $\Pi_{\substack{i=1 \\ i \neq j}}^{n} Q_i M = \Pi_{i=1}^{n} Q_i M = 0$. Hence $\Pi_{\substack{i=1 \\ i \neq j}}^{n} Q_i \subseteq \mathrm{Ann}_R(M) \subseteq Q_j$ which is a contradiction since $Q_1, ..., Q_n$ are distinct maximal ideals of R. Therefore by [1, Lemma 2.1], $(0 :_M Q_i)$ is a simple module for all $i = 1, ..., n$. Hence for all $i = 1, ..., n$ there exists $x_i \in M$ such that $(0 :_M Q_i) = Rx_i$. We shall show that $Rx_j \cap \Sigma_{\substack{i=1 \\ i \neq j}} Rx_i = 0$ for all

$j = 1, ..., n$. Assume that $Rx_j \cap \Sigma_{\substack{i=1 \\ i \neq j}} Rx_i \neq 0$ for some $j = 1, ..., n$. Since Rx_j is simple, $Rx_j \cap \Sigma_{\substack{i=1 \\ i \neq j}} Rx_i = Rx_j$. So, $Rx_j \subseteq \Sigma_{\substack{i=1 \\ i \neq j}} Rx_i$ and hence $Q_j = \mathrm{Ann}_R(Rx_j) \supseteq \mathrm{Ann}_R(\Sigma_{\substack{i=1 \\ i \neq j}} Rx_i) \supseteq \Pi_{i=1}^n Q_i$. This implies that there exists $i \neq j$, $1 \leq i \leq n$, such that $Q_i = Q_j$ which is a contradiction since $Q_1, ..., Q_n$ are distinct maximal ideals of R. Therefore $N = Rx_1 \oplus ... \oplus Rx_n$ is a submodule of M and $\mathrm{Ann}_R(N) = \mathrm{Ann}_R(M)$. Since M is a comultiplication module, by Lemma 1.2, $N = M$. Hence $M = Rx_1 \oplus ... \oplus Rx_n \cong R/Q_1 \oplus ... \oplus R/Q_n$. □

Von Neumann regular ring is a ring R such that for every $a \in R$ there exists an element $b \in R$ such that $a = aba$.

Proposition 1.15. *Let M be a finitely generated comultiplication R-module. If R is a von Neumann regular ring, then $I(M) = Q_1...Q_n$, where Q_i are maximal ideals of R, $1 \leq i \leq n$.*

Proof. By [2, Corollary 1.7], M is a semisimple module. Hence there exist maximal ideals $Q_1, ..., Q_n$ of R such that $M \cong R/Q_1 \oplus ... \oplus R/Q_n$. So $I(M) = Q_1...Q_n$. □

Theorem 1.16. *Let M be a finitely generated comultiplication module. If Fitt$_0(M) = Q_1...Q_n$, where Q_i, $1 \leq i \leq n$, are distinct maximal ideals of R, then M is a semisimple module.*

Proof. Similar to the proof of Theorem 1.14, $(0 :_M Q_i)$ are simple modules for all $i = 1, ..., n$ and for all $1 \leq j \leq n$, $Rx_j \cap \Sigma_{\substack{i=1 \\ i \neq j}} Rx_i = 0$, where $Rx_i = (0 :_M Q_i)$ for all $1 \leq i \leq n$. Put $N = Rx_1 \oplus ... \oplus Rx_n$. We have $Q_1...Q_n = \mathrm{Fitt}_0(M) \subseteq \mathrm{Ann}_R(M) \subseteq \mathrm{Ann}_R(N) = Q_1...Q_n$. Thus by Lemma 1.2, $M = N$. □

Lemma 1.17. *Let M be a finitely generated module. If Ann$_R(M) = \langle e \rangle$, where e is a non-zero idempotent element of R, then $I(M) =$ Ann$_R(M)$.*

Proof. Let $M = \langle x_1, ..., x_n \rangle$. By [9, Proposition 20.7], $(\mathrm{Ann}_R(M))^n \subseteq \mathrm{Fitt}_0(M)$. So $e = e^n \in \mathrm{Fitt}_0(M)$. Hence $\mathrm{Fitt}_0(M) = \langle e \rangle$. □

Theorem 1.18. *Let M be a finitely generated comultiplicatin module. If there is a submodule N of M such that Ann$_R(N) = \langle e \rangle$, where e is an idempotent element of R, then N is a direct summand of M and $I(M) \subseteq \langle e \rangle$.*

Proof. Assume that N is a proper submodule of M and $\mathrm{Ann}_R(N) = \langle e \rangle$. Put $K = \{m \in M : (1 - re)m = 0 \text{ for some } r \in R\}$. It is clear that K is a submodule of M and $N \cap K = 0$. For $m \in M$ we have

$m = (1-e)m + em$ and $em \in K$, $(1-e)m \in N$. So $M = N \oplus K$. By [7, p.174], $I(M) = I(N)I(K)$ and by Lemma 1.17, $I(N) = \langle e \rangle$, so $I(M) \subseteq \langle e \rangle$. □

Corollary 1.19. *Let M be a finitely generated strong comultiplication R-module. If e is an idempotent element of R, then $e \in Ann_R(M)$ or $1-e \in Ann_R(M)$.*

Proof. Let e be an idempotent element of R. If $(0 :_M e) = M$, then $e \in \text{Ann}_R(M)$. If $(0 :_M e) = 0$, then $(0 :_M 1-e) = M$. Hence $1-e \in \text{Ann}_R(M)$. If $(0 :_M e)$ is neither M nor 0, then by Theorem 1.18, $M = (0 :_M e) \oplus L$ where $L = \{m : (1-re)m = 0 \text{ for some } r \in R\}$. If $m \in L$, then there is some $r \in R$ such that $m = rem$. So $em = r(e)^2m = rem = m$. Hence $1-e \in \text{Ann}_R(L)$. Thus $L \subseteq (0 :_M 1-e)$. It's clear that $(0 :_M 1-e) \subseteq L$. Since M is a strong comultiplication, $Ann_R(L) = \langle 1-e \rangle$ and $\text{Ann}_R(0 :_M e) = \langle e \rangle$. By Theorem 1.18, $I(M) \subseteq \langle e \rangle$ and $I(M) \subseteq \langle 1-e \rangle$. So $I(M) = 0$ and it's contradiction. □

Proposition 1.20. *Let M be a finitely generated module over a Prüfer domain R and Q be a maximal ideal of R. Then $Ann_R(M) = Q^n$, for some positive integer n if and only if $Fitt_0(M) = Q^k$, for some $k \in \mathbb{N}$.*

Proof. Let M be generated by m elements. By [9, Proposition 20.7], $Q^{nm} = \text{Ann}_R(M)^m \subseteq \text{Fitt}_0(M)$. So $\text{Fitt}_0(M)$ is a Q-primary ideal of R. By [13, Proposition 6.9], there exists some $k \in \mathbb{N}$ such that $\text{Fitt}_0(M) = Q^k$. Hence $\text{Fitt}_0(M) = Q^k$. Conversely, suppose that $\text{Fitt}_0(M) = Q^k$, for some $k \in \mathbb{N}$. Since $Q^k = \text{Fitt}_0(M) \subseteq \text{Ann}_R(M)$, $\text{Ann}_R(M)$ is a Q-primary ideal of R. Hence By [13, Proposition 6.9], there exists some $n \in \mathbb{N}$ such that $\text{Ann}_R(M) = Q^n$. □

Theorem 1.21. *Let M be a finitely generated comultiplicatin module over a Prüfer domain R. If $Fitt_0(M) = Q^n$, where Q is a maximal ideal of R and n is a positive integer, then M is cyclic.*

Proof. Let $M = \langle x_1, ..., x_n \rangle$. Since $Q^n = \text{Fitt}_0(M) \subseteq \text{Ann}_R(M) \subseteq \text{Ann}_R(x_i)$, $\text{Ann}_R(x_i)$ is a Q-primary ideal of R . By [13, Proposition 6.9], there exist some $k_i \in \mathbb{N}$, $1 \le i \le n$, such that $\text{Ann}_R(x_i) = Q^{k_i}$. Put $k = \max\{k_1, ..., k_n\}$. Let $\text{Ann}_R(x_j) = Q^k$, for some $1 \le j \le n$. So $\text{Ann}_R(x_j) \subseteq \text{Ann}_R(x_i)$ for all $1 \le i \le n$. Hence $Rx_i \subseteq Rx_j$, for all $1 \le i \le n$. This implies that $M = Rx_j$. □

Theorem 1.22. *Let M be a finitely generated comultiplicatin module. Then $R/Fitt_0(M)$ is a semilocal ring.*

Proof. Let M be generated by n elements. By [9, Proposition 20.7], $\text{Ann}_R(M)^n \subseteq \text{Fitt}_0(M)$. If Q is a maximal ideal of R such that

$\mathrm{Fitt}_0(M) \subseteq Q$, then $\mathrm{Ann}_R(M) \subseteq Q$. Since $R/\mathrm{Ann}_R(M)$ is a semilocal ring, $R/\mathrm{Fitt}_0(M)$ is a semilocal ring. □

Corollary 1.23. *Let M be a finitely generated comultiplicatin module. If R is not a semilocal ring, then $I(M) = Fitt_0(M)$.*

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S. Karimzadeh
Department of Mathematics, Vali-e-Asr University of Rafsanjan, P.O.Box 7718897111, Rafsanjan, Iran.
Email:karimzadeh@vru.ac.ir

S. Hadjirezaei
Department of Mathematics, Vali-e-Asr University of Rafsanjan, P.O.Box 7718897111, Rafsanjan, Iran.
Email:s.hajirezaei@vru.ac.ir