Journal of Algebra and Related Topics
Vol. 3, No 1, (2015), pp 1-11

LINE GRAPHS ASSOCIATED TO THE MAXIMAL GRAPH

ARTI SHARMA AND ATUL GAUR *

Abstract. Let R be a commutative ring with identity. Let $G(R)$ denote the maximal graph associated to R, i.e., $G(R)$ is a graph with vertices as the elements of R, where two distinct vertices a and b are adjacent if and only if there is a maximal ideal of R containing both. Let $\Gamma(R)$ denote the restriction of $G(R)$ to non-unit elements of R. In this paper we study the various graphical properties of the line graph associated to $\Gamma(R)$, denoted by $L(\Gamma(R))$ such that diameter, completeness, and Eulerian property. A complete characterization of rings is given for which $diam(L(\Gamma(R))) = diam(\Gamma(R))$ or $diam(L(\Gamma(R))) < diam(\Gamma(R))$ or $diam(L(\Gamma(R))) > diam(\Gamma(R))$. We have shown that the complement of the maximal graph $G(R)$, i.e., the comaximal graph is a Euler graph if and only if R has odd cardinality. We also discuss the Eulerian property of the line graph associated to the comaximal graph.

1. Introduction

Let R be a commutative ring with unity. The maximal graph associated to R, defined by authors in [5], as a simple graph whose vertices are elements of R such that two distinct vertices a and b are adjacent if and only if $a, b \in \mathfrak{m}$ for some maximal ideal $\mathfrak{m}$ of R. It is denoted by $G(R)$. Note that the comaximal graph [7] associated to R and the maximal graph $G(R)$ are complements of each other. In [6], the authors defined $\Gamma(R)$ as the restriction of $G(R)$, whose vertices are the non-unit elements of R, i.e., $\Gamma(R)$ is a simple graph whose vertices are

MSC(2010): Primary: 13M99; Secondary: 05C99, 05C45
Keywords: Maximal graph, line graph, eulerian graph, comaximal graph.
Received: 17 February 2015, Accepted: 11 May 2015.
∗Corresponding author.

non-unit elements of R such that two distinct vertices a and b are adjacent if and only if $a, b \in \mathfrak{m}$ for some maximal ideal $\mathfrak{m}$ of R. We shall call both the graphs as the maximal graph associated to R.

In graph theory, the line graph of a given graph G, is denoted by $L(G)$ and is defined as a graph such that each vertex of $L(G)$ represents an edge of G and any two vertices of $L(G)$ are adjacent if and only if their corresponding edges in G share a common vertex. One important theorem, due to Whitney in [8], about the line graphs is that with one exceptional case, $L(G) = K_3$, the structure of any connected graph can be recovered from its line graph, i.e., there is a one-to-one correspondence between the class of connected graphs and the class of connected line graphs. With the class of maximal graphs at hand, it is natural to keep an eye on the properties of their line graphs and seek any relations between them. Recently, in [2], Chiang-hsieh et al studied the line graphs associated to the zero-divisor graphs of commutative rings. In this paper we study the line graphs associated to the maximal graphs of R.

Given a simple graph G, we let $V = V(G)$ denote its vertex set and $E = E(G)$ its edge set. The degree of $v \in V$, denoted by $deg(v)$, is the number of edges of G which are incident with v. Recall that a walk in a graph G is a finite sequence of vertices $u = v_0, v_1, \ldots, v_n = v$ and edges $a_1, a_2, \ldots, a_n$ of G:

$$v_0, a_1, v_1, a_2, \ldots, a_n, v_n,$$

where the endpoints of a_i are v_{i-1} and v_i for each i. A walk is closed when the first and last vertices, v_0 and v_n, are the same. A path is a walk in which no vertex is repeated. The length of a path is the number of edges in it. We say that G is connected if every pair of distinct vertices $u, v \in V(G)$ are joined by a path. The distance $d(u, v)$ between two distinct vertices u and v in G is the length of a shortest path joining them, and $d(u, u) = 0$. The supremum of $d(u, v)$ among all pairs of u, v of $V(G)$ is called the diameter of G and is denoted by $diam(G)$. A graph G is a complete graph if every vertex of G is adjacent to every other vertex and a complete graph on n vertices is denoted by K_n. A graph G with no edge is called an empty graph and with no vertex is called a null graph. Let $J(R)$ be the Jacobson radical of R.

In section 2, we prove that $diam(L(\Gamma(R))) < diam(\Gamma(R))$ if and only if either $R \cong \mathbb{Z}_4$, $R \cong \mathbb{Z}_2[x]/(x^2)$, or $R \cong \mathbb{Z}_2[x]/(x^2-x)$; $diam(L(\Gamma(R)))$ $= diam(\Gamma(R))$ if and only if either $R \cong \mathbb{Z}_9$, $R \cong \mathbb{Z}_3[x]/(x^2)$, $R \cong \mathbb{Z}_2 \times F$, where F is a field, or $R \cong \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$; for all other finite rings, $diam(L(\Gamma(R))) > diam(\Gamma(R))$.

In section 3, we prove that for any finite ring $R \cong R_1 \times R_2 \times \cdots \times R_n$, if $L(\Gamma(R))$ is Eulerian, then one of the following holds:

(i) Each R_i has odd cardinality.

(ii) $|J(R)|$ is even.

(iii) $|J(R)|$ is odd and atleast two R_i's have even cardinalities.

Converse also holds in the first two cases.

In section 4, we have shown that the complement of $G(R)$, i.e., the comaximal graph associated to R is Eulerian if and only if R has odd cardinality. Also, we prove that $L(G(R)')$ is Eulerian if and only if R has odd cardinality or R is a field with $|R| = 2^n$, where n is an integer.

2. Diameter of $L(\Gamma(R))$

Throughout this section, we are assuming that R is not a field as the line graph $L(\Gamma(R))$ of $\Gamma(R)$, when R is a field, is just a null graph.

Before examining the diameter of $L(\Gamma(G))$, we first recall the following result from [2], for a simple connected graph G.

Proposition 2.1. [2, Proposition 2.1] *Let G be a simple connected graph. Then $diam(L(G)) \leqslant diam(G) + 1$.*

Note that for any ring R, $diam(\Gamma(R)) \leq 2$. Therefore, $diam(L(\Gamma(R)))$ ≤ 3. The following results may be known, but is given for the completeness.

Proposition 2.2. *Let G be a connected graph. Then $L(G)$ is a connected graph.*

Proof. Let $[s,t]$, $[u,v] \in V(L(G))$ and $P = \{t = u_0, u_1, u_2, \ldots, u_n = u\}$ be a path in G from t to u. Put $e_0 = [s,t]$, $e_{n+1} = [u,v]$, and $e_i = [u_{i-1}, u_i]$ for $1 \leq i \leq n$. Then $\{e_0, e_1, \ldots, e_{n+1}\}$ is a path in $L(G)$ from e_0 to e_{n+1}. Therefore, $L(G)$ is connected. □

Remark 2.3. It is trivial to see that the line graph of $K_{1,n}$ is K_n for all integers $n \geq 1$. Therefore, if $K_{1,n}$ is a subgraph of a graph G, then K_n is a subgraph of $L(G)$.

Proposition 2.4. *Let R be a ring. If $\Gamma(R)$ is a star graph, then either $\Gamma(R) = K_{1,1}$ or $\Gamma(R) = K_{1,2}$.*

Proof. Assume $\Gamma(R) = K_{1,n}$. Since R is not a field, we have $|\mathfrak{m}| = 2$ for every maximal ideal $\mathfrak{m}$ in R. Now, if R has atleast three maximal ideals, say $\mathfrak{m}_1$, $\mathfrak{m}_2$, and $\mathfrak{m}_3$, then $\mathfrak{m}_1 \cap \mathfrak{m}_2 = \{0\} \subset \mathfrak{m}_3$ implies that

either $\mathfrak{m}_1 \subset \mathfrak{m}_3$ or $\mathfrak{m}_2 \subset \mathfrak{m}_3$, which is a contradiction. Therefore, R has atmost two maximal ideals of cardinality two. Now, if R has only one maximal ideal of cardinality two, then $\Gamma(R) = K_{1,1}$. Also, if R has two maximal ideals of cardinality two, then $\Gamma(R) = K_{1,2}$ with the central vertex 0. □

In the next proposition, we enquire about the rings R for which $\Gamma(R) = K_{1,1}$ or $K_{1,2}$.

Proposition 2.5. *Let R be a ring. Then*

(i) $\Gamma(R) = K_{1,1}$ *if and only if R is isomorphic to $\mathbb{Z}_4$ or $\mathbb{Z}_2[x]/(x^2)$.*

(ii) $\Gamma(R) = K_{1,2}$ *if and only if R is isomorphic to $\mathbb{Z}_2[x]/(x^2 - x)$.*

Proof. The sufficiency is obvious in both the cases. For necessity, note that R is a finite ring with all maximal ideals of cardinality two. Now, if x is any non-zero non-unit element in R, then Rx and $ann_R(x)$ are maximal ideals, and hence of cardinality two. Therefore, $|R| = |Rx||ann_R(x)| = 4$. Now, if $\Gamma(R) = K_{1,1}$, then R is a local ring of cardinality 4, and hence R is isomorphic to either $\mathbb{Z}_4$ or $\mathbb{Z}_2[x]/(x^2)$. Also, if $\Gamma(R) = K_{1,2}$, then R is a ring with two maximal ideals, and is of cardinality 4. Therefore, R is isomorphic to $\mathbb{Z}_2[x]/(x^2 - x)$. □

Corollary 2.6. *Let R be a ring. Then $\Gamma(R)$ is a star graph if and only if R is isomorphic to either $\mathbb{Z}_4$, $\mathbb{Z}_2[x]/(x^2)$, or $\mathbb{Z}_2[x]/(x^2 - x)$.*

Proof. This follows directly from Propositions 2.4 and 2.5. □

Theorem 2.7. *Let R be a ring. Then $diam(L(\Gamma(R))) < diam(\Gamma(R))$ if and only if R is isomorphic to either $\mathbb{Z}_4$, $\mathbb{Z}_2[x]/(x^2)$, or $\mathbb{Z}_2[x]/(x^2 - x)$.*

Proof. First, assume that $diam(L(\Gamma(R))) < diam(\Gamma(R))$. Since $0 < diam(\Gamma(R)) \leq 2$, we have either $diam(\Gamma(R)) = 1$ or $diam(\Gamma(R)) = 2$. If $diam(\Gamma(R)) = 1$, then $diam(L(\Gamma(R))) = 0$. Since $\Gamma(R)$ is connected, by Proposition 2.2, $L(\Gamma(R))$ is also connected. Therefore, $L(\Gamma(R)) = K_1$. This implies that $\Gamma(R) = K_2 = K_{1,1}$. Now, by Proposition 2.5, $R \cong \mathbb{Z}_4$ or $R \cong \mathbb{Z}_2[x]/(x^2)$.

Now, suppose that $diam(\Gamma(R)) = 2$. This implies that $diam(L(\Gamma(R))) = 1$ as if $diam(L(\Gamma(R))) = 0$, then $\Gamma(R) = K_{1,1}$ which contradicts that $diam(\Gamma(R)) = 2$. Therefore, $L(\Gamma(R))$ is a complete graph. Let $L(\Gamma(R)) = K_n$, $n \neq 3$. Then $\Gamma(R)$ is a star graph and hence by Corollary 2.6, $R \cong \mathbb{Z}_2[x]/(x^2 - x)$. Now if $L(\Gamma(R)) = K_3$, then either $\Gamma(R) = K_3$ or $\Gamma(R) = K_{1,3}$. If $\Gamma(R) = K_3$, then $diam(\Gamma(R)) = 1$, which is a contradiction. Also, by Proposition 2.4, $\Gamma(R) \neq K_{1,3}$.

Conversely, if $R \cong \mathbb{Z}_4$ or $R \cong \mathbb{Z}_2[x]/(x^2)$, then $\Gamma(R) = K_2$, and hence $L(\Gamma(R)) = K_1$. Also, if $R \cong \mathbb{Z}_2[x]/(x^2 - x)$, then $\Gamma(R) = K_{1,2}$, and hence $L(\Gamma(R)) = K_2$. Therefore, in any case, we have $diam(L(\Gamma(R))) < diam(\Gamma(R))$.

□

Proposition 2.8. *Let R be a ring. Then*

$$diam(L(\Gamma(R))) = diam(\Gamma(R)) = 1$$

if and only if either $R \cong \mathbb{Z}_9$ or $R \cong \mathbb{Z}_3[x]/(x^2)$.

Proof. First, assume that $diam(L(\Gamma(R))) = diam(\Gamma(R)) = 1$. This implies that $L(\Gamma(R))$ and $\Gamma(R)$ are both complete graphs, and hence $L(\Gamma(R)) = \Gamma(R) = K_3$. Therefore, R is a local ring with maximal ideal $\mathfrak{m}$ of cardinality three. Now, if x is any non-zero element in $\mathfrak{m}$, then $|Rx| = |ann_R(x)| = 3$. This gives that $|R| = |Rx||ann_R(x)| = 9 = 3^2$. Thus, by [3, p. 687], either $R \cong \mathbb{Z}_9$ or $R \cong \mathbb{Z}_3[x]/(x^2)$.

Conversely, assume that either $R \cong \mathbb{Z}_9$ or $R \cong \mathbb{Z}_3[x]/(x^2)$. Then R is a local ring with maximal ideal of cardinality three. Therefore, $diam(L(\Gamma(R))) = diam(\Gamma(R)) = 1$. □

Remark 2.9. (i) If R is a finite ring with maximal ideals $\mathfrak{m}_1, \mathfrak{m}_2, \ldots, \mathfrak{m}_n$, then $R \cong \prod_{i=1}^{n} R_i$, where R_i is a finite local ring with maximal ideal, say $\mathfrak{n}_i$ for all i. Also, $|R_i| = p_i^{m_i\alpha_i}$ for some prime p_i, where m_i is the length of R_i and $|R_i/\mathfrak{n}_i| = p_i^{\alpha_i}$ for all i. Also, if $\mathfrak{m}_i = R_1 \times \cdots \times R_{i-1} \times \mathfrak{n}_i \times R_{i+1} \times \cdots \times R_n$, then

$$|\mathfrak{m}_i| = p_i^{(m_i-1)\alpha_i} \prod_{\substack{j=1 \\ j\neq i}}^{n} p_j^{m_j\alpha_j} = p_i^{-\alpha_i}|R|$$

for all i and

$$|J(R)| = |\cap_{i=1}^{n} \mathfrak{m}_i| = \prod_{i=1}^{n} p_i^{(m_i-1)\alpha_i}.$$

Therefore, if $|\mathfrak{m}_i|$ is even for some i, then the cardinality of every maximal ideal in R is even except possibly one.

(ii) Note that the total number of units in R, i.e.,

$$|R \setminus \cup_{i=1}^{n} \mathfrak{m}_i| = \prod_{i=1}^{n} p_i^{(m_i-1)\alpha_i}(p_i^{\alpha_i} - 1).$$

Therefore, R has odd number of units
if and only if $\prod_{i=1}^{n} p_i^{(m_i-1)\alpha_i}$ and $\prod_{i=1}^{n} p_i^{\alpha_i} - 1$ are odd,
if and only if $p_i = 2$ for all i and $|J(R)| = 1$,

if and only if R is reduced and $|R| = 2^k$, where $k = \sum_{i=1}^{n} \alpha_i$ as for a finite ring R, the nilradical is same as the Jacobson radical.

Also, for non reduced ring,

$$|\mathfrak{m}_i \setminus \cup_{\substack{j=1 \\ j\neq i}}^{n} \mathfrak{m}_j| = \prod_{i=1}^{n} p_i^{(m_i-1)\alpha_i} \prod_{\substack{j=1 \\ j\neq i}}^{n} (p_j^{\alpha_j} - 1) = |J(R)|. \prod_{\substack{j=1 \\ j\neq i}}^{n} (p_j^{\alpha_j} - 1) \geq 2$$

for all i.

(iii) For a reduced ring R, if $|\mathfrak{m}_i \setminus \cup_{\substack{j=1 \\ j\neq i}}^{n} \mathfrak{m}_j| \geq 2$ for some i, then $\prod_{\substack{j=1 \\ j\neq i}}^{n} (p_j^{\alpha_j} - 1) \geq 2$, and hence $p_j^{\alpha_j} - 1 \geq 2$ for atleast one j. This implies that $|\mathfrak{m}_i \setminus \cup_{\substack{j=1 \\ j\neq i}}^{n} \mathfrak{m}_j| \geq 2$ for all i except possibly when $i = j$.

(iv) For a reduced ring R, if $|\mathfrak{m}_i \setminus \cup_{\substack{j=1 \\ j\neq i}}^{n} \mathfrak{m}_j| = 1$ for all i, then $p_i^{\alpha_i} - 1 = 1$ for all i. This implies that $p_i = 2$ and $\alpha_i = 1$ for all i, and hence $R \cong \mathbb{Z}_2 \times \cdots \times \mathbb{Z}_2$(n times).

(v) For all $m, 1 \leq m \leq n$,

$$|\cap_{i=1}^{m} \mathfrak{m}_i \setminus \cup_{j=m+1}^{n} \mathfrak{m}_j| = \prod_{i=1}^{n} p_i^{(m_i-1)\alpha_i} \prod_{j=m+1}^{n} (p_j^{\alpha_j} - 1).$$

Theorem 2.10. *Let $R = \prod_{i=1}^{n} R_i$ be a ring with $R_i, \mathfrak{n}_i, \mathfrak{m}_i, p_i, m_i, \alpha_i$ as in the Remark 2.9(i). Then*

$$diam(L(\Gamma(R))) = diam(\Gamma(R)) = 2$$

if and only if either $R \cong \mathbb{Z}_2 \times F$, where F is any field other than $\mathbb{Z}_2$, or $R \cong \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$.

Proof. The sufficiency is obvious. For necessity, suppose $diam(L(\Gamma(R))) = diam(\Gamma(R)) = 2$. Then R is not a local ring. First assume that R has two maximal ideals, say $\mathfrak{m}_1$ and $\mathfrak{m}_2$. If $s, t \in \mathfrak{m}_1 \setminus \mathfrak{m}_2$ and $u, v \in \mathfrak{m}_2 \setminus \mathfrak{m}_1$, then $[s,t]-[t,0]-[0,u]-[u,v]$ is a shortest path between $[s,t]$ and $[u,v]$ in $L(\Gamma(R))$, which is a contradiction. Therefore, we may assume that $|\mathfrak{m}_1 \setminus \mathfrak{m}_2| = 1$. Thus, by Remark 2.9(ii), $|J(R)| = 1, p_2^{\alpha_2} - 1 = 1$, i.e., $p_2 = 2, \alpha_2 = 1$. This implies that R_1 is any field and $R_2 \cong \mathbb{Z}_2$. Now, if $R_1 \cong \mathbb{Z}_2$, then $R \cong \mathbb{Z}_2 \times \mathbb{Z}_2 \cong \mathbb{Z}_2[x]/(x^2 - x)$, which is a contradiction, by Theorem 2.7.

Now, suppose that R has three maximal ideals, say $\mathfrak{m}_1$, $\mathfrak{m}_2$, and $\mathfrak{m}_3$. If $|\mathfrak{m}_i \setminus (\mathfrak{m}_j \cup \mathfrak{m}_k)| \geq 2$ for some i, then $|\mathfrak{m}_i \setminus (\mathfrak{m}_j \cup \mathfrak{m}_k)| \geq 2$ for atleast two i, by Remark 2.9(iii). Therefore, by the same argument as above, we will get a shortest path of length three in $L(\Gamma(R))$, which is a contradiction. Thus we may assume that $|\mathfrak{m}_i \setminus \mathfrak{m}_j \cup \mathfrak{m}_k| = 1$ for all i. This implies that $R \cong \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$, by Remark 2.9(iv).

Now, assume that R has n maximal ideals, say $\mathfrak{m}_1, \mathfrak{m}_2, \ldots, \mathfrak{m}_n$, $n \geq 4$. By the same argument as above, we may assume that $|\mathfrak{m}_i \setminus \cup_{\substack{j=1 \\ j\neq i}}^{n} \mathfrak{m}_j| = 1$ for all i. Choose $s \in \mathfrak{m}_1 \setminus \cup_{j=2}^{n} \mathfrak{m}_j, t \in (\mathfrak{m}_1 \cap \mathfrak{m}_2) \setminus \cup_{j=3}^{n} \mathfrak{m}_j$ and $u \in \mathfrak{m}_3 \setminus \cup_{\substack{j=1 \\ j\neq 3}}^{n} \mathfrak{m}_j, v \in (\mathfrak{m}_3 \cap \mathfrak{m}_4) \setminus \cup_{\substack{j=1 \\ j\neq 3,4}}^{n} \mathfrak{m}_j$. Then $[s,t]-[t,0]-[0,u]-[u,v]$ is a shortest path between $[s,t]$ and $[u,v]$ in $L(\Gamma(R))$ of length 3, which is a contradiction. □

Remark 2.11. We see that the inequality $diam(L(\Gamma(R))) \leq diam(\Gamma(R))+1$ is sharp for all the finite rings other than listed in Theorem 2.7, Proposition 2.8, and Theorem 2.10, i.e., $diam(\Gamma(R)) < diam(L(\Gamma(R)))$ for all other finite rings. Therefore, for all finite non-local rings other than listed in Theorems 2.7, 2.10, and Proposition 2.8, $diam(L(\Gamma(R))) = 3$.

Proposition 2.12. *Let R be a ring. Then $L(\Gamma(R))$ is a complete graph if and only if R is isomorphic to one of the following five rings:*

$$\mathbb{Z}_4,\ \mathbb{Z}_2[x]/(x^2),\ \mathbb{Z}_2[x]/(x^2-x),\ \mathbb{Z}_9, \text{ or } \mathbb{Z}_3[x]/(x^2).$$

In particular, $L(\Gamma(R)) = K_n$ for $n = 1, 2$ or 3.

Proof. The sufficiency is obvious. For necessity, assume that $L(\Gamma(R)) = K_n$, $(n \neq 3)$ is a complete graph. Then $\Gamma(R)$ is a star graph. Therefore, $R \cong \mathbb{Z}_4$, $R \cong \mathbb{Z}_2[x]/(x^2)$, or $R \cong \mathbb{Z}_2[x]/(x^2-x)$, by Corollary 2.6. Now, let $L(\Gamma(R)) = K_3$, then either $\Gamma(R) = K_3$ or $\Gamma(R) = K_{1,3}$. Since, by Proposition 2.4, $\Gamma(R) \neq K_{1,3}$, we conclude that $\Gamma(R) = K_3$, and hence $|R| = 3^2$, by Proposition 2.8. Therefore, by [3, p. 687], either $R \cong \mathbb{Z}_9$ or $R \cong \mathbb{Z}_3[x]/(x^2)$. □

3. Eulerian property of $L(\Gamma(R))$

We begin this section with the following definition of a Euler graph from [4].

Definition 3.1. A closed walk running through every edge of the graph G exactly once is called a Euler line and a graph G that contains a Euler line is called a Euler graph.

From [4, Theorem 2.4], recall that a connected graph G is Eulerian if and only if all the vertices of G are of even degree. We conclude that if G is Eulerian, then so is $L(G)$ as $deg([u,v]) = deg(u)+deg(v)-2$ for

any $[u, v] \in V(L(G))$. In this section we shall discuss, when the line graph of $\Gamma(R)$ is Eulerian.

The following lemma has been proved in [2] for zero-divisor graphs. Exactly the same proof will work for maximal graph, i.e., for $\Gamma(R)$.

Lemma 3.2. *Let R be a finite ring. Then $L(\Gamma(R))$ is Eulerian if and only if $deg(v)$ is even for all $v \in V(\Gamma(R))$ or $deg(v)$ is odd for all $v \in V(\Gamma(R))$.*

Proof. The proof follows from [2, Lemma 4.1]. □

Proposition 3.3. *Let $(R, \mathfrak{m})$ be a finite local ring. Then $L(\Gamma(R))$ is Eulerian.*

Proof. Since $(R, \mathfrak{m})$ is a finite local ring, we have $deg(v) = |\mathfrak{m}| - 1$ for all $v \in V(\Gamma(R))$ and hence result follows by Lemma 3.2. □

Theorem 3.4. *Let $R = \prod_{i=1}^{n} R_i$ be a ring with $R_i, \mathfrak{n}_i, \mathfrak{m}_i, p_i, m_i, \alpha_i$ as in the Remark 2.9(i). If $L(\Gamma(R))$ is Eulerian, then exactly one of the following holds:*

(i) *Each R_i has odd cardinality.*

(ii) *$|J(R)|$ is even.*

(iii) *$|J(R)|$ is odd and atleast two R_i's have even cardinalities.*

Converse also holds in the first two cases.

Proof. First, suppose that $L(\Gamma(R))$ is Eulerian. By Lemma 3.2, this implies that either $deg(v)$ is even for all $v \in V(\Gamma(R))$ or $deg(v)$ is odd for all $v \in V(\Gamma(R))$. Let $deg(v)$ be even for all $v \in V(\Gamma(R))$. Choose $v_i \in \mathfrak{m}_i \setminus \cup_{\substack{j=1 \\ j \neq i}}^{n} \mathfrak{m}_j$ for all i. Then, by [6, Lemma 2.2], $deg(v_i) = |\mathfrak{m}_i| - 1$. This implies that every maximal ideal of R is of odd cardinality. Therefore, R has odd cardinality, by [6, Proposition 2.4], and hence each R_i has odd cardinality.

Now suppose that $deg(v)$ is odd for all $v \in V(\Gamma(R))$. By the same argument as above, we conclude that every maximal ideal of R has even cardinality. It follows that R has even cardinality, by [6, Proposition 2.4]. Thus atleast one R_i has even cardinality. If $|J(R)|$ is even, then nothing to prove. Now, assume that $|J(R)|$ is odd. Suppose that there is exactly one R_i, say R_{i_0}, of even cardinality. Now, if $|J(R)| = 1$, then by Remark 2.9(i), $|\mathfrak{m}_{i_0}|$ is odd, a contradiction. Also, if $|J(R)| \neq 1$, then R_{i_0} is a field, and hence $|\mathfrak{m}_{i_0}|$ is odd, again a contradiction.

Conversely, suppose that R_i has odd cardinality for all i. Then R has odd cardinality. By [6, Theorem 2.6], it follows that $\Gamma(R)$ is Eulerian.

Therefore, $L(\Gamma(R))$ is Eulerian. Now, assume that $|J(R)|$ is even. Then $|\mathfrak{m}_i|$ is even for all i. Therefore, $|\cup_{k=1}^{r}\mathfrak{m}_{i_k}|$ is even for any maximal ideals $\mathfrak{m}_{i_1},\ldots,\mathfrak{m}_{i_r}$, since $|J(R)|$ divides every term in the expansion of $|\cup_{k=1}^{r}\mathfrak{m}_{i_k}|$. This implies that $deg(v)$ is odd for any $v\in V(\Gamma(R))$, by [6, Lemma 2.2]. Now, the result follows by Lemma 3.2. □

In the above theorem, converse may not be true in the third case as we have the following example:

Example 3.5. Consider $R=\mathbb{Z}_2\times\mathbb{Z}_2\times\mathbb{Z}_2$. Then R is a reduced ring with three maximal ideals, say, $\mathfrak{m}_1,\mathfrak{m}_2$, and $\mathfrak{m}_3$, and $|\mathfrak{m}_1|=|\mathfrak{m}_2|=|\mathfrak{m}_3|=4$. Also, $|\mathfrak{m}_1\cup\mathfrak{m}_2\cup\mathfrak{m}_3|=7$; so that $deg(0)=|\mathfrak{m}_1\cup\mathfrak{m}_2\cup\mathfrak{m}_3|-1=6$, which is even. Now, if we choose any $v\in\mathfrak{m}_1\setminus(\mathfrak{m}_2\cup\mathfrak{m}_3)$, then $deg(v)=|\mathfrak{m}_1|-1=3$, which is odd. Therefore, by Lemma 3.2, $L(\Gamma(R))$ cannot be Eulerian.

4. Eulerian property of comaximal graph

Throughout this section, we are assuming that R is a finite ring with maximal ideals $\mathfrak{m}_1,\mathfrak{m}_2,\ldots,\mathfrak{m}_n$. We begin this section with the following definition.

Definition 4.1. The complement G' of a graph G is a graph with the same vertex set as G, with the property that two vertices are adjacent in G' if and only if they are not adjacent in G.

Throughout this section, we will denote the maximal graph associated to R as $G(R)$. Observe that the complement of a maximal graph $G(R)$ is nothing but the comaximal graph associated to R defined in [7] and vice versa.

Lemma 4.2. *Let R be a ring and $a\in V(G(R)')$. Then the following holds.*

(i) *If a is unit in R, then $deg(a)=|R|-1$.*

(ii) *Let a be non-unit of R and $\{\mathfrak{m}\mid\mathfrak{m}$ is a maximal ideal of R and $a\in\mathfrak{m}\}=\{\mathfrak{m}_1,\ldots,\mathfrak{m}_k\}$. Then $deg(a)=|U(R)|+|\cup_{i=k+1}^{n}\mathfrak{m}_i\setminus\cup_{j=1}^{k}\mathfrak{m}_j|$, where $U(R)$ is the set of all units in R. In particular, $deg(a)=|U(R)|$ for any $a\in J(R)$.*

Proof. For (i), since a is unit in R, it is an isolated vertex in $G(R)$. Therefore, $deg(a)=|R|-1$ in $G(R)'$.

We now prove (ii). Note that $deg(a)=|\cup_{i=1}^{k}\mathfrak{m}_i|-1$ in $G(R)$, by [6, Lemma 2.2]. Therefore, in $G(R)'$,

$$deg(a)=|R|-|\cup_{i=1}^{k}\mathfrak{m}_i|=|U(R)|+|\cup_{i=k+1}^{n}\mathfrak{m}_i\setminus\cup_{j=1}^{k}\mathfrak{m}_j|.$$

□

Theorem 4.3. *Let R be a ring. Then the comaximal graph $G(R)'$ is Eulerian if and only if R has odd cardinality.*

Proof. First, assume that $G(R)'$ is Eulerian. Then, by [4, Theorem 2.4], $deg(1)$ is even and hence by Lemma 4.2, $|R| = deg(1) + 1$, is odd.

Conversely, suppose that R has odd cardinality. Then $|\mathfrak{m}_i|$ is odd for all i, and hence $|U(R)|$ is even, by Remark 2.9(i). Let a be a unit in R. Then, by Lemma 4.2(i) $deg(a)$ is even. Now let a be a non-unit in R, and $\{\mathfrak{m} \mid \mathfrak{m}$ is a maximal ideal of R and $a \in \mathfrak{m}\} = \{\mathfrak{n}_1, \mathfrak{n}_2, \ldots, \mathfrak{n}_k\}$. Then, by Lemma 4.2(ii),

$$d(a) = |U(R)| + |\cup_{i=k+1}^{n} \mathfrak{n}_i \setminus \cup_{j=1}^{k} \mathfrak{n}_j| = |U(R)| + |\cup_{i=k+1}^{n} \mathfrak{n}_i| - |\cup_{i=k+1,j=1}^{n,\ k} \mathfrak{n}_i \cap \mathfrak{n}_j|.$$

As $|\cup_{i=k+1}^{n} \mathfrak{n}_i|$ and $|\cup_{i=k+1,j=1}^{n,\ k} (\mathfrak{n}_i \cap \mathfrak{n}_j)|$ are odd as every maximal ideal is of odd cardinality and number of terms after expansion is odd. Thus $deg(a)$ is even for all $a \in R$. Therefore, $G(R)'$ is Eulerian. □

Theorem 4.4. *Let R be a ring. Then $L(G(R)')$ is Eulerian if and only if either R has odd cardinality or R is a field with $|R| = 2^n$, where n is an integer.*

Proof. First, suppose that $L(G(R)')$ is Eulerian. Then, by Lemma 3.2, $deg(a)$ is even for all $a \in V(G(R)')$ or $deg(a)$ is odd for all $a \in V(G(R)')$. If $deg(a)$ is even for all $a \in V(G(R)')$, then $G(R)'$ is Eulerian, by [4, Theorem 2.4]. Therefore, by Theorem 4.3, R has odd cardinality. Now, assume that $deg(a)$ is odd for all $a \in V(G(R)')$. Then, by Lemma 4.2(ii), $|U(R)|$ is odd. This implies that R is reduced and $|R| = 2^n$, by Remark 2.9(ii). We assert that (0) is the maximal ideal in R. If not, then every maximal ideal in R is of even cardinality, and hence $|\cup_{i=1}^{n} \mathfrak{m}_i|$ is even. But this is a contradiction since, by Lemma 4.2(i), $|\cup_{i=1}^{n} \mathfrak{m}_i| = deg(a) + 1 - |U(R)|$ for any unit $a \in R$. Therefore, (0) is the maximal ideal in R, and hence R is a field, by [1, Proposition 1.2].

Conversely, if R has odd cardinality, then by Theorem 4.3, $G(R)'$ is Eulerian, and hence $L(G(R)')$ is Eulerian. Now, assume that R is a field with $|R| = 2^n$. Then $G(R)$ is an empty graph, and hence $G(R)' = K_{2^n}$, a complete graph with 2^n vertices. This implies that all the vertices in $G(R)'$ are of the same degree. Therefore, by Lemma 3.2, $L(G(R)')$ is Eulerian. □

Acknowledgments

The first author was supported by a grant from CSIR India, No. 09/045 (1142)/2011-EMR-I and the second author was supported by

R & D grant, University of Delhi, Delhi. The authors would like to thank the referee for the careful review and the valuable comments, which provided insights that helped improve the paper.

References

1. M. F. Atiyah and I. G. Macdonald, *Introduction to Commutative Algebra*, Addison-Wesley, 1969.
2. H. J. Chiang-Hsieh, P. F. Lee and H. J. Wang, *On the line graphs associated to the zero-divisor graphs of commutative rings*, Houston J. Math., (1) **39** (2013), 61-72.
3. B. Corbas and G. D. Williams, *Rings of order p^5. I. nonlocal rings*, J. Algebra, **231** (2000), 677-690.
4. N. Deo, *Graph Theory with Applications to Engineering and Computer Science*, Prentice Hall of India, 2001.
5. A. Gaur and A. Sharma, *Maximal graph of a commutative ring*, Int. J. Algebra, (12) **7** (2013), 581-588.
6. A. Gaur and A. Sharma, *Eulerian graphs and automorphisms of a maximal graph*, Preprint submitted to Indian J. Pure Appl. Math.
7. P. K. Sharma and S. M. Bhatwadekar, *A note on graphical representation of rings*, J. Algebra, **176** (1995), 124-127.
8. H. Whitney, *Congruent graphs and connectivity of graphs*, Amer. J. Math., **54** (1932), 150-168.

Arti Sharma
Department of Mathematics, University of Delhi, Delhi, India.
Email: anjanaarti@gmail.com

Atul Gaur
Department of Mathematics, University of Delhi, Delhi, India.
Email: gaursatul@gmail.com