

Ordinal sums in the class of L -algebras

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Abstract. This paper initiates a systematic investigation of ordinal sum constructions within the variety of L -algebras. We first introduce the concept of an ordinal sum for a family of L -algebras and explore its fundamental properties. Our study focuses on the transfer of algebraic properties between the component algebras and their ordinal sum. Specifically, we analyze the behavior of various important classes of L -algebras under this construction, including K -algebras, CL -algebras, and lattices. To deepen the structural analysis, we introduce the notion of an ideal ordinal sum, extending the construction to the level of ideals. We then investigate the ideal theory of ordinal sums, providing a characterization of prime and maximal ideals in terms of the ideals of the constituent algebras. Finally, we explore the global properties of the ordinal sum. We derive necessary and sufficient conditions for the ordinal sum to be sub-directly irreducible, contributing to the understanding of the equational theory of L -algebras. Furthermore, we examine the conditions under which the ordinal sum exhibits properties such as semi-simplicity, simplicity, locality, and perfectness.

Keywords: L -algebras, Ordinal sum, Ideal, Sub-directly irreducible, Semi-simple.

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1 Introduction

The set-theoretic Yang-Baxter equation plays a central role in mathematical physics, low-dimensional topology, and the theory of braid groups. Its algebraic study has led to the development of several structures that encode solutions in an intrinsic and tractable way. Among these, the notion of L -algebras, introduced by Wolf Rump, provides a fundamental algebraic framework for studying non-degenerate set-theoretic solutions of the Yang-Baxter equation. These

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algebras provide a clean and intrinsic algebraic setting in which the combinatorial data of non-degenerate set-theoretic solutions are translated into algebraic identities. This translation is not merely formal: it enables the application of algebraic tools such as congruence theory, ideal theory, and decomposition methods to problems that might otherwise remain intractable by purely combinatorial means ([7, 9, 10, 14–18, 20, 22]). The evolution of the theory of L -algebras has been shaped by its interplay with related algebraic structures. Notably, connections have emerged with braces, which themselves were introduced to study set-theoretic solutions and have since become a vibrant area of research in their own right. The relationship between L -algebras and braces has led to a cross-fertilization of ideas: techniques developed in one setting often find analogs in the other, and structural insights such as the characterization of ideals, quotients, and homomorphisms carry over and enrich both theories. This interconnection underscores a broader phenomenon. Actually, the Yang-Baxter equation serves as a unifying thread that ties together seemingly disparate algebraic formalisms, from rings and modules to hoop algebras and beyond. Notwithstanding their broad utility of L -algebras and the considerable attention given to other algebras, the construction of new L -algebras from existing ones, particularly through ordinal sums, remains an area that invites further investigation.

The concept of ordinal sum in algebra, is driven by its usefulness in building new ordered structures from existing ones, all while maintaining important properties like the order relations and the integrity of algebraic operations. This idea is especially valuable in fields such as lattice theory [2], order theory, and algebraic systems that possess a built-in hierarchy or ordering, such as ordered groups and semi-groups. One of the primary motivations behind the ordinal sum, is its ability to construct more extensive ordered systems by “stacking” smaller ordered components together. For example, given two ordered sets A and B , the ordinal sum creates a new ordered set $A + B$ where every element of A is less than every element of B (Construction of Larger Structures). In addition, the ordinal sum respects the order of the components. If A and B are totally ordered sets, the ordinal sum ensures the resulting structure is also totally ordered. Moreover, in algebraic systems with order (e.g., lattices or semigroups), the ordinal sum is used to construct new systems while preserving algebraic and order-theoretic properties. For instance: Lattices: The ordinal sum of lattices preserves lattice properties while combining the underlying order structures. Semigroups: Ordinal sum helps in defining ordered semigroups or other similar structures. Finally, since many real-world phenomena and mathematical problems involve hierarchical or layered structures, the ordinal sum is a natural way to model such systems, where one part is “below” another in a strict sense.

In the context of L -algebras, the ordinal sum construction is particularly appealing because L -algebras carry a natural order structure that is integral to their definition. The compatibility between the algebraic operations and the order, suggests that an ordinal sum, if properly defined, could yield new L -algebras while preserving key properties of the components. This raises a number of natural questions. Some of the questions are: Under what conditions does the ordinal sum of two L -algebras remain an L -algebra? The same question can be asked about specific L -algebras, such as K -algebras or CL -algebras.

How do the ideals, congruences, and other internal structures of the components behave under the ordinal sum? Motivated by these questions, the present paper undertakes a systematic investigation of the ordinal sum construction for L -algebras. Our goals are twofold. First, we establish the foundational properties of the construction, showing that the ordinal sum of a

family of L -algebras indexed by a chain is again an L -algebra under natural conditions. This provides a robust framework for generating new examples and for understanding the structure of more complex L -algebras in terms of simpler ones. Second, we extend the construction to the level of ideals, introducing the notion of an ideal ordinal sum. This extension is far from routine: ideals in L -algebras play a role analogous to that of ideals in rings, serving as kernels of homomorphisms and providing a compact way to generate congruences. Understanding how ideals behave under the ordinal sum is essential for a complete structural description of the resulting algebras. The study of ideals in L -algebras has attracted considerable attention in recent years. Gran and his colleagues [6] investigated ideals and congruence relations, establishing foundational results that connect the ideal structure to the overall algebraic structure. Yun et al. [21] examined prime and maximal ideals in commutative L -algebras, while Borzooei and Khajeh-Nasir [3] extended this work to primary ideals and explored their properties. More recently, Shirvani Bourojeni [19] introduced roughness ideals, adding yet another dimension to the ideal theory of L -algebras. Despite this wealth of research, the question of how ideals behave under ordinal sums has remained largely unaddressed. Our work fills this gap by providing a complete description of the ideals, prime ideals, and maximal ideals of an ordinal sum in terms of the corresponding ideals of its components. The applications of our results extend beyond the theory of L -algebras themselves. Since L -algebras encode solutions to the Yang-Baxter equation, any systematic construction of new L -algebras and any understanding of their internal structure has the potential to yield new insights into the Yang-Baxter equation itself. In particular, the ordinal sum construction may provide a way to build complex solutions from simpler ones, mirroring the way that ordinal sums in other algebraic settings have been used to construct complex structures from elementary building blocks. Moreover, the ideal theory developed here may have analogs in related structures such as braces and cycle sets, suggesting avenues for future cross-fertilization. The paper is structured as follows: In Section 2, we recall preliminary definitions and key properties of L -algebras. Section 3 introduces the ordinal sum construction and proves its fundamental properties. Section 4 concludes with open problems and directions for future research.

2 Preliminaries

This section outlines the predefined contents that will be utilized later on.

An algebraic structure $(\mathbf{A}, \rightarrow, 1)$ is called an **L -algebra** if, for any $x, y, z \in \mathbf{A}$, the following conditions hold:

- (L_1) $x \rightarrow x = x \rightarrow 1 = 1$ and $1 \rightarrow x = x$,
- (L_2) $(x \rightarrow y) \rightarrow (x \rightarrow z) = (y \rightarrow x) \rightarrow (y \rightarrow z)$,
- (L_3) if $x \rightarrow y = y \rightarrow x = 1$, then $x = y$.

The L -algebra $\mathbf{A} = (\mathbf{A}, \rightarrow, 1)$ is called trivial if $\mathbf{A} = \{1\}$. By above conditions, the relation $x \leq y$ if and only if $x \rightarrow y = 1$, for any $x, y \in \mathbf{A}$, defines a partial order on an L -algebra $\mathbf{A}$. (see [11]).

An L -algebra $\mathbf{A}$ is called: (i) **bounded L -algebra**, if $\mathbf{A}$ admits a smallest element 0,

(ii) **semiregular** [11] if the equation

$$((x \rightarrow y) \rightarrow z) \rightarrow ((y \rightarrow x) \rightarrow z) = ((x \rightarrow y) \rightarrow z) \rightarrow z, \quad (1)$$

holds in $\mathbf{A}$,

(iii) **K-algebra**, if for any $x, y \in \mathbf{A}$, satisfies in condition (K), that is $x \rightarrow (y \rightarrow x) = 1$,

(iv) **CL-algebra**, if for any $x, y, z \in \mathbf{A}$, satisfies in condition (C), that is $x \rightarrow (y \rightarrow z) = y \rightarrow (x \rightarrow z)$.

Note: From now on, we let $(\mathbf{A}, \rightarrow, 1)$ or $\mathbf{A}$ for short, be an L -algebra, unless otherwise state.

If $(\mathbf{A}, \rightarrow, 0, 1)$ is bounded, then for any element $x \in \mathbf{A}$, we introduce a unary operation $'$ on $\mathbf{A}$ (called **negation**), where $x' = x \rightarrow 0$.

Proposition 1. [12] *Let $\mathbf{A}$ be a semiregular L -algebra with negation. Then*

$$x \leq y \iff y' \leq x', \quad (2)$$

holds for $x, y \in \mathbf{A}$.

Proposition 2. [12] *For an L -algebra $\mathbf{A}$, the following are equivalent:*

$$x \leq y \rightarrow x, \quad (3)$$

$$\text{if } x \leq y, \text{ then } y \rightarrow z \leq x \rightarrow z, \quad (4)$$

$$((x \rightarrow y) \rightarrow y) \rightarrow y = x \rightarrow y. \quad (5)$$

For an L -algebra with negation, we set

$$x \wedge y := ((x \rightarrow y) \rightarrow x'), \quad x \vee y := (x' \rightarrow y') \rightarrow x. \quad (6)$$

Proposition 3. [12] *Let $\mathbf{A}$ be an L -algebra with negation which satisfies (2) and (3). Then $\mathbf{A}$ is a lattice with meet and join given by (6).*

Proposition 4. [5] *Every CL -algebra is a K -algebra.*

A non-empty subset I of $\mathbf{A}$ is called an **ideal of $\mathbf{A}$** if for any $x, y \in \mathbf{A}$: (I_1) : $1 \in I$, (I_2) : if $x, x \rightarrow y \in I$, then $y \in I$, (I_3) : if $x \in I$, then $(x \rightarrow y) \rightarrow y \in I$, and (I_4) : if $x \in I$, then $y \rightarrow x \in I$ and $y \rightarrow (x \rightarrow y) \in I$.

The set of all ideals of $\mathbf{A}$ is denoted by $\mathcal{I}(\mathbf{A})$. $I \in \mathcal{I}(\mathbf{A})$ is called **proper** if $I \neq \mathbf{A}$ and a non-trivial L -algebra $\mathbf{A}$ is called **simple** if $\mathcal{I}(\mathbf{A}) = \{\{1\}, \mathbf{A}\}$ (see [11]).

Remark 1. [5] *We have to notice that if $\mathbf{A}$ is a K -algebra, then (I_3) and if $\mathbf{A}$ is a CL -algebra, then (I_3) and (I_4) can be omitted.*

Proposition 5. [1] *Let $I \in \mathcal{I}(\mathbf{A})$. Then I is an upset.*

Let $I \in \mathcal{I}(\mathbf{A})$. Then for any $x, y \in \mathbf{A}$, define a binary relation θ_I on $\mathbf{A}$ by: $(x, y) \in \theta_I$ if and only if $\{x \rightarrow y, y \rightarrow x\} \subseteq I$. In [11, Proposition 1], the author proved that the relation θ_I is a congruence relation on $\mathbf{A}$. The set of all congruence relations on $\mathbf{A}$ is denoted by $\text{Con}(\mathbf{A})$. There is a bijective correspondence between ideals of $\mathbf{A}$ and congruence relation $\sim$ for which $\mathbf{A}/\sim$ is an L -algebra (see [11]).

Let $\emptyset \neq X \subseteq \mathbf{A}$. The smallest ideal of $\mathbf{A}$ containing X is called the **generated ideal** by X in $\mathbf{A}$ which is denoted by $\langle X \rangle$. Indeed, $\langle X \rangle = \bigcap_{X \subseteq I \in \mathcal{I}(\mathbf{A})} I$ (see [5]). Let P and M be two

proper ideals of $\mathbf{A}$. Then P is called **prime**, if for any ideal I either $I \subseteq P$ or $I \rightarrow P \subseteq P$. The set of all prime ideals of $\mathbf{A}$ is denoted by $\text{Prime}(\mathbf{A})$. Moreover, M is called a **maximal** ideal of $\mathbf{A}$ if for any $N \in \mathcal{I}(\mathbf{A})$ such that $M \subseteq N \subseteq \mathbf{A}$, we have $M = N$ or $N = \mathbf{A}$. The set of all maximal ideals of $\mathbf{A}$ is denoted by $\text{Max}(\mathbf{A})$. $\mathbf{A}$ is said to be **local** if it has a unique maximal ideal. (see [13, 21]).

In a bounded L -algebra $\mathbf{A}$, the **order** of an element $x \in \mathbf{A}$ is the smallest integer n such that $x^n \rightarrow 0 = 1$, is denoted by $\text{ord}(x) = n$, and if no such n exists, then $\text{ord}(x) = \infty$. A bounded L -algebra $\mathbf{A}$ is called **perfect** if for all $x \in \mathbf{A}$, $\text{ord}(x) < \infty$ if and only if $\text{ord}(x') = \infty$ (see [8]).

3 Ordinal sum of L -algebras

In this section, the idea of ordinal sums within the framework of L -algebras is investigated. Using ordinal sums, we can build new L -algebras from a particular class, like chain L -algebras, by combining existing structures in a systematic way. Next, we will illustrate that the ordinal sum of linear L -algebras preserves linearity.

Definition 6. Let $(I, \leq)$ be a linearly ordered set and $\{(\mathbf{A}_i, \rightarrow_i, 1)\}_{i \in I}$ be a family of L -algebras such that $\mathbf{A}_i \cap \mathbf{A}_j = \{1\}$, for all $i \neq j \in I$. Define the binary operation $\rightarrow$ on $\mathbf{A} = \bigcup_{i \in I} \mathbf{A}_i$ as follows:

$$x \rightarrow y = \begin{cases} x \rightarrow_i y, & \text{if } x, y \in \mathbf{A}_i \\ 1, & \text{if } x \in \mathbf{A}_i \setminus \{1\}, y \in \mathbf{A}_j, i < j \\ y, & \text{if } x \in \mathbf{A}_i, y \in \mathbf{A}_j, i > j \end{cases} \quad (7)$$

Then $(\mathbf{A}, \rightarrow, 1)$ is called the **ordinal sum** of family $\{(\mathbf{A}_i, \rightarrow_i, 1)\}_{i \in I}$ and is denoted by $\mathbf{A} = \bigoplus_{i \in I} \mathbf{A}_i$. If I is the finite index family $\{1, \dots, n\}$, then we simply denote the ordinal sum of the family by $\mathbf{A} = \mathbf{A}_1 \oplus \dots \oplus \mathbf{A}_n$.

Proposition 7. Let $\{(\mathbf{A}_i, \rightarrow_i, 1)\}_{i \in I}$ be as Definition 6, and $\mathbf{A} = \bigoplus_{i \in I} \mathbf{A}_i$. Then, $(\mathbf{A}, \rightarrow, 1)$ is an L -algebra.

Proof. (L_1): According to Definition 6, clearly (L_1) holds.

(L_2): Let $x, y, z \in \mathbf{A}$. By using Definition 6, we consider 4 cases:

Case 1: Let $x, y, z \in \mathbf{A}_i$. Then the proof is clear.

Case 2: Let $x, y \in \mathbf{A}_i$, and $z \in \mathbf{A}_j$:
If $x, y \in \mathbf{A}_i \setminus \{1\}$, $z \in \mathbf{A}_j$, and $i < j$, then

$$(x \rightarrow y) \rightarrow (x \rightarrow z) = (x \rightarrow_i y) \rightarrow 1 = 1 = (y \rightarrow_i x) \rightarrow 1 = (y \rightarrow x) \rightarrow (y \rightarrow z).$$

If $x, y \in \mathbf{A}_i, z \in \mathbf{A}_j$, and $i > j$, then

$$(x \rightarrow y) \rightarrow (x \rightarrow z) = (x \rightarrow_i y) \rightarrow z = z = (y \rightarrow_i x) \rightarrow z = (y \rightarrow x) \rightarrow (y \rightarrow z).$$

Case 3: Let $x, z \in \mathbf{A}_i$ and $y \in \mathbf{A}_j$.

If $x, z \in \mathbf{A}_i \setminus \{1\}, y \in \mathbf{A}_j$ and $i < j$, then

$$(x \rightarrow y) \rightarrow (x \rightarrow z) = 1 \rightarrow (x \rightarrow_i z) = x \rightarrow_i z = (y \rightarrow x) \rightarrow (y \rightarrow z).$$

If $x, z \in \mathbf{A}_i, y \in \mathbf{A}_j$, and $i > j$, then

$$(x \rightarrow y) \rightarrow (x \rightarrow z) = y \rightarrow (x \rightarrow_i z) = 1 = 1 \rightarrow 1 = (y \rightarrow x) \rightarrow (y \rightarrow z).$$

Case 4: Let $x \in \mathbf{A}_i$ and $y, z \in \mathbf{A}_j$.

If $x \in \mathbf{A}_i \setminus \{1\}, y, z \in \mathbf{A}_j$ and $i < j$, then

$$(x \rightarrow y) \rightarrow (x \rightarrow z) = 1 \rightarrow 1 = 1 = x \rightarrow (y \rightarrow_j z) = (y \rightarrow x) \rightarrow (y \rightarrow z).$$

If $x \in \mathbf{A}_i, y, z \in \mathbf{A}_j$ and $i > j$, then

$$(x \rightarrow y) \rightarrow (x \rightarrow z) = y \rightarrow_j z = 1 \rightarrow (y \rightarrow_j z) = (y \rightarrow x) \rightarrow (y \rightarrow z).$$

Hence, in all above cases (L_2) holds.

(L_3) Let $x, y \in \mathbf{A}$ and $x \rightarrow y = y \rightarrow x = 1$, but $x \neq y$, by contrary. We consider 3 cases:

Case 1: If $x, y \in \mathbf{A}_i$, then by definition $x = y$.

Case 2: If $x \in \mathbf{A}_i \setminus \{1\}$ and $y \in \mathbf{A}_j$ such that $i < j$, then $1 = x \rightarrow y \neq y \rightarrow x = x$, which is a contradiction.

Case 3: If $x \in \mathbf{A}_j \setminus \{1\}$ and $y \in \mathbf{A}_i$ such that $i < j$, then, $y = x \rightarrow y = y \rightarrow x = 1$, but it is impossible, because if $y = 1$, then $1 = x \rightarrow 1 = 1 \rightarrow x = x$, which is a contradiction.

Hence, in all above cases (L_3) holds. Therefore, $(\mathbf{A}, \rightarrow, 1)$ is an L -algebras. $\square$

Example 1. Let $\mathbf{A}_1 = \{0, a, b, 1\}$ be a chain where $0 < a < b < 1$, $\mathbf{A}_2 = \{x, y, z, 1\}$ be a poset, where $x < y, z < 1$, $\mathbf{A}_3 = \{\gamma, \alpha, \beta, 1\}$ be a chain where $\gamma < \alpha < \beta < 1$ and operations $\rightarrow_1, \rightarrow_2$ and $\rightarrow_3$ on $\mathbf{A}_1, \mathbf{A}_2$ and $\mathbf{A}_3$ are defined as shown in Tables 1a, 1b and 1c, respectively.

Table 1: Tables of operations on $\mathbf{A}_1, \mathbf{A}_2$ and $\mathbf{A}_3$

(a) The implication of $\mathbf{A}_1$

$\rightarrow_1$	0	a	b	1
0	1	1	1	1
a	b	1	1	1
b	a	b	1	1
1	0	a	b	1

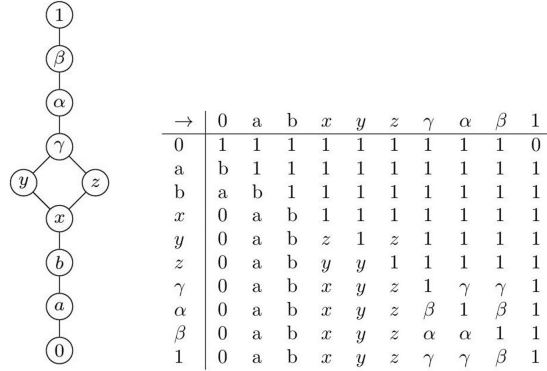
(b) The implication of $\mathbf{A}_2$

$\rightarrow_2$	x	y	z	1
x	1	1	1	1
y	z	1	z	1
z	y	y	1	1
1	x	y	z	1

(c) The implication of $\mathbf{A}_3$

$\rightarrow_3$	γ	α	β	1
γ	1	1	1	1
α	γ	1	1	1
β	γ	α	1	1
1	γ	α	β	1

Then $(\mathbf{A}_1, \rightarrow_1, 1)$, $(\mathbf{A}_2, \rightarrow_2, 1)$ and $(\mathbf{A}_3, \rightarrow_3, 1)$ are L -algebras. It is easy to see that $(\mathbf{A} = \mathbf{A}_1 \oplus \mathbf{A}_2 \oplus \mathbf{A}_3, \rightarrow, 1)$ with the Hasse diagram and table in which contain in Figure 1, is an L -algebra.

**Figure 1:** The Hasse diagram and implication of $(\mathbf{A} = \mathbf{A}_1 \oplus \mathbf{A}_2 \oplus \mathbf{A}_3, \rightarrow, 1)$

Remark 2. Let $\{(\mathbf{A}_i, \rightarrow_i, 1)\}_{i \in I}$ be a family of L -algebras and $\mathbf{A} = \bigoplus_{i \in I} \mathbf{A}_i$. If I is bounded from below with minimal element i_0 and $\mathbf{A}_{i_0}$ is bounded, since for any $i \in I \setminus \{i_0\}$, we have $i_0 < i$, then by Definition 6, all elements of $\mathbf{A}_{i_0}$ are less than other elements in $\mathbf{A}_i$, so the smallest element in $\mathbf{A}_{i_0}$ is the smallest element of $\mathbf{A} = \bigoplus_{i \in I} \mathbf{A}_i$. Hence $\mathbf{A}$ is bounded, too. Moreover, it is straightforward that $\mathbf{A} = \bigoplus_{i \in I} \mathbf{A}_i$ is associative. In addition, the sub- L -algebra of $\mathbf{A} = \bigoplus_{i \in I} \mathbf{A}_i$ is exactly the L -algebra of the form $L = \bigoplus_{i \in I} L_i$, where any L_i is (possibly trivial) a sub- L -algebra of $\mathbf{A}_i$. It is straightforward to see that any $\mathbf{A}_i$ is a sub-algebra of $(\bigoplus_{i \in I} \mathbf{A}_i, \rightarrow, 1)$. In addition and in general case, $\mathbf{A}_i \oplus \mathbf{A}_j \neq \mathbf{A}_j \oplus \mathbf{A}_i$, for any $i, j \in I$ such that $i \neq j$ and both are non-trivial.

Example 2. According to Example 1, $(\mathbf{A}_1 \oplus \mathbf{A}_2, \rightarrow_{12}, 1)$ and $(\mathbf{A}_1 \oplus \mathbf{A}_2, \rightarrow_{21}, 1)$ are two L -algebras with the Tables 2a, 2b:

Table 2: Two tables for showing that $\mathbf{A}_1 \oplus \mathbf{A}_2 \neq \mathbf{A}_2 \oplus \mathbf{A}_1$ **(a)** The implication of $\mathbf{A}_1 \oplus \mathbf{A}_2$

$\rightarrow_{12}$	0	a	b	x	y	z	1
0	1	1	1	1	1	1	1
a	b	1	1	1	1	1	1
b	a	b	1	1	1	1	1
x	0	a	b	1	1	1	1
y	0	a	b	z	1	z	1
z	0	a	b	y	y	1	1
1	0	a	b	x	y	z	1

(b) The implication of $\mathbf{A}_1 \oplus \mathbf{A}_2$

$\rightarrow_{21}$	x	y	z	0	a	b	1
x	1	1	1	1	1	1	1
y	z	1	z	1	1	1	1
z	y	y	1	1	1	1	1
0	x	y	z	1	1	1	1
a	x	y	z	b	1	1	1
b	x	y	z	a	b	1	1
1	x	y	z	0	a	b	1

Clearly, $\mathbf{A}_1 \oplus \mathbf{A}_2 \neq \mathbf{A}_2 \oplus \mathbf{A}_1$.

Theorem 1. Let $\{\mathbf{A}_i\}_{i \in I}$ be a family of L -algebras and $\mathbf{A} = \bigoplus_{i \in I} \mathbf{A}_i$.

- (i) If for any $i \in I$, $\mathbf{A}_i$ is a K -algebra, then $\mathbf{A}$ is a K -algebra, too.
- (ii) If for any $i \in I$, $\mathbf{A}_i$ is a CL -algebra, then $\mathbf{A}$ is a CL -algebra, too.

(iii) If for any $i \in I$, $\mathbf{A}_i$ is a chain L -algebra, then $\mathbf{A}$ is a chain L -algebra, too.

Proof. (i) Let for any $i \in I$, $\mathbf{A}_i$ be a K -algebra. We have to show that for any $x, y \in \mathbf{A}$, $x \rightarrow (y \rightarrow x) = 1$. For this, suppose $x, y \in \mathbf{A}$. Then

$$x \rightarrow (y \rightarrow x) = \begin{cases} x \rightarrow_i (y \rightarrow_i x) = 1 & \text{if } x, y \in \mathbf{A}_i \\ x \rightarrow 1 = 1 & \text{if } y \in \mathbf{A}_i \setminus \{1\}, x \in \mathbf{A}_j, i < j \\ x \rightarrow x = 1 & \text{if } y \in \mathbf{A}_i, x \in \mathbf{A}_j, i > j \end{cases}$$

Hence, in any cases, $x \rightarrow (y \rightarrow x) = 1$, and so $\mathbf{A}$ is a K -algebra.

(ii) Let for any $i \in I$, $\mathbf{A}_i$ be a CL -algebra. Then we should prove that for any $x, y, z \in \mathbf{A}$,

$$x \rightarrow (y \rightarrow z) = y \rightarrow (x \rightarrow z).$$

If for any $i \in I$, $x, y, z \in \mathbf{A}_i$, then the proof is clear.

If $x, y \in \mathbf{A}_i \setminus \{1\}$, $z \in \mathbf{A}_j$ and $i < j$, then

$$x \rightarrow (y \rightarrow z) = x \rightarrow 1 = 1 = y \rightarrow 1 = y \rightarrow (x \rightarrow z).$$

If $x, y \in \mathbf{A}_i$, $z \in \mathbf{A}_j$ and $i > j$, then

$$x \rightarrow (y \rightarrow z) = x \rightarrow z = z = y \rightarrow z = y \rightarrow (x \rightarrow z).$$

If $x \in \mathbf{A}_i \setminus \{1\}$, $y, z \in \mathbf{A}_j$ and $i < j$, then

$$x \rightarrow (y \rightarrow z) = 1 = y \rightarrow 1 = y \rightarrow (x \rightarrow z).$$

If $x \in \mathbf{A}_i$, $y, z \in \mathbf{A}_j$ and $i > j$, then

$$x \rightarrow (y \rightarrow z) = y \rightarrow_j z = y \rightarrow (x \rightarrow z).$$

If $x, z \in \mathbf{A}_i \setminus \{1\}$, $y \in \mathbf{A}_j$ and $i < j$, then

$$x \rightarrow (y \rightarrow z) = x \rightarrow_i z = y \rightarrow (x \rightarrow z).$$

If $x, z \in \mathbf{A}_i$, $y \in \mathbf{A}_j$ and $i > j$, then

$$x \rightarrow (y \rightarrow z) = x \rightarrow 1 = 1 = y \rightarrow (x \rightarrow z).$$

Therefore, for any cases, $x \rightarrow (y \rightarrow z) = y \rightarrow (x \rightarrow z)$, and so $\mathbf{A}$ is a CL -algebra.

(iii) Let for any $i \in I$, $\mathbf{A}_i$ be a chain. We have to show that for any $x, y \in \mathbf{A}$, $x \leq y$ or $y \leq x$.

For this, for any $x, y \in \mathbf{A}$, we have.

$$x \rightarrow y = \begin{cases} x \rightarrow_i y & \text{if } x, y \in \mathbf{A}_i \\ 1 & \text{if } x \in \mathbf{A}_i \setminus \{1\}, y \in \mathbf{A}_j, i < j \\ y & \text{if } x \in \mathbf{A}_i, y \in \mathbf{A}_j, i > j \end{cases}$$

If $x, y \in \mathbf{A}_i$, since for any $i \in I$, $\mathbf{A}_i$ is a chain, we know that $x \rightarrow_i y = 1$ or $y \rightarrow_i x = 1$. So by above definition, $x \rightarrow y = x \rightarrow_i y = 1$ or $y \rightarrow x = y \rightarrow_i x = 1$. If $x \in \mathbf{A}_i \setminus \{1\}$, $y \in \mathbf{A}_j$ and $i < j$, then by above definition, $x \rightarrow y = 1$, and so $x \leq y$. Thus $\mathbf{A}$ is a chain. If $x \in \mathbf{A}_i$, $y \in \mathbf{A}_j$ and $i > j$, since by above definition, $x \rightarrow y = y$, if $y = 1$, the proof is complete. Suppose $y \neq 1$. Since $y \in \mathbf{A}_j \setminus \{1\}$, $x \in \mathbf{A}_i$, and $i > j$, by the definition of ordinal sum, $y \rightarrow x = 1$, and so $y \leq x$. Therefore, $\mathbf{A} = \bigoplus_{i \in I} \mathbf{A}_i$ is a chain L -algebra. $\square$

Proposition 8. *Let $\{\mathbf{A}_i\}_{i \in I}$ be a family of lattice L -algebras such that $\mathbf{A}_k$ is bounded, where $k = \max\{i \in I\}$. Then $\mathbf{A} = \bigoplus_{i \in I} \mathbf{A}_i$ is a lattice L -algebra.*

Proof. According to Definition 6, $\mathbf{A}$ is an L -algebra and $(\mathbf{A}, \leq)$ is a poset where $x \leq y$ if and only if $x \rightarrow y = 1$. So it is enough to show that for any $x, y \in \mathbf{A}$, $\sup\{x, y\}$ and $\inf\{x, y\}$ exist. If $x, y \in \mathbf{A}_i$, for any $i \in I$, since by assumption $\mathbf{A}_i$ is a lattice, so $\sup\{x, y\}$ and $\inf\{x, y\}$ exist. If $x \in \mathbf{A}_i$ and $y \in \mathbf{A}_j$, where $i < j$, then by Definition 6, $x \rightarrow y = 1$, and so $x \leq y$. Thus, $\sup\{x, y\} = y$ and $\inf\{x, y\} = x$. If $x \in \mathbf{A}_i$ and $y \in \mathbf{A}_j$, where $i > j$, then $y \rightarrow x = 1$, and so $\sup\{x, y\} = x$ and $\inf\{x, y\} = y$. Therefore, $\mathbf{A}$ is a lattice L -algebra. $\square$

Theorem 2. *Let $\mathbf{A}_1$ and $\mathbf{A}_2$ be two L -algebras and $\mathbf{A} = \mathbf{A}_1 \oplus \mathbf{A}_2$. Then $\mathbf{A}_2$ is an ideal of $\mathbf{A}$.*

Proof. We have to show that $\mathbf{A}_2$ satisfies identities $(I_1), \dots, (I_4)$.

(I_1) : Obviously, $1 \in \mathbf{A}_2$.

(I_2) : Suppose $x, x \rightarrow y \in \mathbf{A}_2$. Then by definition of ordinal sum, we have

$$x \rightarrow y = \begin{cases} x \rightarrow_2 y & \text{if } x, y \in \mathbf{A}_2, \\ y & \text{if } x \in \mathbf{A}_2, y \in \mathbf{A}_1 \end{cases}$$

If $y \in \mathbf{A}_2$, then the proof is complete. If $y \notin \mathbf{A}_2$, then $y \in \mathbf{A}_1$, and so $y = x \rightarrow y \in \mathbf{A}_2$, which is impossible. So, $\mathbf{A}_2$ satisfies identity (I_2) .

(I_3) : Let $x \in \mathbf{A}_2$ and $y \in \mathbf{A}$. If $y \in \mathbf{A}_1$, then $(x \rightarrow y) \rightarrow y = y \rightarrow y = 1 \in \mathbf{A}_2$. If $y \in \mathbf{A}_2$, then $(x \rightarrow y) \rightarrow y = (x \rightarrow_2 y) \rightarrow_2 y \in \mathbf{A}_2$.

(I_4) : Let $x \in \mathbf{A}_2$ and $y \in \mathbf{A}$. If $y \in \mathbf{A}_1$, then $y \rightarrow x = 1 \in \mathbf{A}_2$ and $y \rightarrow (x \rightarrow y) = y \rightarrow y = 1 \in \mathbf{A}_2$. If $y \in \mathbf{A}_2$, then clearly, $y \rightarrow x \in \mathbf{A}_2$ and $y \rightarrow (x \rightarrow y) \in \mathbf{A}_2$.

Therefore, $\mathbf{A}_2$ is an ideal of $\mathbf{A}$. $\square$

Remark 3. *By Proposition 5, $\mathbf{A}_1$ is not an ideal of $\mathbf{A}$, since $\mathbf{A}_1$ is not an upset in $\mathbf{A}$.*

Example 3. According to Example 2, $(\mathbf{A} = \mathbf{A}_1 \oplus \mathbf{A}_2, \rightarrow_{12}, 1)$ is an L -algebra, but $\mathbf{A}_1$ is not an ideal of $\mathbf{A}$, since $a \rightarrow x = 1 \in \mathbf{A}_1$ and $a \in \mathbf{A}_1$, but $x \notin \mathbf{A}_1$.

Corollary 9. *Let $\{\mathbf{A}_i\}_{i=1}^n$ be a family of L -algebras and $\mathbf{A} = \mathbf{A}_1 \oplus \mathbf{A}_2 \oplus \dots \oplus \mathbf{A}_n$. Then $\mathbf{A}_n$ is an ideal of $\mathbf{A}$.*

Proof. The proof is similar to the proof of Theorem 2. $\square$

Theorem 3. *Let $\mathbf{A} = \mathbf{A}_1 \oplus \mathbf{A}_2$ and $I \in \mathcal{I}(\mathbf{A}_2)$. Then $I \in \mathcal{I}(\mathbf{A})$.*

Proof. (I_1): Clearly $1 \in I$.

(I_2): Let $x, y \in \mathbf{A}$ and $x, x \rightarrow y \in I$. Since $I \subseteq \mathbf{A}_2$, we get $x \in \mathbf{A}_2$.

If $y \in \mathbf{A}_2$, then $x \rightarrow y = x \rightarrow_2 y \in \mathbf{A}_2$ and since $I \in \mathcal{I}(\mathbf{A}_2)$, we have $y \in I$. If $y \in \mathbf{A}_1$, then $x \rightarrow y = y \in I \subseteq \mathbf{A}_2$, which is impossible. Thus, $y = 1 \in I$. Therefore, I satisfies (I_2).

(I_3): Let $x \in I \subseteq \mathbf{A}_2$. Then we have two cases. If $y \in \mathbf{A}_1$, then $(x \rightarrow y) \rightarrow y = y \rightarrow y = 1 \in I$. If $y \in \mathbf{A}_2$, then $(x \rightarrow y) \rightarrow y = (x \rightarrow_2 y) \rightarrow_2 y$ and since $I \in \mathcal{I}(\mathbf{A}_2)$, we get $(x \rightarrow y) \rightarrow y \in I$. Thus, (I_3) holds.

(I_4): Let $x \in I$. Then $x \in \mathbf{A}_2$. If $y \in \mathbf{A}_1$, then $y \rightarrow (x \rightarrow y) = y \rightarrow y = 1 \in I$. If $y \in \mathbf{A}_2$, then $y \rightarrow (x \rightarrow y) = y \rightarrow_2 (x \rightarrow_2 y) \in I$, since $I \in \mathcal{I}(\mathbf{A}_2)$. Hence, (I_4) holds.

The above discussion shows that $I \in \mathcal{I}(\mathbf{A})$. $\square$

Example 4. According to Example 2, $(\mathbf{A}_1 \oplus \mathbf{A}_2, \rightarrow_{12}, 1)$ is an L -algebra and $I = \{b, x, y, z, 1\}$ is an ideal of $\mathbf{A}$, but is not an ideal of $\mathbf{A}_2$, since $b \notin \mathbf{A}_2$.

Corollary 10. Let $\{\mathbf{A}_i\}_{i=1}^n$ be a family of L -algebras and $\mathbf{A} = \mathbf{A}_1 \oplus \mathbf{A}_2 \oplus \cdots \oplus \mathbf{A}_n$. If $I \in \mathcal{I}(\mathbf{A}_n)$, then $I \in \mathcal{I}(\mathbf{A})$.

Proposition 11. Let $\mathbf{A}_1$ and $\mathbf{A}_2$ be two L -algebras and $\mathbf{A} = \mathbf{A}_1 \oplus \mathbf{A}_2$. Then the following statements holds:

- (i) If $I \in \mathcal{I}(\mathbf{A})$, then $I_i = I \cap \mathbf{A}_i \in \mathcal{I}(\mathbf{A}_i)$, for $i = 1, 2$.
- (ii) If $I_1 \in \mathcal{I}(\mathbf{A}_1)$, then $I = I_1 \cup \mathbf{A}_2 \in \mathcal{I}(\mathbf{A})$.

Proof. (i) Let $I \in \mathcal{I}(\mathbf{A})$ and $I_1 = I \cap \mathbf{A}_1$. Since $I \in \mathcal{I}(\mathbf{A})$, we have $1 \in I \cap \mathbf{A}_1 = I_1$. So, identity (I_1) holds. Now, suppose $x, x \rightarrow y \in I_1$, for $x, y \in \mathbf{A}_1$. Thus $x, x \rightarrow y \in \mathbf{A}_1$ and $x, x \rightarrow y \in I$. Since $I \in \mathcal{I}(\mathbf{A})$, we get $y \in I$, and so $y \in I \cap \mathbf{A}_1 = I_1$. To investigate identities (I_3), suppose $x \in I_1$. Then for any $y \in \mathbf{A}_1$, since $x \in I$ and $I \in \mathcal{I}(\mathbf{A})$, we get $(x \rightarrow y) \rightarrow y \in I$. On the other hand, since $x, y \in \mathbf{A}_1$, we have

$$(x \rightarrow y) \rightarrow y = (x \rightarrow_1 y) \rightarrow_1 y \in \mathbf{A}_1.$$

Thus, $(x \rightarrow y) \rightarrow y \in I \cap \mathbf{A}_1 = I_1$. So, (I_3) holds. Finally, let $x \in I \cap \mathbf{A}_1$ and $y \in \mathbf{A}_1$. Then $y \rightarrow x = y \rightarrow_1 x \in \mathbf{A}_1$. Moreover, since $x \in I$, $I \in \mathcal{I}(\mathbf{A})$, and $y \in \mathbf{A}_1 \subseteq \mathbf{A}$, we get $y \rightarrow x \in I$. Thus, $y \rightarrow x \in \mathbf{A}_1 \cap I = I_1$. On the other hand, for every $x, y \in \mathbf{A}_1$, $y \rightarrow (x \rightarrow y) \in \mathbf{A}_1$. Since $x \in I_1 = I \cap \mathbf{A}_1$, then $x \in I$. From $I \in \mathcal{I}(\mathbf{A})$, we have $y \rightarrow (x \rightarrow y) \in I$. Hence,

$$y \rightarrow (x \rightarrow y) \in \mathbf{A}_1 \cap I = I_1.$$

Therefore, $I_1 = I \cap \mathbf{A}_1 \in \mathcal{I}(\mathbf{A}_1)$. The proof of the other case is similar.

(ii) Consider $I_1 \in \mathcal{I}(\mathbf{A}_1)$. We show that $I = I_1 \cup \mathbf{A}_2 \in \mathcal{I}(\mathbf{A})$. Since $1 \in I_1$, we get $1 \in I$. Now, let $x, y \in \mathbf{A}$ and $x, x \rightarrow y \in I$. We have the following four cases:

Case 1: Let $x, x \rightarrow y \in I_1$. Clearly, $x \in \mathbf{A}_1$. If $y \in \mathbf{A}_1$, then $x \rightarrow y = x \rightarrow_1 y \in I_1$. Since $I_1 \in \mathcal{I}(\mathbf{A}_1)$, we get $y \in I_1 \subseteq I$. If $y \in \mathbf{A}_2$ and $y \in I_1$, then $\mathbf{A}_1 \cap \mathbf{A}_2 \neq \{1\}$, which is impossible. In addition, if $y \in \mathbf{A}_2$, then $y \in I$.

Case 2: Let $x, x \rightarrow y \in \mathbf{A}_2$. If $y \in \mathbf{A}_2$, then $y \in I$. If $y \in \mathbf{A}_1$, then $x \rightarrow y = y \in \mathbf{A}_2$. So,

$\mathbf{A}_1 \cap \mathbf{A}_2 \neq \{1\}$, which is impossible.

Case 3: Let $x \in I_1$ and $x \rightarrow y \in \mathbf{A}_2$. Then $x \in I_1 \subseteq \mathbf{A}_1$. If $y \in \mathbf{A}_1$, then $x \rightarrow y = x \rightarrow_1 y \in \mathbf{A}_1$, that is impossible by assumption that $x \rightarrow y \in \mathbf{A}_2$. If $y \in \mathbf{A}_2$, then $y \in I_1 \cup \mathbf{A}_2 = I$, and so $y \in I$.

Case 4: Let $x \in \mathbf{A}_2$ and $x \rightarrow y \in I_1$. If $y \in \mathbf{A}_1$, then $x \rightarrow y = y \in I_1$, and so $y \in I$. If $y \in \mathbf{A}_2$, then $x \rightarrow y \in \mathbf{A}_2$, which is impossible by assumption that $x \rightarrow y \in I_1 \subseteq \mathbf{A}_1$. So, identity (I_2) is met.

Now, for proving identities (I_3) and (I_4) , suppose $x \in I$. If $x \in I_1 \subseteq \mathbf{A}_1$ and $y \in \mathbf{A}_1$, since $I_1 \in \mathcal{I}(\mathbf{A}_1)$, then we have $(x \rightarrow y) \rightarrow y \in I_1 \subseteq I$. Also, we get $y \rightarrow x = y \rightarrow_1 x \in I_1$, so $y \rightarrow x \in I$. Similarly, $y \rightarrow (x \rightarrow y) = y \rightarrow_1 (x \rightarrow_1 y) \in I_1$. If $x \in I_1 \subseteq \mathbf{A}_1$ and $y \in \mathbf{A}_2$, then

$$(x \rightarrow y) \rightarrow y = 1 \rightarrow y = y \in \mathbf{A}_2 \subseteq I.$$

Also, $y \rightarrow x = x \in I_1$ and so $y \rightarrow x \in I$. Furthermore, $y \rightarrow (x \rightarrow y) = y \rightarrow 1 = 1 \in I_1$ and ultimately belongs to I . In addition, if $x \in \mathbf{A}_2$ and $y \in \mathbf{A}_1$, then $(x \rightarrow y) \rightarrow y = y \rightarrow y = 1 \in I_1 \subseteq I$ and $y \rightarrow x = 1 \in I_1$, and $y \rightarrow (x \rightarrow y) = y \rightarrow y = 1 \in I_1$. Finally, if $x, y \in \mathbf{A}_2$, then $(x \rightarrow y) \rightarrow y = (x \rightarrow_2 y) \rightarrow_2 y \in \mathbf{A}_2 \subseteq I$ and $y \rightarrow x = y \rightarrow_2 x \in \mathbf{A}_2$, and

$$y \rightarrow (x \rightarrow y) = y \rightarrow_2 (x \rightarrow_2 y) \in \mathbf{A}_2.$$

Thus, all cases show that I satisfies identities (I_3) and (I_4) and the proof is completed. $\square$

Theorem 4. Let $\{\mathbf{A}_i\}_{i=1}^n$ be a family of L -algebras and $\mathbf{A} = \mathbf{A}_1 \oplus \mathbf{A}_2 \oplus \cdots \oplus \mathbf{A}_n$.

(i) If I_j is an ideal of $\mathbf{A}_j$, for some $j \in \{1, 2, \dots, n\}$, then $I = I_j \cup (\cup_{k>j} \mathbf{A}_k)$ is an ideal of $\mathbf{A}$.

(ii) If I is a non-trivial ideal of $\mathbf{A}$ and

$$k_j = \min\{i \in \{1, 2, \dots, n\} \mid (I \setminus \{1\}) \cap \mathbf{A}_i \neq \emptyset\},$$

then $I_{k_j} = I \cap \mathbf{A}_{k_j}$ is an ideal of $\mathbf{A}_{k_j}$.

Proof. (i) Similar to the Proposition 11, for $x, x \rightarrow y \in I$, if $x, x \rightarrow y \in I_j$, since $I_j \in \mathcal{I}(\mathbf{A}_j)$, clearly $y \in I_j$. In other cases, since I_j is an upset, by Corollary 9, $\cup_{k>j} \mathbf{A}_k$ is an ideal of $\oplus \mathbf{A}_i$, we get that $y \in \bigcup_{k>j} \mathbf{A}_k \subseteq I$. Hence, I is an ideal of $\oplus \mathbf{A}_i$.

(ii) Since $i : \mathbf{A}_i \rightarrow \oplus \mathbf{A}_i$ is an embedding, it is straightforward that if I is an ideal of $\oplus \mathbf{A}_i$, then $i^{-1}(I) = I \cap \mathbf{A}_i$ is an ideal of $\mathbf{A}_i$. $\square$

Proposition 12. Let $\mathbf{A}_1$ and $\mathbf{A}_2$ be two L -algebras and $\mathbf{A} = \mathbf{A}_1 \oplus \mathbf{A}_2$. Then the following statements hold:

(i) If $P \in \text{Prime}(\mathbf{A})$ and $P_1 = P \cap \mathbf{A}_1$, then $P_1 \in \text{Prime}(\mathbf{A}_1)$.

(ii) If $P_1 \in \text{Prime}(\mathbf{A}_1)$, then $P = P_1 \cup \mathbf{A}_2 \in \text{Prime}(\mathbf{A})$.

Proof. (i) By the Proposition 11(i), if P is an ideal of $\mathbf{A}$, then $P_1 = P \cap \mathbf{A}_1$ is an ideal of $\mathbf{A}_1$. P_1 is proper, otherwise if $P_1 = \mathbf{A}_1$, then we have $P_1 = P \cap \mathbf{A}_1 = \mathbf{A}_1$, so $\mathbf{A}_1 \subseteq P$. On the other hand, $P \in \mathcal{I}(\mathbf{A})$, so by Proposition 5, it is an upset, thus $\mathbf{A}_2 \subseteq P$. Therefore, $P = \mathbf{A}_1 \cup \mathbf{A}_2 = \mathbf{A}$, which is a contradiction. Now, let $I_1 \in \mathcal{I}(\mathbf{A}_1)$ such that $I_1 \rightarrow P_1 \not\subseteq P_1$. We prove $I_1 \subseteq P_1$. Suppose $I_1 \not\subseteq P_1$. Then there exists $x \in I_1$ such that $x \notin P_1 = P \cap \mathbf{A}_1$. So, $x \notin \mathbf{A}_1$, which is a contradiction. Thus $I_1 \subseteq P_1$, and so $P_1 \in \text{Prime}(\mathbf{A}_1)$.

(ii) By Proposition 11(ii), if $P_1 \in \text{Prime}(\mathbf{A}_1)$, then $P = P_1 \cup \mathbf{A}_2$ is a proper ideal of $\mathbf{A}$. Let $I \in \mathcal{I}(\mathbf{A})$, arbitrary. We have to prove $I \subseteq P$ or $I \rightarrow P \subseteq P$. By Proposition 11(i), we know $I_1 = I \cap \mathbf{A}_1 \in \mathcal{I}(\mathbf{A}_1)$. Since $P_1 \in \text{Prime}(\mathbf{A}_1)$, $I_1 \subseteq P_1$ or $I_1 \rightarrow P_1 \subseteq P_1$. If $I_1 \subseteq P_1$, then

$$I \subseteq I \cup \mathbf{A}_2 = (I \cup \mathbf{A}_2) \cap (\mathbf{A}_1 \cup \mathbf{A}_2) = (I \cap \mathbf{A}_1) \cup \mathbf{A}_2 = I_1 \cup \mathbf{A}_2 \subseteq P_1 \cup \mathbf{A}_2 = P$$

Hence, $P \in \text{Prime}(\mathbf{A})$.

If $I_1 \rightarrow P_1 \subseteq P_1$, then we have:

$$I \rightarrow P = \{x \in \mathbf{A} \mid \langle x \rangle \cap I \subseteq P\} = \{x \in \mathbf{A}_1 \mid \langle x \rangle \cap I \subseteq P\} \cup \{x \in \mathbf{A}_2 \mid \langle x \rangle \cap I \subseteq P\}.$$

Suppose $\mathbf{U} := \{x \in \mathbf{A}_1 \mid \langle x \rangle \cap I \subseteq P\}$ and $\mathbf{V} := \{x \in \mathbf{A}_2 \mid \langle x \rangle \cap I \subseteq P\}$. Now, we prove that $\mathbf{U} \subseteq P$ and $\mathbf{V} \subseteq P$. According to definition of $\mathbf{V}$, clearly $\mathbf{V} \subseteq \mathbf{A}_2$ and since $\mathbf{A}_2 \subseteq P$, we get $\mathbf{V} \subseteq P$. Also, by assumption,

$$\{x \in \mathbf{A}_1 \mid \langle x \rangle \cap I_1 \subseteq P_1\} = I_1 \rightarrow P_1 \subseteq P_1.$$

Since $P_1 \subseteq P$, we get $\{x \in \mathbf{A}_1 \mid \langle x \rangle \cap I_1 \subseteq P\} \subseteq P$. In addition, $I_1 = I \cap \mathbf{A}_1$, so we have $\{x \in \mathbf{A}_1 \mid \langle x \rangle \cap I \cap \mathbf{A}_1 \subseteq P\} \subseteq P$. From $x \in \mathbf{A}_1$, we get $\{x \in \mathbf{A}_1 \mid \langle x \rangle \cap I \subseteq P\} \subseteq P$. Hence, $\mathbf{U} \subseteq P$, and so $\mathbf{U} \cup \mathbf{V} \subseteq P$. Therefore, $I \rightarrow P \subseteq P$. $\square$

Example 5. According to Examples 1 and 2, $(\mathbf{A}_1 \oplus \mathbf{A}_2, \rightarrow_{12}, 1)$ is an L -algebra and $P = \{b, x, y, z, 1\} \in \text{Prime}(\mathbf{A})$. Clearly, $P_2 = P \cap \mathbf{A}_2 = \mathbf{A}_2 \notin \text{Prime}(\mathbf{A}_2)$, since it is not proper. In addition, $Q = \{z, 1\} \in \text{Prime}(\mathbf{A}_2)$. By Proposition 5, $Q \cup \mathbf{A}_1 \notin \mathcal{I}(\mathbf{A})$.

Note: In the following, we have some results about L -algebras and maximal ideals of them. Before it, we have to notice that every L -algebra does not have a maximal ideal, in general. In the following, we give an example of unbounded L -algebra which has not any maximal ideal.

Example 6. Let $\mathbf{A} = (0, 1]$. Then, $(\mathbf{A}, \rightarrow, 1)$ is an L -algebra where

$$x \rightarrow y = \begin{cases} 1 & \text{if } x \leq y \\ y & \text{if } x > y \end{cases}$$

Easy calculation shows that $\mathcal{I}(\mathbf{A}) = \{I_a\}_{a \leq 1}$, where $I_a = (a, 1]$. We claim that $\mathbf{A}$ has no maximal ideal. For any $a \in \mathbf{A} \setminus \{1\}$, there exists $b \in \mathbf{A}$, such that $0 < b \leq a$, in which $(a, 1] \subseteq (b, 1]$. Thus, $\mathbf{A}$ does not have any maximal ideal.

Proposition 13. Let $\mathbf{A} = \mathbf{A}_1 \oplus \mathbf{A}_2$ and $\text{Max}(\mathbf{A}) \neq \emptyset$. Then, we have the following statements:

- (i) If $M \in \text{Max}(\mathbf{A})$ and $M_i = M \cap \mathbf{A}_i$, then $M_i \in \text{Max}(\mathbf{A}_i)$, for $i = 1, 2$.

(ii) If $M_1 \in \text{Max}(\mathbf{A}_1)$, then $M = M_1 \cup \mathbf{A}_2 \in \text{Max}(\mathbf{A})$.

Proof. (i) Since $i : \mathbf{A}_1 \rightarrow \mathbf{A}_1 \oplus \mathbf{A}_2$ is an embedding and $i^{-1}(M) = M \cap \mathbf{A}_1$, we get $M_1 \in \text{Max}(\mathbf{A}_1)$. The proof of other case is similar.

(ii) Let $M_1 \in \text{Max}(\mathbf{A}_1)$. Take $M := M_1 \cup \mathbf{A}_2$. Then by Proposition 11(ii), M is a proper ideal of $\mathbf{A}$. Now, we want to show that $M \in \text{Max}(\mathbf{A})$. For this, suppose that $M \subseteq N \subsetneq \mathbf{A}$. Then, $M_1 = M \cap \mathbf{A}_1 \subseteq N \cap \mathbf{A}_1 \subsetneq \mathbf{A} \cap \mathbf{A}_1 = \mathbf{A}_1$. Since $M_1 \in \text{Max}(\mathbf{A}_1)$, we get $M_1 = N \cap \mathbf{A}_1$. Thus,

$$M = M_1 \cup \mathbf{A}_2 = (N \cap \mathbf{A}_2) \cup \mathbf{A}_2 = (N \cup \mathbf{A}_2) \cap (\mathbf{A}_1 \cup \mathbf{A}_2) = N \cup \mathbf{A}_2.$$

Hence, $M = N \cup \mathbf{A}_2$. By assumption, $M \subseteq N$. Since $M = N \cup \mathbf{A}_2$, we get $N \cup \mathbf{A}_2 \subseteq N \subseteq N \cup \mathbf{A}_2$. So, $N = N \cup \mathbf{A}_2 = M$. Thus, $M \in \text{Max}(\mathbf{A})$. $\square$

Example 7. According to Examples 1 and 2, $(\mathbf{A}_1 \oplus \mathbf{A}_2, \rightarrow_{12}, 1)$ is an L -algebra and the set $M_2 = \{z, 1\} \in \text{Max}(\mathbf{A}_2)$, but $M_2 \cup \mathbf{A}_1 \notin \mathcal{I}(\mathbf{A})$, since it is not an upset. In addition, $M_2 \cup \mathbf{A}_2 = \mathbf{A}_2 \in \mathcal{I}(\mathbf{A})$ but $\mathbf{A}_2 \notin \text{Max}(\mathbf{A})$ since $N = \{b, x, y, z, 1\} \in \mathcal{I}(\mathbf{A})$ and $\mathbf{A}_2 \subseteq N$.

Theorem 5. Let $\mathbf{A} = \mathbf{A}_1 \oplus \mathbf{A}_2$, $\text{Max}(\mathbf{A}) \neq \emptyset$ and $\mathbf{A}_1$ be simple. Then $M \in \text{Max}(\mathbf{A})$ if and only if $M = \mathbf{A}_2$.

Proof. Let $M \in \text{Max}(\mathbf{A})$. By Proposition 13(i), $M \cap \mathbf{A}_1 \in \text{Max}(\mathbf{A}_1)$. Since $\mathbf{A}_1$ is simple, we get $M \cap \mathbf{A}_1 = \{1\}$. Also, $\mathbf{A}_1 \cap \mathbf{A}_2 = \{1\}$, thus $M \subseteq \mathbf{A}_2$.

On the other hand, let $x \in \mathbf{A}_2 \setminus M$, we get $M \subsetneq \langle M \cup \{x\} \rangle \subseteq \mathbf{A}$.

From $M \in \text{Max}(\mathbf{A})$, we have $\langle M \cup \{x\} \rangle = \mathbf{A}$. Since $M \subseteq \mathbf{A}_2$ and $x \in \mathbf{A}_2$, we obtain $\mathbf{A}_1 \cup \mathbf{A}_2 = \mathbf{A} = \langle M \cup \{x\} \rangle \subseteq \mathbf{A}_2$, thus $\mathbf{A}_1 \subseteq \mathbf{A}_2$, which is a contradiction. Hence, there exists no $x \in \mathbf{A}_2 \setminus M$. So, $\mathbf{A}_2 = M$.

Conversely, consider $M = \mathbf{A}_2$. Then we show that $\mathbf{A}_2 \in \text{Max}(\mathbf{A})$. Suppose that $N \in \mathcal{I}(\mathbf{A})$ such that $\mathbf{A}_2 \subseteq N \subseteq \mathbf{A}$. Then by Proposition 11(i), $N \cap \mathbf{A}_1 \in \mathcal{I}(\mathbf{A}_1)$. By assumption $\mathbf{A}_1$ is simple, so $N \cap \mathbf{A}_1 = \{1\}$ or $N \cap \mathbf{A}_1 = \mathbf{A}_1$.

If $N \cap \mathbf{A}_1 = \{1\}$, since $\mathbf{A}_1 \cap \mathbf{A}_2 = \{1\}$, we get $N \subseteq \mathbf{A}_2$. Also, we know $\mathbf{A}_2 \subseteq N$. Hence, $N = \mathbf{A}_2$.

If $N \cap \mathbf{A}_1 = \mathbf{A}_1$, then $\mathbf{A}_1 \subseteq N$. Also, we know that $\mathbf{A}_2 \subseteq N$. Thus, $\mathbf{A} = \mathbf{A}_1 \cup \mathbf{A}_2 \subseteq N \subseteq \mathbf{A}$, and so $N = \mathbf{A}$. Therefore, $\mathbf{A}_2 \in \text{Max}(\mathbf{A})$. $\square$

Set $\text{Rad}(\mathbf{A}) = \bigcap_{M \in \text{Max}(\mathbf{A})} M$. If $\mathbf{A}$ does not have any maximal ideal, we set $\text{Rad}(\mathbf{A}) = \mathbf{A}$.

Corollary 14. Let $\mathbf{A} = \mathbf{A}_1 \oplus \mathbf{A}_2$ such that $\text{Max}(\mathbf{A}) \neq \emptyset$. If $\mathbf{A}_1$ is simple, then

(i) $\text{Rad}(\mathbf{A}) = \mathbf{A}_2$.

(ii) $\mathbf{A}$ is local.

Proof. By Theorem 5, the proof is clear. $\square$

Theorem 6. [8] Let $\mathbf{A}$ be a bounded CL -algebra and $\text{Rad}(\mathbf{A})' = \{x | x' \in \text{Rad}(\mathbf{A})\}$. Then the following statements are equivalent:

(i) $\mathbf{A} = \text{Rad}(\mathbf{A}) \cup \text{Rad}(\mathbf{A})'$.

(ii) $\mathbf{A}$ is perfect.

Theorem 7. *Let $\mathbf{A} = \mathbf{A}_1 \oplus \mathbf{A}_2$ be a bounded CL-algebra and $\text{Max}(\mathbf{A}) \neq \emptyset$. Then $\mathbf{A}$ is perfect and $\mathbf{A}_1$ is simple if and only if $\mathbf{A}_1 = \{0, 1\}$.*

Proof. Let $\mathbf{A}$ be perfect and $\mathbf{A}_1$ be simple. Then by Corollary 14(i), $\text{Rad}(\mathbf{A}) = \mathbf{A}_2$. Also, by Theorem 6, $\mathbf{A} = \text{Rad}(\mathbf{A}) \cup \text{Rad}(\mathbf{A})'$, thus $\mathbf{A} = \mathbf{A}_2 \cup \text{Rad}(\mathbf{A})'$. Since $\text{Rad}(\mathbf{A}) \cap \text{Rad}(\mathbf{A})' = \emptyset$, we get $\text{Rad}(\mathbf{A})' = \mathbf{A}_1 \setminus \{1\}$. Now, we prove $\text{Rad}(\mathbf{A})' = \{0\}$. Since $1' = 0$, we have $\{0\} \subseteq \text{Rad}(\mathbf{A})'$. If $x \in \text{Rad}(\mathbf{A})'$, then there exists $y \in \text{Rad}(\mathbf{A})$ such that $x = y'$. By definition of the operation $\rightarrow$, we have $x = y' = y \rightarrow 0 = 0$. Hence, $x = 0$, and so $\text{Rad}(\mathbf{A})' \subseteq \{0\}$. Therefore, $\text{Rad}(\mathbf{A})' = \{0\}$ and so $\mathbf{A}_1 = \{0, 1\}$.

Conversely, let $\mathbf{A}_1 = \{0, 1\}$. Clearly, $\mathbf{A}_1$ is simple, $\mathbf{A} = \{0, 1\} \cup \mathbf{A}_2$ and $\text{Rad}(\mathbf{A}) \cup \text{Rad}(\mathbf{A})' \subseteq \mathbf{A}$. If $x \in \mathbf{A}$, then $x = 0$ or $x \in \mathbf{A}_2$. If $x = 0$, then $0' = 1 \in \text{Rad}(\mathbf{A})$, and so $0 \in \text{Rad}(\mathbf{A})'$. Also, since $\mathbf{A}_1$ is simple, by Corollary 14(i), $x \in \mathbf{A}_2 = \text{Rad}(\mathbf{A})$. Thus, $\mathbf{A} \subseteq \text{Rad}(\mathbf{A}) \cup \text{Rad}(\mathbf{A})'$. Hence, $\mathbf{A} = \text{Rad}(\mathbf{A}) \cup \text{Rad}(\mathbf{A})'$. Therefore, by Theorem 6, $\mathbf{A}$ is perfect. $\square$

Definition 15. [4] *An algebra $\mathbf{A}$ is **sub-directly irreducible** if for every sub-direct embedding $\alpha : \mathbf{A} \rightarrow \prod_{i \in I} \mathbf{A}_i$ there is an $i \in I$ such that $\pi_i \circ \alpha : \mathbf{A} \rightarrow \mathbf{A}_i$ is an isomorphism.*

Theorem 8. [4] *An algebra $\mathbf{A}$ is sub-directly irreducible iff $\mathbf{A}$ is trivial or there is a minimum congruence in $\text{Con} \mathbf{A} - \{\Delta\}$.*

Proposition 16. *Let $\mathbf{A} = \mathbf{A}_1 \oplus \mathbf{A}_2$ and $\mathbf{A}_1$ be simple. Then*

(i) $\mathcal{I}(\mathbf{A}) \setminus \{\mathbf{A}\} = \mathcal{I}(\mathbf{A}_2)$.

(ii) $\mathbf{A}$ is sub-directly irreducible if and only if $\mathbf{A}_2$ is sub-directly irreducible.

Proof. (i) By Proposition 4(i), $\mathcal{I}(\mathbf{A}_2) \subseteq \mathcal{I}(\mathbf{A})$. For the other side of inclusion, let $I \in \mathcal{I}(\mathbf{A}) \setminus \{\mathbf{A}\}$, then we have two cases:

Case 1: If $I \cap (\mathbf{A}_1 \setminus \{1\}) = \emptyset$, then $I \subseteq \mathbf{A}_2$ and so $I \in \mathcal{I}(\mathbf{A}_2)$.

Case 2: If $I \cap (\mathbf{A}_1 \setminus \{1\}) \neq \emptyset$, then by Proposition 11(i), $I \cap \mathbf{A}_1 \in \mathcal{I}(\mathbf{A}_1)$, since $\mathbf{A}_1$ is simple, we get $I \cap \mathbf{A}_1 = \{1\}$ or $I \cap \mathbf{A}_1 = \mathbf{A}_1$. If $I \cap \mathbf{A}_1 = \mathbf{A}_1$, then $\mathbf{A}_1 \subseteq I$. Since I is an upset, we get $\mathbf{A}_2 \subseteq I$ and so $\mathbf{A} = \mathbf{A}_1 \cup \mathbf{A}_2 \subseteq I$, i.e. $I = \mathbf{A}$, which is a contradiction by assumption. Hence, $I \cap \mathbf{A}_1 = \{1\}$ and so $I \subseteq \mathbf{A}_2$. Therefore, $I \in \mathcal{I}(\mathbf{A}_2)$.

(ii) Since every ideal gives rise to a congruence relation, by (i) it is clear that if two L -algebras have the same ideal, then they are equivalent in terms of sub-direct irreducibility. $\square$

An L -algebra $\mathbf{A}$ such that $\text{Max}(\mathbf{A}) \neq \emptyset$ is called **semi-simple** if $\text{Rad}(\mathbf{A}) = \{1\}$. By Corollary 14, if $\mathbf{A}_1$ is simple, then $\mathbf{A}$ is not semi-simple.

Theorem 9. *Let $\mathbf{A} = \mathbf{A}_1 \oplus \mathbf{A}_2$ such that $\text{Max}(\mathbf{A}) \neq \emptyset$. If $\mathbf{A}$ is semi-simple, then $\mathbf{A}_i$ is semi-simple, for $i = 1, 2$.*

Proof. Let $a \in \text{Rad}(\mathbf{A}_1)$. Then for any $M_1 \in \text{Max}(\mathbf{A}_1)$, $a \in M_1$. Also, if $M \in \text{Max}(\mathbf{A})$, then by Proposition 13(i), $M \cap \mathbf{A}_1 \in \text{Max}(\mathbf{A}_1)$. Hence, $a \in M \cap \mathbf{A}_1$ and so $a \in M$. Thus, for any $M \in \text{Max}(\mathbf{A})$, $a \in M$, and so $a \in \text{Rad}(\mathbf{A})$. Hence $\text{Rad}(\mathbf{A}_1) \subseteq \text{Rad}(\mathbf{A})$. Since $\mathbf{A}$ is semi-simple, we get $\text{Rad}(\mathbf{A}) = \{1\}$, and so $\{1\} \subseteq \text{Rad}(\mathbf{A}_1) \subseteq \{1\}$. Hence, $\text{Rad}(\mathbf{A}_1) = \{1\}$. Therefore, $\mathbf{A}_1$ is semi-simple. $\square$

Note: We have to notice that if $\mathbf{A}_1$ is semi-simple, $\mathbf{A}$ may not be semi-simple, where $\text{Max}(\mathbf{A}) \neq \emptyset$. Since for any $M \in \text{Max}(\mathbf{A}_1)$, by Proposition 13(ii), $M \cup \mathbf{A}_2 \in \text{Max}(\mathbf{A})$, and so the set of all maximal ideals of $\mathbf{A}_1$ is equivalent to the set of all maximal ideals of $\mathbf{A}$, i.e. $\text{Max}(\mathbf{A}_1) \subseteq \text{Max}(\mathbf{A})$. Thus, $\text{Rad}(\mathbf{A}) \subseteq \text{Rad}(\mathbf{A}_1) \cup \mathbf{A}_2 = \{1\} \cup \mathbf{A}_2 = \mathbf{A}_2$. Hence, $\text{Rad}(\mathbf{A}) \neq \{1\}$, in general.

4 Conclusion and future works

In this paper, we have undertaken a comprehensive study of the ordinal sum construction within the framework of L -algebras. We began by formally introducing the ordinal sum of a family of L -algebras and established its basic structural properties. A key focus of our investigation was the transfer of algebraic identities and properties between component algebras and their sum. We provided characterizations for the ordinal sum to belong to important sub-classes of L -algebras, specifically K -algebras, CL -algebras, and lattices, thereby clarifying how these varieties behave under the ordinal sum operation. Moving beyond the algebra itself, we introduced the concept of an ideal ordinal sum, extending the construction to the level of ideals. This allowed for a deeper structural analysis, leading to a complete description of the ideals of an ordinal sum. In particular, we characterized the prime and maximal ideals of the ordinal sum in terms of the ideals of its components. At the global level, we investigated the conditions under which an ordinal sum inherits or exhibits significant algebraic properties. We established necessary and sufficient conditions for the ordinal sum to be sub-directly irreducible, a result that contributes to the understanding of the equational theory and the classification of sub-directly irreducible members of the variety. Furthermore, we explored the criteria for semi-simplicity, simplicity, locality, and perfectness. The results presented in this work lay a foundation for the study of complex L -algebras built from simpler components and highlight the ordinal sum as a fundamental tool for constructing new examples with tailored properties. Several avenues for future research remain open. It would be natural to investigate the ordinal sum in the context of state L -algebras or lattice-ordered L -algebras. Furthermore, exploring the categorical aspects of the ordinal sum—for instance, determining whether it can be viewed as a coproduct or a pushout in a suitable category—could yield deeper insights. Finally, the characterization of prime ideals established here invites a study of the Zariski topology on the prime spectrum of an ordinal sum, potentially connecting this work to sheaf representations of L -algebras.

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