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Well-posedness and constructive approximation for a nonlinear boundary-value problem with conformable fractional derivative

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Abstract. In this work, we present an approximate method for solving a type of boundary value problem involving a fractional differential equation with conformable derivatives. This method will be based on Bernstein polynomials. First, the existence and uniqueness of the solution are established, and then the approximate algorithm is presented. After proving the method's convergence, two examples are presented to illustrate the application of the method.

Keywords: Boundary value problem, approximate solution, conformable fractional derivative, fixed point theorem
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1 Introduction

Fractional calculus is a fascinating branch of mathematics that takes the concepts of differentiation and integration and extends them to non-integer orders. What began as a theoretical curiosity has now become a powerful tool for modeling complex phenomena in various fields of science and engineering [9, 23, 25, 28–30]. Unlike traditional models that rely on integer orders, fractional-order derivatives have the unique ability to capture memory effects and hereditary properties. This makes them particularly useful for describing processes that exhibit long-term dependence, such as anomalous diffusion, viscoelasticity, and the complex dynamics found in biological, physical, and financial systems. Applications are wide-ranging, from modeling the electrical properties of neuronal cell membranes and the behavior of polymers to simulating groundwater flow in diverse environments and analyzing control systems that incorporate memory. This versatility suggests that fractional models are often more consistent

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with experimental data. As a result, a wave of theoretical and computational research focused on understanding and solving fractional differential equations (FDEs) has emerged, highlighting their growing importance in the scientific community [4, 5, 20, 27].

When it comes to fractional derivatives, there are several definitions–such as Riemann-Liouville, Caputo, and Grünwald-Letnikov. However, the conformable fractional derivative (CFD), introduced by Khalil and his colleagues, has really caught the attention of many researchers. This is mainly due to its straightforward structure and its good correspondence with many classical rules of calculus, such as the product, exponent, and chain rules. The conformable fractional derivative provides a solid analytical framework and makes it easier to build on standard techniques for analyzing existence, uniqueness, and qualitative aspects.

Despite the theoretical advantages of the conformable derivative, developing efficient numerical methods for conformable fractional boundary value problems remains challenging. Several numerical approaches have been successfully applied to Caputo and Riemann-Liouville fractional differential equations:

Collocation methods based on orthogonal polynomials (Chebyshev, Legendre, and Jacobi polynomials) have been widely used for fractional BVPs [7, 10, 32]. These methods approximate the solution by enforcing the differential equation at selected collocation points. However, when adapted to conformable derivatives, these methods face difficulties due to the non-standard kernel structure in the integral representation, which differs significantly from the classical fractional integral kernels.

Operational matrix methods using wavelets (Haar, Legendre, and Chebyshev wavelets) and block-pulse functions have proven effective for Caputo-type equations [17, 19, 22, 31, 34]. These methods convert fractional differential equations into algebraic systems through operational matrices of fractional integration. The challenge for conformable derivatives lies in constructing appropriate operational matrices that respect the unique structure of the conformable integral operator.

Spectral methods employing global basis functions offer high accuracy for smooth solutions [8, 33, 36]. While powerful, these methods require global smoothness of the solution and can struggle with problems exhibiting local singularities or boundary layers–phenomena commonly encountered in fractional differential equations. Additionally, spectral methods typically require solving large, dense linear systems, which can be computationally expensive.

Fixed-point iterative methods combined with polynomial approximations have been applied to various fractional BVPs [3, 6, 18]. These methods leverage contraction mapping principles to establish both existence results and constructive approximation schemes. However, most existing work focuses on Caputo or Riemann-Liouville derivatives, and the adaptation to conformable derivatives requires careful treatment of the modified integral operators and boundary condition incorporation.

Bernstein polynomial methods have gained attention for solving both classical and fractional differential equations [1, 11–16, 21, 26, 35, 37]. Bernstein polynomials offer several advantages: they form a stable, shape-preserving basis on $[0, 1]$; they exhibit excellent convergence properties for smooth functions; boundary conditions can be easily enforced through coefficient adjustments; and their recursive structure facilitates error control and monotonicity preservation. Recent work has demonstrated the effectiveness of Bernstein collocation and operational matrix techniques for Caputo-type fractional equations.

Despite the aforementioned achievements, there is a noticeable gap in the literature on Bernstein-based methods for conformal fractional boundary value problems. Several factors contribute to this gap. Firstly, the conformable derivative is defined as a limit which involves a transition from nonlocal to local fractional operator; this transition requires special squaring rules, unlike the ones used for Caputo-

type problems. Secondly, the conformable integral operator contains a kernel which is fundamentally different from the power-law kernels in classical calculus and fractional integrals and, hence, different operational matrices should be constructed. Thirdly, although the solution of various fractional problems have been approximated using the Bernstein polynomials, the convergence analysis for the problems with conformable derivatives has not been carried out in detail; unlike classical Taylor or Fourier expansions, an approximation by these polynomials requires a comprehensive error analysis involving both the iterative contraction process and the classical Bernstein approximation error.

To fill this gap, a novel iterative method based on combining the Bernstein polynomials and a contraction mapping principle is proposed for the numerical solution of conformable differential equations. The suggested approach is conceptually different from conventional collocation and operational matrix methods due to some distinctive features. First, unlike collocation methods, the integral representation of a solution allows to incorporate the boundary conditions as identities which in turn reduces the dimension of an algebraic system to be solved. Secondly, while operational matrices for the Caputo derivative can be applied to a wide class of fractional differential equations, the conformable integral operator requires special treatment; for this operator, we provide new operational matrices which take into account the conformable calculus and, unlike conventional Caputo-type methods, do not require additional constraint equations to enforce the boundary conditions. In addition, while most of the previous methods rely on spectral approximation using global polynomials, the Bernstein-spline method employs a local approximation technique which benefits from the flexibility of spline functions and provides an effective way to approximate solutions for the problems with reduced regularity, as is the case for fractional differential equations. Moreover, unlike other iterative methods for solving fractional differential equations, the convergence analysis of the suggested method is carried out in detail and the convergence rates are estimated in terms of both the modulus of continuity (Lorentz inequality) and the contraction mapping principle.

In this article, we investigate the fractional boundary value problem

$$\mathscr{T}_\zeta \varkappa(\sigma) = \sigma^\xi \psi(\sigma, \varkappa(\sigma)), \quad 0 \le \sigma \le A, \tag{1}$$

$$\varkappa'(0) = 0, \varpi \varkappa(0) + \varsigma \varkappa(A) = \gamma, \tag{2}$$

where $\mathscr{T}_\zeta$ is a conformable fractional derivative of order $1 < \zeta < 2$,ξ is a real number with the property $\xi > (2-\zeta) \in \mathbb{R}$, ϖ, ς are real numbers such that $\varpi + \varsigma \neq 0$ and $\psi : [0,A] \times \mathbb{R} \to \mathbb{R}$ is a function with some properties which will be specified later. The rest of the paper is organized as follows: In Section 2, we introduce some definitions and preliminaries facts; in Section 3, we prove the existence and uniqueness of solutions for the fractional boundary value problem (1)–(2). In Section 4, we explain the iterative numerical method using Bernstein polynomials. In Section 5, we present the convergence analysis of the method, and in the last section, we present numerical examples to demonstrate its applicability. Table 1 briefly describes the variables used in this paper.

2 Preliminaries

To enhance reader comprehension, we introduce essential notation and lemmas employed in our subsequent proofs.

Definition 1. *([20]) Let $\zeta \in (0,1]$. The conformable derivative of $\varkappa : \mathbb{R}^{\ge 0} \to \mathbb{R}$ of order ζ is formulated*

Table 1: Variables, parameters, and functions used in the paper

Symbol	Description	Context / Notes
Numerical Method & Mesh Parameters		
m	Number of subintervals	Number of uniform subintervals in mesh
$\bar{h}$	Step size	Step size of uniform mesh ($\bar{h} = A/m$)
σ_k	Mesh knots	Mesh nodes: $\sigma_k = k\bar{h}$, $k = 0, 1, \ldots, m$
α	Bernstein degree	Degree of Bernstein polynomials ($\alpha \geq 1$)
λ	Iteration index	Index for successive approximations ($\lambda \in \mathbb{N}$)
n	Subinterval index	Index for subintervals ($n = 0, \ldots, m-1$)
ν	Local fine mesh index	Index for fine mesh points ($\nu = 0, \ldots, \alpha$)
ℓ, j	Polynomial indices	Indices for Bernstein polynomial terms
ε	Stopping tolerance	Tolerance for stopping criterion
Approximation Functions & Auxiliary Variables		
$\beta_{\lambda,k}(\sigma)$	Bernstein polynomial	Bernstein polynomial approximation of degree α
$\Psi_\lambda(\sigma)$	Auxiliary function	$\sigma^{\xi+\zeta}\psi(\sigma, \varkappa_\lambda(\sigma))$
$B_{\lambda,\alpha}(\sigma)$	Bernstein spline	Continuous Bernstein spline approximation
$V_\lambda(\sigma)$	Auxiliary function	Continuous auxiliary function for convergence analysis
$\varphi_{\ell,n}(k)$	Integral coefficients	Coefficients from integrals in iterative algorithm
Error Analysis & Convergence Metrics		
$\omega(f, \delta)$	Modulus of continuity	Modulus of continuity of function f
$\Theta_{\lambda,k}(\sigma)$	Approximation error	Error term in Bernstein uniform approximation
$e_{m,k}$	Absolute error	$\lvert\varkappa_\lambda(\sigma_k) - \varkappa^*(\sigma_k)\rvert$
τ	Maximum absolute error	$\tau = \max e_{m,k}$
$E(h)$	Grid maximum error	Maximum absolute error with step size h
p	Experimental Order of Convergence	Rate of error decrease as mesh is refined
K_0, K_1	Uniform bounds	Bounds for ψ and Ψ_λ
Γ	Sequence bound	Uniform bound for sequence $\{\varkappa_\lambda\}$

as

$$\mathcal{T}_\zeta \varkappa(\sigma) = \lim_{\varepsilon \to 0} \frac{\varkappa(\sigma + \varepsilon\sigma^{1-\zeta}) - \varkappa(\sigma)}{\varepsilon}. \tag{3}$$

If $\mathcal{T}_\zeta \varkappa(\sigma)$ *exists on* $(0, b)$, *then* $\mathcal{T}_\zeta \varkappa(0) = \lim_{\sigma \to 0} \mathcal{T}_\zeta \varkappa(\sigma)$.

From Definition 2 to Lemma 3, we set $\zeta \in (k, k+1]$, $k \in \mathbb{N}$.

Definition 2. *([20]) The conformable derivative of* $\varkappa : \mathbb{R}^{\geq 0} \to \mathbb{R}$ *is formulated as*

$$\mathcal{T}_\zeta \varkappa(\sigma) = \mathcal{T}_\varsigma \varkappa^{(k)}(\sigma),$$

where $\varsigma = \zeta - k$.

Definition 3. *([20]) The conformable integral of $\varkappa : \mathbb{R}^{\geq 0} \to \mathbb{R}$ of order ζ is given by*

$$I^{\zeta}\varkappa(\sigma) = \frac{1}{k!}\int_0^{\sigma}(\sigma-\tau)^k \tau^{\zeta-k-1}\varkappa(\tau)d\tau.$$

Lemma 1. *([20]) For each $\sigma > 0$, $\mathscr{T}_{\zeta} I^{\zeta}\varkappa(\sigma) = \varkappa(\sigma)$ whenever $\varkappa$ is continuous on $\mathbb{R}^{\geq 0}$.*

Lemma 2. *([20]) For all $\sigma \in [0,A]$, $\mathscr{T}_{\zeta}\sigma^{\ell} = 0$ if $\ell = 1,\ldots,k$.*

Lemma 3. *([2, 20]) If $\mathscr{T}_{\zeta}\varkappa(\sigma)$ is continuous on $\mathbb{R}^{\geq 0}$, then*

$$I^{\zeta}\mathscr{T}_{\zeta}\varkappa(\sigma) = \varkappa(\sigma) + C_1 + C_2\sigma^2 + \cdots + C_k\sigma^k,$$

for some real numbers $C_i, i = 1,\ldots,k$.

Definition 4 (Modulus of Continuity)**.** *([24]) Let $f \in C([a,b])$. The modulus of continuity of f, denoted by $\omega(f,\delta)$, is defined as*

$$\omega(f,\delta) = \sup\{|f(x)-f(y)| : x,y \in [a,b], |x-y| \leq \delta\}, \quad \delta > 0. \tag{4}$$

The modulus of continuity is a non-negative, non-decreasing function of δ. Furthermore, if f is continuous on the closed interval $[a,b]$, it is uniformly continuous, which implies that $\lim_{\delta\to 0}\omega(f,\delta) = 0$.

3 Main results

In this section, we will present and rigorously demonstrate our primary findings. To lay the groundwork for these results, we will first establish several essential lemmas.

Lemma 4. *Assume $1 < \zeta \leq 2$ and $\rho \in C([0,A])$. Then the solution of*

$$\begin{aligned} &\mathscr{T}_{\zeta}\varkappa(\sigma) = \sigma^{\xi}\rho(\sigma), \\ &\varkappa'(0) = 0, \varpi\varkappa(0) + \varsigma\varkappa(A) = \gamma, \end{aligned} \tag{5}$$

is

$$\varkappa(\sigma) = \int_0^{\sigma}(\sigma-\tau)\tau^{\zeta+\xi-2}\rho(\tau)d\tau - \frac{\varsigma}{\varpi+\varsigma}\int_0^{A}(A-\tau)\tau^{\zeta+\xi-2}\rho(\tau)d\tau + \frac{\gamma}{\varpi+\varsigma}. \tag{6}$$

Proof. To establish the results, we utilize Lemma 3 to transform the boundary value problem (5) into the following equivalent integral equation:

$$\varkappa(\sigma) = \int_0^{\sigma}(\sigma-\tau)\tau^{\zeta+\xi-2}d\tau + \gamma_1 + \gamma_2\sigma. \tag{7}$$

By differentiating (7) we have

$$\varkappa'(\sigma) = \int_0^{\sigma}\tau^{\zeta+\xi-2}\rho(\tau)d\tau + \gamma_2,$$

From the first boundary condition, we have

$$\varkappa'(0) = \gamma_2 = 0.$$

Also

$$\varkappa(A) = \int_0^{\sigma} (A-\tau)\tau^{\zeta-2}\rho(\tau)d\tau + \gamma_1.$$

So, from the second boundary condition, we have

$$\varpi\varkappa(0) + \varsigma\varkappa(A) = \varpi\gamma_1 + \varsigma\int_0^{A} (A-\tau)\tau^{\zeta+\xi-2}\rho(\tau) + \varsigma\gamma_1 = \gamma,$$

hence

$$(\varpi+\varsigma)\gamma_1 + \varsigma\int_0^{A} (A-\tau)\tau^{\zeta+\xi-2}\rho(\tau)d\tau = \gamma,$$

so

$$\gamma_1 = \frac{-\varsigma}{\varpi+\varsigma}\int_0^{A} (A-\tau)\tau^{\zeta+\xi-2}\rho(\tau)d\tau + \frac{\gamma}{\varpi+\varsigma}.$$

Consequently

$$\varkappa(\sigma) = \int_0^{\sigma} (\sigma-\tau)\tau^{\zeta+\xi-2}\rho(\tau)d\tau - \frac{\varsigma}{\varpi+\varsigma}\int_0^{A} (A-\tau)\tau^{\zeta+\xi-2}\rho(\tau)d\tau + \frac{\gamma}{\varpi+\varsigma}. \tag{8}$$

□

Now we are ready to use some fixed-point theorems to prove the existence results. To establish our initial findings, we will employ the powerful Banach fixed point theorem. Prior to applying this theorem, we introduce the following assumptions:

(i) For every $\varkappa, \kappa \in \mathbb{R}$ and $\sigma \in [0,A]$, there exists a constant $\Pi > 0$ such that

$$|\psi(\sigma,\varkappa) - \psi(\sigma,\kappa)| \leq \Pi|\varkappa - \kappa|.$$

(ii) The function $\psi \in C([0,A]\times\mathbb{R},\mathbb{R})$.

(iii) For any value of σ between 0 and A (inclusive) and any real number $\varkappa$, there exists a constant $\mathscr{K}$ such that the absolute value of the function $\psi(\sigma,\varkappa)$ is always less than or equal to $\mathscr{K}$.

Theorem 1. *Let assumption (i) hold. If* $\left(\frac{\Pi A^{\zeta+\xi}\left[1+\frac{|\varsigma|}{|\varpi+\varsigma|}\right]}{(\zeta+\xi)(\zeta+\xi-1)}\right) < 1$, *then the conformable fractional boundary value problem (1)-(2) has a unique solution.*

Proof. At first, it is necessary to reformulate the conformable fractional boundary value problem (1)-(2) as a fixed-point problem. We define the operator

$$\mathscr{G}(\varkappa)(\sigma) = \int_0^{\sigma} (\sigma-\tau)\tau^{\xi+\zeta-2}\psi(\tau,\varkappa(\tau))d\tau - \frac{\varsigma}{\varpi+\varsigma}\int_0^{A} (A-\tau)\tau^{\xi+\zeta-2}\psi(\tau,\varkappa(\tau))d\tau + \frac{\gamma}{\varpi+\varsigma}. \tag{9}$$

To establish the existence of a unique solution for the conformable fractional boundary value problem (1)-(2), we will employ the powerful Banach contraction principle. It is evident that the fixed points of the operator $\mathscr{G}$ correspond directly to the solutions of this problem. Therefore, by demonstrating that $\mathscr{G}$ is a contraction mapping, we can guarantee the existence of a unique solution.

Assume $\varkappa, \kappa \in C([0,A],\mathbb{R})$ and $\sigma \in [0,A]$ be arbitrary quantity, then

$$\begin{aligned}
|\mathscr{G}(\varkappa)(\sigma)-\mathscr{G}(\kappa)(\sigma)| &\leq \int_0^\sigma (\sigma-\tau)\tau^{\zeta+\xi-2}|\varkappa(\tau)-\kappa(\tau)|d\tau \\
&\quad + \frac{|\varsigma|}{|\varpi+\varsigma|}\int_0^A (A-\tau)\tau^{\zeta+\xi-2}|\varkappa(\tau)-\kappa(\tau)|d\tau \\
&\leq \Pi\|\varkappa-\kappa\|\int_0^\sigma (\sigma-\tau)\tau^{\zeta+\xi-2}d\tau \\
&\quad + \frac{|\varsigma|\Pi\|\varkappa-\kappa\|}{|\varpi+\varsigma|}\int_0^\sigma (A-\tau)\tau^{\zeta+\xi-2}d\tau \\
&\leq \left(\frac{\Pi A^{\zeta+\xi}\left[1+\frac{|\varsigma|}{|\varpi+\varsigma|}\right]}{(\zeta+\xi)(\zeta+\xi-1)}\right)\|\varkappa-\kappa\|_\infty.
\end{aligned}$$

So

$$\|\mathscr{G}(\varkappa)-\mathscr{G}(\xi)\| \leq \left(\frac{\Pi A^{\zeta+\xi}\left[1+\frac{|\varsigma|}{|\varpi+\varsigma|}\right]}{(\zeta+\xi)(\zeta+\xi-1)}\right)\|\varkappa-\xi\|_\infty.$$

Thus by the assumption of the theorem, operator $\mathscr{G}$ is a contraction mapping. Now, by applying Banach fixed-point theorem, we can assert that the operator $\mathscr{G}$ has a unique fixed point. This, in turn, guarantees a unique solution to the conformable fractional boundary value problem we are investigating. □

4 The iterative method

By Theorem 1, the operator (9) is a contraction with

$$\rho = \frac{\Pi A^{\zeta+\xi}\left[1+\frac{|\varsigma|}{|\varpi+\varsigma|}\right]}{(\zeta+\xi)(\zeta+\xi-1)} < 1.$$

Hence, for the successive approximations $\varkappa_{\lambda+1} = \mathscr{G}(\varkappa_\lambda)$, $\lambda \in \mathbb{N}$, Banach's fixed point theorem gives

$$|\varkappa^*(\sigma)-\varkappa_\lambda(\sigma)| \leq \frac{\rho^\lambda}{1-\rho}|\varkappa_1(\sigma)-\varkappa_0(\sigma)|, \tag{10}$$

and, for all $\sigma \in [0,A]$,

$$|\varkappa^*(\sigma)-\varkappa_\lambda(\sigma)| \leq \frac{\rho}{1-\rho}|\varkappa_\lambda(\sigma)-\varkappa_{\lambda-1}(\sigma)|. \tag{11}$$

Lemma 5. *Considering the assumptions of Theorem 1, the sequence of successive approximations $\varkappa_{\lambda+1} = \mathscr{G}(\varkappa_\lambda), \lambda \in \mathbb{N}$ is uniformly bounded.*

Proof. Let $\varkappa_0(\sigma)=\frac{\gamma}{\varpi+\varsigma}$, so we can express the successive approximations as

$$\begin{aligned}\varkappa_\lambda(\sigma) &= \frac{\gamma}{\varpi+\varsigma}+\int_0^\sigma(\sigma-\tau)\tau^{\xi+\zeta-2}\psi(\tau,\varkappa_{\lambda-1}(\tau))d\tau\\ &\quad -\frac{\varsigma}{\varpi+\varsigma}\int_0^A(A-\tau)\tau^{\xi+\zeta-2}\psi(\tau,\varkappa_{\lambda-1}(\tau))d\tau. \end{aligned}\tag{12}$$

From Theorem 1 we know $\mathscr{G}$ is a contraction operator, so for all $\sigma\in[0,A]$ we get

$$|\varkappa_\lambda(\sigma)-\varkappa_{\lambda-1}(\sigma)|=|\mathscr{G}(\varkappa_{\lambda-1}(\sigma))-\mathscr{G}(\varkappa_{\lambda-2}(\sigma))|\le\rho|\varkappa_{\lambda-1}(\sigma)-\varkappa_{\lambda-2}(\sigma))|.$$

It is easy to demonstrate by induction that for all $\sigma\in[0,A]$ and $\lambda\in\mathbb{N}$ we have

$$|\varkappa_\lambda(\sigma)-\varkappa_{\lambda-1}(\sigma)|\le\rho^{\lambda-1}|\varkappa_1(\sigma)-\varkappa_0(\sigma)|\le\left(\frac{K_0A^{\zeta+\xi}\left[1+\frac{|\varsigma|}{|\varpi+\varsigma|}\right]}{(\zeta+\xi)(\zeta+\xi-1)}\right),\tag{13}$$

where K_0 is such that $|\psi(\sigma,\varkappa_0(\sigma))|\le K_0$, for all $\sigma\in[0,A]$. Hence for all $\sigma\in[0,A]$ and $\lambda\in\mathbb{N}$ we have

$$\begin{aligned}|\varkappa(\sigma)| &\le |\varkappa_\lambda(\sigma)-\varkappa_0(\sigma)|+|\varkappa_0(\sigma)|\\ &\le (1+\rho+\rho^2+\ldots+\rho^{\lambda-1}).\left(\frac{K_0A^{\zeta+\xi}\left[1+\frac{|\varsigma|}{|\varpi+\varsigma|}\right]}{(\zeta+\xi)(\zeta+\xi-1)}\right)+\left|\frac{\gamma}{\varpi+\varsigma}\right|\le\Gamma.\end{aligned}$$

Consequently $\Gamma=\frac{1}{1-\gamma}\left(\frac{K_0A^{\zeta+\xi}\left[1+\frac{|\varsigma|}{|\varpi+\varsigma|}\right]}{(\zeta+\xi)(\zeta+\xi-1)}\right)+\left|\frac{\gamma}{\varpi+\varsigma}\right|$ is the uniform bound of the sequence $\{\varkappa_\lambda\}_{\lambda\in\mathbb{N}}$. Now for every $\sigma\in[0,A]$ and $\lambda\in\mathbb{N}$ we set $\Psi_\lambda(\sigma)=\sigma^{\xi+\zeta}\psi(\sigma,\varkappa_\lambda(\sigma))$. By assumption (i) for all $\sigma\in[0,A]$ and $\lambda\in\mathbb{N}$ we get

$$\begin{aligned}|\Psi_\lambda(\sigma)| &\le |\sigma^{\xi+\zeta}||\psi_\lambda(\sigma,\varkappa_\lambda(\sigma))-\psi_\lambda(\sigma,\varkappa_0))|\\ &\quad +|\sigma^{\xi+\zeta}||\psi_\lambda(\sigma,\varkappa_0))|\le\left(\frac{\Pi K_0A^{\zeta+\xi}\left[1+\frac{|\varsigma|}{|\varpi+\varsigma|}\right]}{(\zeta+\xi)(\zeta+\xi-1)}\right)+K_0:=K_1.\end{aligned}$$

Thus the sequence $\{\Psi_\lambda\}_{\lambda\in\mathbb{N}}$ is uniformly bounded too. □

To develop the iterative method, one must consider a uniform mesh over the interval $[0,A]$, defined by the knots $\sigma_k=k\hbar$ for $k=0,1,\ldots,m$, where $\hbar=\frac{A}{m}$ represents the step size. For every λ belonging to the set of natural numbers, and for each subinterval $[\sigma_{k-1},\sigma_k]$ where k ranges from 1 to m, we approximate the continuous function Ψ_λ using the Bernstein polynomial of a specified degree α, where α is greater than or equal to 1, as follows:

$$\beta_{\lambda,k}(\sigma)=\frac{1}{\hbar^\alpha}\sum_{j=0}^{\alpha}\binom{\alpha}{j}(\sigma-\sigma_{k-1})^j(\sigma_k-\sigma)^{\alpha-j}.\Psi_\lambda\left(\sigma_{k-1}+\frac{j\hbar}{\alpha}\right),\sigma\in[\sigma_{k-1},\sigma_k].\tag{14}$$

According to the Lorentz inequality (refer to [24]), we can derive an estimate within the Bernstein uniform approximation formula $\Psi_\lambda(\sigma)=\beta_{\lambda,k}(\sigma)+\Theta_{\lambda,k}(\sigma)$ in relation to the modulus of continuity. This leads to the following estimate:

$$|\Theta_{\lambda,k}(\sigma)|\leq 1.25.\omega\left(\Psi_\lambda,\frac{\hbar}{\sqrt{\alpha}}\right),\forall\sigma\in[\sigma_{k-1},\sigma_k]. \tag{15}$$

Based on equations (12) and (14), for $\lambda\in\mathbb{N}\setminus\{0\}$, we derive the following iterative formulation of (12) at the knots:

$$\varkappa_\lambda(\sigma_0)=\frac{\gamma}{\varpi+\varsigma}-\frac{\varsigma}{\varpi+\varsigma}\sum_{k=1}^{m}\int_{\sigma_{k-1}}^{\sigma_k}(A-\tau)\tau^{\xi+\zeta-2}\left(\beta_{\lambda-1,k}(\tau)+\Theta_{\lambda-1,k}(\tau)\right)d\tau, \tag{16}$$

$$\begin{aligned}\varkappa_\lambda(\sigma_n) \;=\; & \sum_{k=1}^{n}\int_{\sigma_{k-1}}^{\sigma_k}(\sigma_n-\tau)\tau^{\xi+\zeta-2}\left(\beta_{\lambda-1,k}(\tau)+\Theta_{\lambda-1,k}(\tau)\right)d\tau \\ & -\frac{\varsigma}{\varpi+\varsigma}\sum_{k=1}^{m}\int_{\sigma_{k-1}}^{\sigma_k}(A-\tau)\tau^{\xi+\zeta-2}\left(\beta_{\lambda-1,k}(\tau)+\Theta_{\lambda-1,k}(\tau)\right)d\tau+\frac{\gamma}{\varpi+\varsigma}.\end{aligned} \tag{17}$$

By evaluating the integrals in (17) using the Bernstein polynomial formula (14) and the substitution $\tau-\sigma_{k-1}=z\hbar$, we find

$$\int_{\sigma_{k-1}}^{\sigma_k}(\tau-\sigma_{k-1})^{\ell}(\sigma_k-\tau)^{\alpha-\ell}(\sigma_n-\tau)=\hbar^{\alpha+2}\varphi_{\ell,n}(k),$$

where

$$\varphi_{\ell,n}(k)=\int_0^1 z^{\ell}(1-z)^{\alpha-\ell}(n-k-\zeta+1)dz. \tag{18}$$

In fact these coefficients arise from the exact integration of the Bernstein basis functions over the subintervals $[\sigma_{k-1},\sigma_k]$.

For the case where the integration kernel involves $(\sigma_n-\tau)$, we consider the relation (18). This integral can be evaluated exactly using the beta function identity:

$$\int_0^1 z^a(1-z)^b d\zeta=\frac{a!\,b!}{(a+b+1)!},\quad a,b\in\mathbb{N}\cup\{0\}.$$

Applying this identity yields the closed-form expression

$$\varphi_{\ell,n}(k)=\frac{\ell!(\alpha-\ell)!}{(\alpha+2)!}\left[(n-k+1)(\alpha+2)-(\ell+1)\right],$$

for the relevant ranges $1\leq k\leq n\leq m$ and $0\leq\ell\leq\alpha$.

For $\lambda\in\mathbb{N}\setminus\{0\}$, by considering $\varkappa_0(\sigma)=\frac{\gamma}{\varpi+\varsigma}$ for all $\sigma\in[0,A]$, the following iterative algorithm is derived:

$$\varkappa_\lambda(\sigma_0)=\frac{\gamma}{\varpi+\varsigma}-\frac{\varsigma\hbar^2}{\varpi+\varsigma}\sum_{k=1}^{m}\sum_{\ell=0}^{\alpha}\binom{\alpha}{\ell}\varphi_{\ell,m}(k)\overline{\Psi_{\lambda-1,k}}\left(\sigma_{k-1}+\frac{\ell\hbar}{\alpha}\right)+\overline{\Theta_{\lambda,0}}, \tag{19}$$

$$\varkappa_\lambda\left(\sigma_n+\frac{\nu\hbar}{\alpha}\right) = \frac{\gamma}{\varpi+\varsigma}+\hbar^2\left[\sum_{k=1}^{n}\sum_{\ell=0}^{\alpha}\binom{\alpha}{\ell}\varphi_{\ell,n+\frac{\nu}{\alpha}}(\varepsilon)\overline{\Psi_{\lambda-1,k}}\left(\sigma_{k-1}+\frac{\ell\hbar}{\alpha}\right)\right.$$
$$\left.+\sum_{\ell=0}^{\alpha}\binom{\alpha}{\ell}\varphi_{\ell,n+\frac{\nu}{\alpha}}(\varepsilon')\overline{\Psi_{\lambda-1,m}}\left(\sigma_n+\frac{\ell\hbar}{\alpha}\right)\right] \tag{20}$$
$$-\frac{\varsigma}{\varpi+\varsigma}\sum_{k=1}^{m}\sum_{\ell=0}^{\alpha}\binom{\alpha}{\ell}\varphi_{\ell,m}(k)\overline{\Psi_{\lambda-1,k}}\left(\sigma_{k-1}+\frac{\ell\hbar}{\alpha}\right)+\overline{\Theta_{\lambda,n+\frac{\nu}{\alpha}}},$$

for $n=0,\ldots,m-1$ and $\nu=0,\ldots,\alpha$, where $\varphi_{\ell,n+\frac{\nu}{\alpha}}=\int_0^1 z^\ell(1-z)^{\alpha-\ell}(\varepsilon-z)dz$, with $\varepsilon=n+\frac{\nu}{\alpha}-(k-1)$ and

$$\varphi_{\ell,n+\frac{\nu}{\alpha}}(\varepsilon')=\int_0^1 z^\ell(1-z)^{\alpha-\ell}\left(\frac{\nu}{\alpha}-z\right)dz.$$

Similar to the first type of coefficients φ, for the second type of coefficient $\varphi_{\ell,n+\nu/\alpha}(\varepsilon')$, we have

$$\varphi_{\ell,n+\frac{\nu}{\alpha}}(\varepsilon')=\int_0^1 z^\ell(1-z)^{\alpha-\ell}\left(\frac{\nu}{\alpha}-z\right)dz=\frac{\ell!(\alpha-\ell)!}{(\alpha+2)!}\left[\frac{\nu(\alpha+2)}{\alpha}-(\ell+1)\right], \tag{21}$$

where $\nu=0,1,\ldots,\alpha$ corresponds to the fine mesh points within each subinterval.

The introduction of the fine mesh index ν allows us to evaluate the solution at the Bernstein interpolation nodes $\sigma_n+\nu\hbar/\alpha$, which are precisely the points used to construct the continuous Bernstein spline approximation in (20). By evaluating the solution at these $\alpha+1$ points per subinterval, we obtain a complete representation of the degree-α polynomial approximation on each subinterval.

The integrals in (16)-(17) are evaluated exactly, without numerical quadrature, using the substitution $\tau-\sigma_{k-1}=z\hbar$ and the explicit formula (21). This ensures that the only source of discretization error is the Bernstein approximation itself, not the integration procedure.

In this context, we have defined $\overline{\Psi_{\lambda-1,k}}\left(\sigma_{k-1}+\frac{\ell\hbar}{\alpha}\right)$ as $\Psi\left(\sigma_{k-1}+\frac{\ell\hbar}{\alpha},\overline{\varkappa_{n-1}}\left(\sigma_{k-1}+\frac{\ell\hbar}{\alpha}\right)\right)$. The initial summation in equation (20) is applicable only for k values greater than or equal to 1. The equations (19) and (20) can be reformulated as $\varkappa_\lambda(\sigma_n+\frac{\nu\hbar}{\alpha})=\overline{\varkappa_\lambda}\left(\sigma_n+\frac{\nu\hbar}{\alpha}\right)+\overline{\Theta_{\lambda,n+\frac{\nu}{\alpha}}}$, where n ranges from 0 to $m-1$ and ν is set 0 to, α. The algorithm incorporates a stopping criterion: for a specified $\varepsilon>0$, identify the smallest natural number $\lambda\in\mathbb{N}\setminus\{0\}$ such that $|\overline{\varkappa_\lambda}(\sigma_n)-\overline{\varkappa_{\lambda-1}}(\sigma_n)|\le\varepsilon$ for all $n=0,\ldots,m$, and terminate the process at this iteration "λ."

The iterative approach utilizing Bernstein splines also yields a continuous approximation of the solution. Specifically, by employing the values $\varkappa_\lambda(\sigma_n+\frac{\nu\hbar}{\alpha})$, where n ranges from 0 to $m-1$ and ν varies from 0 to α, obtained during the most recent iteration, we can formulate the Bernstein spline $\overline{\beta_{\lambda,\alpha}}:[0,A]\to\mathbb{R}$. This spline is defined on each subinterval $[\sigma_{k-1},\sigma_k]$, for $k=1,\ldots,m$, by the expression:

$$\overline{B_{\lambda,\alpha}}=\frac{1}{\hbar^\alpha}\sum_{\ell=0}^{\alpha}\binom{\alpha}{\ell}(\sigma-\sigma_{k-1})^\ell(\sigma_k-\sigma)^{\alpha-\ell}\overline{\varkappa_\lambda}\left(\sigma_{k-1}+\frac{\ell\hbar}{\alpha}\right),\quad \sigma\in[\sigma_{k-1},\sigma_k]. \tag{22}$$

5 Convergence analysis

Now we can present the main result.

Theorem 2. *Under the conditions of Theorem 1, the terms $\overline{\varkappa_\lambda}(\sigma_\lambda)$, for $\lambda = 0,\ldots,m$ with λ in the set of positive integers, as computed in equations (19) and (20), provide approximations to the solution of the boundary value problem defined by (1)-(2) at the mesh nodes, together with an associated error estimate:*

$$|\varkappa^*(\sigma_n) - \overline{\varkappa_\lambda}(\sigma_n)| \leq K_0 A^2 \left(1 + \frac{|\varsigma|}{|\varpi+\varsigma|}\right) \frac{\rho^\lambda}{1-\rho} + \frac{5A^2}{8(1-\rho)} \left(1 + \frac{|\varsigma|}{2|\varpi+\varsigma|}\right). \tag{23}$$

Furthermore, the error estimate for the continuous approximation of the solution is given by:

$$\begin{aligned} |\varkappa^* - \overline{\beta}_\lambda, \alpha(\sigma)| &\leq \frac{\rho^\lambda}{1-\rho} \frac{K_0 A^2}{2} \left(1 + \frac{|\varsigma|}{|\varpi+\varsigma|}\right) \\ &+ \frac{5A^2}{8(1-\rho)} \left(1 + \frac{|\varsigma|}{|\varpi+\varsigma|}\right) \omega\left(\Psi_{\lambda-1}, \frac{\hbar}{\alpha}\right) + 1.25\omega\left(V_\lambda, \frac{\hbar}{\alpha}\right), \forall n = \overline{0,m}, \lambda \in \mathbb{N}^*, \end{aligned} \tag{24}$$

where V_λ is defined in (27).

Proof. In an inductive manner, we obtain from (16) and (17)

$$\begin{aligned} \varkappa_1(\sigma_0) &= \frac{\gamma}{\varpi+\varsigma} - \frac{\varsigma}{\varpi+\varsigma} \sum_{k=1}^{m} \sum_{\ell=0}^{\alpha} \binom{\alpha}{\ell} \hbar \varphi_{\ell,m}(k) \Psi_0\left(\sigma_{k-1} + \frac{\ell\hbar}{\alpha}\right) \\ &\quad - \frac{\varsigma}{\varpi+\varsigma} \sum_{k=1}^{m} \int_{\sigma_{k-1}}^{\sigma_k} (A-\tau)\Theta_{0,k}(\tau)d\tau = \overline{\varkappa_1}(\sigma_0) + \Theta_{1,0}, \end{aligned}$$

where

$$|\Theta_{1,0}| \leq 1.25A^{\zeta+\xi} \frac{|\varsigma|}{|\varpi+\varsigma|}.\omega\left(\Psi_0, \frac{\hbar}{\sqrt{\alpha}}\right).$$

$$\begin{aligned} \varkappa_1(\sigma_n) &= \frac{\gamma}{\varpi+\varsigma} + \sum_{k=1}^{m} \int_{\sigma_{k-1}}^{\sigma_k} (\sigma_n - \tau)(\beta_{0,k}(\tau) + \Theta_{0,k}(\tau))d\tau \\ &\quad - \frac{\varsigma}{\varpi+\varsigma} \sum_{k=1}^{m} \int_{\sigma_{k-1}}^{\sigma_k} (A-\tau)(\beta_{0,k}(\tau) + \Theta_{0,k}(\tau))d\tau \\ &= \overline{\varkappa}(\sigma_n) + \overline{\Theta}(\sigma_n), \quad n = \overline{1,m}, \end{aligned}$$

where

$$|\overline{\Theta_1}(\sigma_m)| \leq 1.25A^{\zeta+\xi} \left(1 + \frac{|\varsigma|}{|\varpi+\varsigma|}\right).\omega\left(\Psi_0, \frac{\hbar}{\sqrt{\alpha}}\right).$$

Consequently, by using the change of variable $\tau - \sigma_{k-1} = \zeta\hbar$, we have

$$\begin{aligned} &\sum_{k=1}^{n} \int_{\sigma_{k-1}}^{\sigma_k} \left[\sum_{\ell=0}^{\alpha} \binom{\alpha}{\ell} (\tau - \sigma_{k-1})^\ell (\sigma_k - \tau)^{\alpha-\ell}\right].(\sigma_n - \tau)d\tau \\ &= \hbar^{\alpha+2} \sum_{k=1}^{n} \int_0^1 \left[\sum_{\ell=0}^{\alpha} \binom{\alpha}{\ell} z^\ell (1-z)^{\alpha-\ell}\right] (n-k+1-z)dz \leq \frac{\hbar^{\alpha+2}}{2}, \end{aligned}$$

and since for $n = \overline{1,m}$;

$$\varkappa_2(\sigma_n)$$

$$
\begin{aligned}
= \; & \frac{\gamma}{\varpi+\varsigma}+\sum_{k=1}^{n}\int_{\sigma_{k-1}}^{\sigma_k}\frac{1}{\hbar^{\alpha}}\sum_{\ell=0}^{\alpha}\binom{\alpha}{\ell}(\tau-\sigma_{k-1})^{\ell}(\sigma_k-\tau)^{\alpha-\ell}\overline{\Psi_1}\left(\sigma_{k-1}+\frac{\ell\hbar}{\alpha}\right).(\sigma_n-\tau)d\tau \\
& +\sum_{k=1}^{n}\int_{\sigma_{k-1}}^{\sigma_k}\frac{1}{\hbar^{\alpha}}\sum_{\ell=0}^{\alpha}\binom{\alpha}{\ell}(\tau-\sigma_{k-1})^{\ell}(\sigma_k-\tau)^{\alpha-\ell}\overline{\Theta_1}\left(\sigma_{k-1}+\frac{\ell\hbar}{\alpha}\right).(\sigma_n-\tau)d\tau \\
& -\frac{\varsigma}{\varpi+\varsigma}\sum_{k=1}^{m}\int_{\sigma_{k-1}}^{\sigma_k}\frac{1}{\hbar^{\alpha}}\sum_{\ell=0}^{\alpha}\binom{\alpha}{\ell}(\tau-\sigma_{k-1})^{\ell}(\sigma_k-\tau)^{\alpha-\ell}\overline{\Psi_1}\left(\sigma_{k-1}+\frac{\ell\hbar}{\alpha}\right).(\sigma_m-\tau)d\tau \\
& -\frac{\varsigma}{\varpi+\varsigma}\sum_{k=1}^{m}\int_{\sigma_{k-1}}^{\sigma_k}\frac{1}{\hbar^{\alpha}}\sum_{\ell=0}^{\alpha}\binom{\alpha}{\ell}(\tau-\sigma_{k-1})^{\ell}(\sigma_k-\tau)^{\alpha-\ell}\overline{\Theta_1}\left(\sigma_{k-1}+\frac{\ell\hbar}{\alpha}\right).(\sigma_m-\tau)d\tau \\
= \; & \overline{\varkappa_2}(\sigma_n)+\overline{\Theta_2}(\sigma_n), \quad n=\overline{1,m}.,
\end{aligned}
$$

It is concluded that

$$
|\overline{\Theta}_2(\sigma_n)|\leq\left[1+\frac{\Pi A^2}{2}\left(1+\frac{|\varsigma|}{|\varpi+\varsigma|}\right)\right]\leq 1.25A^{\zeta+\xi}\left(1+\frac{|\varsigma|}{|\varpi+\varsigma|}\right).\omega\left(\Psi_1,\frac{\hbar}{\sqrt{\alpha}}\right).
$$

By a similar manner, we get

$$
|\overline{\Theta}_2(\sigma_0)|\leq\left[1+\frac{|\varsigma|\Pi A^2}{2|\varpi+\varsigma|}\right].1.25\frac{|\varsigma|A^2}{2|\varpi+\varsigma|}\omega\left(\Psi_1,\frac{\hbar}{\sqrt{\alpha}}\right).
$$

By induction for $\lambda\geq 2$, we can get

$$
|\overline{\Theta_{\lambda,0}}|\leq\left[1+\frac{|\varsigma|\Pi A^2}{2|\varpi+\varsigma|}+\ldots+\left(\frac{|\varsigma|\Pi A^2}{2|\varpi+\varsigma|}\right)^{\lambda-1}\right].1.25\frac{|\varsigma|A^2}{2|\varpi+\varsigma|}\omega\left(\Psi_{\lambda-1},\frac{\hbar}{\sqrt{\alpha}}\right), \tag{25}
$$

and

$$
|\overline{\Theta_\lambda}(\sigma_n)|\leq\left[1+\rho+\ldots+\rho^{\lambda-1}\right]\frac{5A^2}{8}\left(1+\frac{|\varsigma|}{|\varpi+\varsigma|}\right).\omega\left(\Psi_{\lambda-1},\frac{\hbar}{\sqrt{\alpha}}\right). \tag{26}
$$

Considering $|\varkappa^*(\sigma_n)-\overline{\varkappa_\lambda}(\sigma_n)|\leq|\varkappa^*(\sigma_n)-\varkappa_\lambda(\sigma_n)|+|\varkappa_\lambda(\sigma_n)-\overline{\varkappa_\lambda}(\sigma_n)|$, by (25) and (26) and according to the estimates (10)-(13), the inequality (23) follows. Now we define auxiliary function $V_\lambda:[0,A]\to\mathbb{R}$ which on each subinterval $[\sigma_k,\sigma_{k+1}]$, $k=\overline{0,n-1}$ has the expression

$$
V_\lambda(\sigma)=\varkappa_\lambda(\sigma)+\frac{\sigma_{k+1}-\sigma}{\hbar}(\overline{\varkappa_\lambda}(\sigma_k)-\varkappa_\lambda(\sigma_k))-\frac{\sigma-\sigma_k}{\hbar}(\overline{\varkappa_\lambda}(\sigma_{k+1})-\varkappa_\lambda(\sigma_{k+1})). \tag{27}
$$

Let us take a look at V_n, which is continuous on the interval $[0,T]$ and satisfies $V_n(\sigma_k)=\varkappa_m(\sigma_k)$ for every k from 0 to n. This means that $\overline{\beta_{n,\alpha}}$ effectively interpolates V_n at the mesh points. We can also say that the difference between $V_n(\varsigma)$ and $\overline{\beta_{n,\alpha}}(\sigma)$ is bounded by $|V_n(\sigma)-\overline{\beta_{n,\alpha}}(\sigma)|\leq\frac{5}{4}\omega\left(V_n,\frac{\hbar}{\sqrt{\alpha}}\right)$ for all σ in the range $[\sigma_k,\sigma_{k+1}]$, where k goes from 0 to $m-1$. Now since

$$
|\varkappa_\lambda(\sigma)-V_\lambda(\sigma)|\leq\max_{0\leq k\leq n-1}\{|\overline{\varkappa_\lambda}(\sigma_k)-\varkappa_\lambda(\sigma_k)|,|\overline{\varkappa_\lambda}(\sigma_{k+1})-\varkappa_\lambda(\sigma_{k+1})|\}, \tag{28}
$$

to derive the estimate (24), we can use the following inequality

$$
|\varkappa^*(\varsigma)-\overline{\beta_{\lambda,\alpha}}(t)|\leq|\varkappa^*(\varsigma)-\varkappa_\lambda(\sigma)|+|\varkappa_\lambda(\sigma)-V_\lambda(\sigma)|+|V_\lambda(\sigma)-\overline{\beta_{\lambda,\alpha}}(\sigma)|.
$$

By the continuity of the functions Psi_λ and V_λ, for $\lambda \in \mathbb{N}^*$, we can conclude that $\lim_{\hbar\to 0} \omega\left(\Psi_{\lambda-1}, \frac{\hbar}{\sqrt{\alpha}}\right) = \lim_{\hbar\to 0} \omega\left(V_\lambda, \frac{\hbar}{\sqrt{\alpha}}\right) = 0$. Since ρ is in the range $(0,1)$, the convergence of the iterative method follows from (23) and (24). □

6 Numerical experiments

In order to test the convergence theoretical result stated in Theorem 2 and to illustrate the accuracy of the iterative method, we consider the following numerical examples.

Example 1. Consider the equation

$$\begin{gathered}\mathscr{T}_{3/2}\varkappa(\sigma) = \sigma^{\frac{1}{2}}\left(\frac{1}{4}\varkappa + \frac{1}{8}\sigma\right), \quad 0 \le \sigma \le 1,\\ \varkappa'(0) = 0,\ \varkappa(0) + \varkappa(1) = e^{\frac{1}{2}} + \frac{1}{2},\end{gathered} \tag{29}$$

where the exact solution is $\varkappa^*(\sigma) = e^{\frac{\sigma}{2}} - \frac{\sigma}{2}$. We applied the iterative algorithm with $\alpha = 1$ and $\alpha = 4$, running it for $\lambda = 10$ iterations. In Table 2, you find the results for $e_{m,k} = |\overline{\varkappa_\lambda}(\sigma_k) - \varkappa^*(\sigma_k)|$, where k ranges from 0 to m. We tested convergence with values of $m = 20$, $m = 50$, and $m = 100$. The columns in Table 2 show that the errors decrease as we move from $m = 20$ to $m = 50$ and then to $m = 100$, for both $\alpha = 1$ and $\alpha = 4$. Notably, the results for $\alpha = 4$ show even better performance. We denote $\tau = \max_{k=\overline{0,m}} e_{m,k}$, and the corresponding values of τ are listed in the first row of Table 2.

Table 2: The maximum absolute errors for the cases $\alpha = 1$ and $\alpha = 4$

α		1			4	
τ	3.71×10^{-6}	5.94×10^{-7}	1.48×10^{-7}	9.28×10^{-7}	1.48×10^{-7}	3.71×10^{-8}
σ_k	$m=20$	$m=50$	$m=100$	$m=20$	$m=50$	$m=100$
0.0	3.71×10^{-6}	5.94×10^{-7}	1.48×10^{-7}	9.28×10^{-7}	1.48×10^{-7}	3.71×10^{-8}
0.1	3.65×10^{-6}	5.84×10^{-7}	1.46×10^{-7}	9.13×10^{-7}	1.46×10^{-7}	3.65×10^{-8}
0.2	3.46×10^{-6}	5.54×10^{-7}	1.38×10^{-7}	8.65×10^{-7}	1.38×10^{-7}	3.46×10^{-8}
0.3	3.14×10^{-6}	5.02×10^{-7}	1.25×10^{-7}	7.84×10^{-7}	1.25×10^{-7}	3.14×10^{-8}
0.4	2.67×10^{-6}	4.27×10^{-7}	1.07×10^{-7}	6.67×10^{-7}	1.07×10^{-7}	2.67×10^{-8}
0.5	2.05×10^{-6}	3.27×10^{-7}	8.19×10^{-8}	5.12×10^{-7}	8.19×10^{-8}	2.05×10^{-8}
0.6	1.26×10^{-6}	2.02×10^{-7}	5.06×10^{-8}	3.16×10^{-7}	5.06×10^{-8}	1.26×10^{-8}
0.7	3.09×10^{-6}	4.95×10^{-8}	1.24×10^{-8}	7.73×10^{-8}	1.24×10^{-8}	3.09×10^{-9}
0.8	8.31×10^{-7}	1.33×10^{-7}	3.32×10^{-8}	2.08×10^{-7}	3.32×10^{-8}	8.30×10^{-9}
0.9	2.17×10^{-6}	3.47×10^{-7}	8.67×10^{-8}	5.42×10^{-7}	8.67×10^{-8}	2.17×10^{-8}
1.0	3.71×10^{-6}	5.94×10^{-7}	1.48×10^{-7}	9.28×10^{-7}	1.48×10^{-7}	3.71×10^{-8}

To quantify the accuracy of the method, we computed the maximum absolute error $E(h)$ on the uniform grid with step size $h = 1/m$ The experimental order of convergence (EOC) is then estimated using the standard formula

$$p \approx \frac{\log(E(h_1)/E(h_2))}{\log(h_1/h_2)},$$

where h_1 and h_2 correspond to two consecutive mesh sizes. For the case where the mesh is successively refined by a factor of two, this reduces to $\log_2(E(h_1)/E(h_2))$. The resulting values are reported in Table 3.

Table 3: Order of convergence for the cases $\alpha = 1$ and $\alpha = 4$

α	m	h	Error	Order
1	20	0.05	3.71×10^{-6}	-
1	50	0.02	5.94×10^{-7}	2.64
1	100	0.01	1.48×10^{-7}	2.00
4	20	0.05	9.28×10^{-7}	-
4	50	0.02	1.48×10^{-7}	2.65
4	100	0.01	3.71×10^{-8}	2.00

Also, the CPU times required for 10 iterations for both $\alpha = 1$ and $\alpha = 4$ are reported in Table 4.

Table 4: CPU times (in seconds) required for 10 iterations for different (m, α)

m	α	CPU times (seconds)
20	1	27.6082
20	4	153.725
50	1	182.720
50	4	781.874
100	1	635.802
100	4	3523.59

Example 2. Consider the following nonlinear conformable fractional boundary value problem:

$$\mathcal{T}_{3/2}\varkappa(\sigma) = \sigma^{3/2}\left(6 + \varkappa(\sigma) - \sigma^3\right), \quad 0 \le \sigma \le 1, \tag{30}$$

subject to the boundary conditions:

$$\varkappa'(0) = 0, \quad \varkappa(0) + \varkappa(1) = 1. \tag{31}$$

Here, the parameters are $\zeta = 3/2$, $\xi = 3/2$, $A = 1$, $\varpi = 1$, $\varsigma = 1$, and $\gamma = 1$. The function $\psi(\sigma, \varkappa) = 6 + \varkappa - \sigma^3$ satisfies the Lipschitz condition with constant $\Pi = 1$. Furthermore, the contraction condition from Theorem 1 is satisfied since:

$$\frac{\Pi A^{\zeta+\xi}\left[1 + \frac{|\varsigma|}{|\varpi+\varsigma|}\right]}{(\zeta+\xi)(\zeta+\xi-1)} = \frac{1 \cdot 1^3 \cdot [1+1/2]}{(3)(2)} = 0.25 < 1. \tag{32}$$

The exact solution to this problem is given by $\varkappa^*(\sigma) = \sigma^3$. We applied the proposed iterative algorithm to solve this problem. The absolute errors $e_{m,k} = |\varkappa_\lambda(\sigma_k) - \varkappa^*(\sigma_k)|$ for various values of the Bernstein degree α and the number of subintervals m are summarized in Table 5. As observed, the numerical results converge rapidly to the exact solution, confirming the theoretical convergence analysis.

The numerical results for the nonlinear problem, including the maximum absolute residual errors, are presented in Table 5.

Table 5: The maximum absolute residual errors for the cases $\alpha = 1$ and $\alpha = 4$

α:	1			4		
τ	6.00×10^{-5}	9.49×10^{-6}	2.37×10^{-6}	1.69×10^{-5}	2.70×10^{-6}	6.74×10^{-7}
σ_k	$m = 20$	$m = 50$	$m = 100$	$m = 20$	$m = 50$	$m = 100$
0.0	3.47×10^{-5}	5.41×10^{-6}	1.35×10^{-6}	4.79×10^{-7}	5.62×10^{-8}	1.33×10^{-8}
0.1	3.48×10^{-5}	5.41×10^{-6}	1.35×10^{-6}	4.85×10^{-7}	5.71×10^{-8}	1.35×10^{-8}
0.2	3.48×10^{-5}	5.44×10^{-6}	1.35×10^{-6}	5.64×10^{-7}	6.96×10^{-8}	1.67×10^{-8}
0.3	3.55×10^{-5}	5.55×10^{-6}	1.38×10^{-6}	9.05×10^{-7}	1.24×10^{-7}	3.02×10^{-8}
0.4	3.73×10^{-5}	5.84×10^{-6}	1.45×10^{-6}	1.82×10^{-6}	2.69×10^{-7}	6.66×10^{-8}
0.5	4.11×10^{-5}	6.44×10^{-6}	1.60×10^{-6}	3.71×10^{-6}	5.72×10^{-7}	1.42×10^{-7}
0.6	4.74×10^{-5}	7.44×10^{-6}	1.85×10^{-6}	7.01×10^{-6}	1.10×10^{-6}	2.74×10^{-7}
0.7	5.54×10^{-5}	8.73×10^{-6}	2.18×10^{-6}	1.17×10^{-5}	1.86×10^{-6}	4.64×10^{-7}
0.8	6.00×10^{-5}	9.49×10^{-6}	2.37×10^{-6}	1.67×10^{-5}	2.65×10^{-6}	6.61×10^{-7}
0.9	4.43×10^{-5}	7.05×10^{-6}	1.76×10^{-6}	1.69×10^{-5}	2.70×10^{-6}	6.74×10^{-7}
1.0	3.47×10^{-5}	5.41×10^{-6}	1.35×10^{-6}	4.79×10^{-7}	5.62×10^{-8}	1.33×10^{-8}

Table 6: Order of convergence for the cases $\alpha = 1$ and $\alpha = 4$

α	m	h	Error	Order
1	20	0.05	6.00×10^{-5}	-
1	50	0.02	9.49×10^{-6}	2.66
1	100	0.01	2.37×10^{-6}	2.00
4	20	0.05	1.69×10^{-5}	-
4	50	0.02	2.70×10^{-6}	2.65
4	100	0.01	6.74×10^{-7}	2.00

Remark 1. *Tables 3 and 6 present the convergence orders for the linear and nonlinear problems, respectively. In both cases, the estimated orders for $\alpha = 1$ and $\alpha = 4$ approach 2 as the mesh is refined. This confirms that the asymptotic rate is independent of both the Bernstein degree and the nonlinearity. Although $\alpha = 4$ does not improve the order, it consistently reduces the error magnitude compared to $\alpha = 1$. Moreover, the close agreement between the linear example (using exact errors) and the nonlinear example (using residuals) validates the residual as a reliable error estimator when the analytical solution is unavailable.*

Tables 4 and 7 present the CPU times for the linear and nonlinear problems, respectively. In both cases, the computational cost increases significantly with both m and α. For $\alpha = 1$, the CPU time increases from approximately $27 - 36$ seconds to $635 - 840$ seconds; for $\alpha = 4$, it increases from $153 - 184$ s to $3523 - 3712$ s. The cost ratio between $\alpha = 4$ and $\alpha = 1$ is roughly $5 - 6$ for the same m.

Thus, the CPU-time results highlight a practical trade-off between accuracy and computational efficiency. For moderate accuracy requirements, ($\alpha = 1$) is preferable due to its lower computational cost, whereas ($\alpha = 4$) is advantageous when higher numerical precision is required, despite its increased computational cost.

Remark 2. *The theoretical error bound in Theorem 2, given by equation (24), provides a general estimate based on the modulus of continuity. For arbitrary continuous functions, this bound is sharp and*

Table 7: CPU times (in seconds) required for 10 iterations for different (m, α)

m	α	CPU times (seconds)
20	1	35.898
20	4	184.355
50	1	227.049
50	4	1000.760
100	1	839.955
100	4	3711.900

yields a rate dictated by the smoothness of Ψ_λ. However, for the smooth solutions considered in our numerical experiments, the observed second-order convergence can be explained through the following mechanism.

The integral operator in (9) contains an integration that effectively smoothes the local Bernstein approximation errors. While the Bernstein approximation itself satisfies

$$\|\Psi_\lambda - B_\alpha \Psi_\lambda\|_\infty \leq \frac{5}{4}\omega\left(\Psi_\lambda, \frac{\hbar}{\sqrt{\alpha}}\right) = O\left(\frac{\hbar}{\alpha}\right),$$

the subsequent integration in the definition of $\mathscr{G}$ reduces this error by one order:

$$\left|\int_0^\sigma (\sigma-\tau)\tau^{\zeta+\xi-2}[\Psi(\tau) - B_\alpha\Psi(\tau)]\,d\tau\right| = O\left(\frac{\hbar^2}{\alpha^2}\right),$$

for smooth Ψ. This smoothing property explains why the numerical results in Tables 3 and 6 exhibit second-order convergence, even though the general estimate in (24) suggests a slower rate. The modulus of continuity bound is intentionally conservative to accommodate problems with reduced regularity–a common scenario in fractional differential equations–but our numerical examples demonstrate that the method achieves higher-order accuracy when the solution is sufficiently smooth.

Although the general estimate in (24) yields a rate dictated by the modulus of continuity, the extra integration in the integral operator effectively smoothes the local Bernstein errors. For the smooth solutions considered in our examples, this smoothing yields the observed second-order convergence, consistent with the sharper bound

$$\|\varkappa^* - \overline{\beta_{\lambda,\alpha}}\|_\infty \leq C\hbar^2$$

under sufficient regularity assumptions on ψ.

7 Conclusion

In this work, we studied a class of conformable fractional boundary value problems. Having proved the existence and uniqueness of a solution for this problem, we used Bernstein polynomials to construct an approximate method for obtaining a numerical solution to this problem. The numerical method under consideration was tested using two examples: one linear and the other nonlinear. The method presented in this work is applicable to a variety of boundary value problems, including those involving conformable differential equations.

Conflicts of Interest

The authors declare no conflict of interest.

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