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Hyper-Fitting ideals of Krasner hypercomultiplication hypermodules

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Abstract. Let R be a commutative hyperring with identity and let M be an R-hypermodule. For Krasner hypermodules admitting finite free presentations, we define hyper-Fitting ideals via an internal determinantal construction and prove that, after fundamental saturation, they coincide with the pullbacks of the classical Fitting ideals of the fundamental module. This gives a presentation-independent notion of hyper-Fitting ideals. We also introduce hypercomultiplication hypermodules, show that their fundamental modules are classical comultiplication modules, and obtain criteria for fundamental cyclicity, simplicity, and semisimplicity.

Keywords: Krasner hypermodules, Hyper-Fitting ideals, Fundamental modules, Comultiplication modules, Hypercomultiplication hypermodules.
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1 Introduction

Fitting ideals are classical invariants associated with finitely generated modules over commutative rings. Defined from finite presentations, they play an important role in the study of annihilators, supports, generators, and projectivity; see, for example, [4, 7, 12, 13]. The first nonzero Fitting ideal has also proved useful in the study of structural properties of modules; see [8–10].

The aim of this paper is twofold. First, we define hyper-Fitting ideals for Krasner hypermodules admitting finite free presentations via an internal determinantal construction. We then show that, after fundamental saturation, these hyperideals agree with the pullbacks of the classical Fitting ideals of the fundamental module. Second, motivated by the theory of comultiplication modules, we introduce hypercomultiplication hypermodules and study their hyper-Fitting ideals through the fundamental module.

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Comultiplication modules form the dual counterpart of multiplication modules. The classical theory shows that finitely generated comultiplication modules satisfy strong structural properties: over integral domains, valuation rings, Dedekind domains, and Prüfer domains one obtains sharp conclusions on the first nonzero Fitting ideal, simplicity, cyclicity, and semisimplicity; see [2,3,11]. Our goal is to obtain hypermodule-theoretic analogues of these results by passing through the fundamental module. The resulting statements are naturally formulated in terms of fundamental cyclicity, fundamental simplicity, and fundamental semisimplicity.

The paper is organized as follows. In Section 2 we recall the fundamental ring and the fundamental module. In Section 3 we introduce the internal determinantal construction of hyper-Fitting ideals and compare it with the pullback of the classical Fitting ideals of the fundamental module. In Section 4 we define hypercomultiplication hypermodules and show that the fundamental module of such a hypermodule is a classical comultiplication module. In Section 5 we derive hypermodule-theoretic analogues of several results on Fitting ideals of finitely generated comultiplication modules.

2 Preliminaries

We use the standard notions of commutative hyperrings with identity, hyperideals, and hypermodules; see, for example, [5,6]. We only recall here the fundamental equivalence relations and the associated fundamental ring and module, which are central to our approach.

Let R be a commutative hyperring with identity and let M be an R-hypermodule. We denote by γ^* the fundamental equivalence relation on R, namely the smallest equivalence relation such that the quotient R/γ^* is a commutative ring. Likewise, we denote by ε^* the corresponding fundamental equivalence relation on M, namely the smallest equivalence relation such that the quotient M/ε^* becomes a module over R/γ^*; see [1,14].

The quotient ring

$$\bar{R} := R/\gamma^*$$

is called the fundamental ring of R, and

$$\bar{M} := M/\varepsilon^*$$

is called the fundamental module of M.

3 Hyper-Fitting ideals

In this section, we first introduce an internal determinantal construction of hyper-Fitting ideals for Krasner hypermodules admitting finite free presentations. We then show that the fundamental saturation of this construction coincides with the pullback of the classical Fitting ideals of the fundamental module. This yields a presentation-independent notion of hyper-Fitting ideals and shows that, in the Krasner finitely presented setting, the pullback construction is the fundamental saturation of an internal determinantal construction.

Let R be a commutative hyperring with identity and let M be an R-hypermodule. Set

$$\bar{R} := R/\gamma^*, \qquad \bar{M} := M/\varepsilon^*, \tag{1}$$

and let

$$\pi : R \to \bar{R}, \qquad \rho : M \to \bar{M} \tag{2}$$

be the canonical quotient maps.

3.1 Internal determinantal construction

Definition 1. *An R-hypermodule M is called a* Krasner R-hypermodule *if the scalar multiplication*

$$R \times M \longrightarrow M, \qquad (r, x) \longmapsto rx$$

is single-valued.

Throughout this subsection, assume that M is a Krasner R-hypermodule admitting a finite free presentation

$$\mathcal{P} : \qquad R^m \xrightarrow{\varphi} R^n \twoheadrightarrow M \to 0. \tag{3}$$

Let

$$A = (a_{ij}) \in M_{n\times m}(R) \tag{4}$$

be the matrix of φ with respect to the standard bases of R^m and R^n.

Definition 2. *Let $B = (b_{rs})$ be a $t \times t$ matrix over R. We define the* hyperdeterminant *of B by*

$$\operatorname{hdet}(B) := \boxplus_{\sigma\in S_t} \operatorname{sgn}(\sigma) \prod_{r=1}^{t} b_{r,\sigma(r)}. \tag{5}$$

Here $\boxplus$ denotes the iterated hyperaddition in R. Since multiplication in R is single-valued, each term

$$\operatorname{sgn}(\sigma) \prod_{r=1}^{t} b_{r,\sigma(r)}$$

is a well-defined element of R, and therefore $\operatorname{hdet}(B)$ *is a well-defined nonempty subset of R.*

Definition 3. *For each integer $i \geq 0$, define the* i-th determinantal hyper-Fitting hyperideal associated with the presentation $\mathcal{P}$ *by*

$$\operatorname{HFitt}_i^{\mathcal{P}}(M) := \left\langle \bigcup_{B\in \operatorname{Min}_{n-i}(A)} \operatorname{hdet}(B) \right\rangle_h, \tag{6}$$

where $\operatorname{Min}_{n-i}(A)$ *denotes the set of all $(n-i) \times (n-i)$ submatrices of A, and $\langle\cdot\rangle_h$ denotes the hyperideal generated by a subset of R.*

By convention, if $n - i > m$, then

$$\operatorname{HFitt}_i^{\mathcal{P}}(M) := \gamma^*(0), \tag{7}$$

and if $i \geq n$, then

$$\operatorname{HFitt}_i^{\mathcal{P}}(M) := R. \tag{8}$$

Remark 1. *The hyperideal* $\mathrm{HFitt}_i^{\mathcal{P}}(M)$ *is defined intrinsically from a finite free presentation of* M *and does not involve the fundamental module in its construction. A priori, however, it may depend on the chosen presentation* $\mathcal{P}$.

Definition 4. *For any hyperideal* $I \subseteq R$*, define its* fundamental saturation *by*

$$\mathrm{Sat}_{\gamma^*}(I) := \pi^{-1}(\pi(I)). \tag{9}$$

For a Krasner hypermodule M *admitting a finite free presentation* $\mathcal{P}$*, define*

$$\mathrm{HFitt}_i(M) := \mathrm{Sat}_{\gamma^*}\big(\mathrm{HFitt}_i^{\mathcal{P}}(M)\big). \tag{10}$$

Lemma 1. *Let* $B = (b_{rs})$ *be a* $t \times t$ *matrix over* R*, and let*

$$\bar{B} = (\pi(b_{rs})) \in M_t(\bar{R}). \tag{11}$$

Then

$$\pi(\mathrm{hdet}(B)) = \{\det(\bar{B})\}. \tag{12}$$

Proof. For each $\sigma \in S_t$, set

$$u_\sigma := \mathrm{sgn}(\sigma) \prod_{r=1}^{t} b_{r,\sigma(r)}. \tag{13}$$

Then, by Definition 2,

$$\mathrm{hdet}(B) = \boxplus_{\sigma \in S_t} u_\sigma.$$

Let $x \in \mathrm{hdet}(B)$. Passing to the fundamental ring $\bar{R}$, the hyperaddition becomes ordinary addition, so we obtain

$$\pi(x) = \sum_{\sigma \in S_t} \pi(u_\sigma).$$

Since π preserves multiplication and additive inverses, it follows from (13) that

$$\pi(x) = \sum_{\sigma \in S_t} \mathrm{sgn}(\sigma) \prod_{r=1}^{t} \pi(b_{r,\sigma(r)}) = \det(\bar{B}).$$

Thus every element of $\mathrm{hdet}(B)$ has the same image in $\bar{R}$, namely $\det(\bar{B})$. Therefore (12) holds. $\square$

Lemma 2. *Let* $X \subseteq R$*. Then*

$$\pi(\langle X \rangle_h) = (\pi(X)) \tag{14}$$

as ideals of $\bar{R}$.

Proof. Since $\langle X \rangle_h$ is a hyperideal of R containing X, its image $\pi(\langle X \rangle_h)$ is an ideal of $\bar{R}$ containing $\pi(X)$. Hence,

$$(\pi(X)) \subseteq \pi(\langle X \rangle_h).$$

Conversely, let

$$I := \pi^{-1}\big((\pi(X))\big).$$

Then I is a hyperideal of R containing X. By the minimality of $\langle X\rangle_h$, we obtain

$$\langle X\rangle_h \subseteq I.$$

Applying π yields

$$\pi(\langle X\rangle_h) \subseteq \pi(I) = (\pi(X)).$$

Therefore (14) holds. □

Proposition 1. *For every integer $i \geq 0$,*

$$\pi\big(\mathrm{HFitt}_i^{\mathcal{P}}(M)\big) = \mathrm{Fitt}^i_{\bar{R}}(\bar{M}). \tag{15}$$

Proof. Applying the canonical quotient maps to the finite free presentation

$$R^m \xrightarrow{\varphi} R^n \twoheadrightarrow M \to 0,$$

we obtain an induced finite presentation of the $\bar{R}$-module $\bar{M}$,

$$\bar{R}^m \xrightarrow{\bar{\varphi}} \bar{R}^n \twoheadrightarrow \bar{M} \to 0,$$

whose presentation matrix is

$$\bar{A} = (\pi(a_{ij})).$$

Assume first that $0 \leq i < n$. Set

$$X_i := \bigcup_{B \in \mathrm{Min}_{n-i}(A)} \mathrm{hdet}(B).$$

Then, by Definition 3,

$$\mathrm{HFitt}_i^{\mathcal{P}}(M) = \langle X_i\rangle_h.$$

Using Lemma 2 and Lemma 1, we obtain

$$\pi\big(\mathrm{HFitt}_i^{\mathcal{P}}(M)\big) = \pi(\langle X_i\rangle_h) = (\pi(X_i)) = \big(\det(\bar{B}) : B \in \mathrm{Min}_{n-i}(A)\big),$$

which is exactly the ideal of $\bar{R}$ generated by all $(n-i)$-minors of $\bar{A}$. Hence

$$\pi\big(\mathrm{HFitt}_i^{\mathcal{P}}(M)\big) = \mathrm{Fitt}^i_{\bar{R}}(\bar{M}).$$

If $n - i > m$, then by convention (7),

$$\mathrm{HFitt}_i^{\mathcal{P}}(M) = \gamma^*(0),$$

and therefore

$$\pi\big(\mathrm{HFitt}_i^{\mathcal{P}}(M)\big) = 0 = \mathrm{Fitt}^i_{\bar{R}}(\bar{M}).$$

If $i \geq n$, then by convention (8),

$$\mathrm{HFitt}_i^{\mathcal{P}}(M) = R,$$

so

$$\pi\big(\mathrm{HFitt}_i^{\mathcal{P}}(M)\big) = \bar{R} = \mathrm{Fitt}^i_{\bar{R}}(\bar{M}).$$

This completes the proof. □

Theorem 1. *Let M be a Krasner R-hypermodule admitting a finite free presentation $\mathcal{P}$. Then for every integer $i \geq 0$,*

$$\mathrm{HFitt}_i(M) = \pi^{-1}\big(\mathrm{Fitt}^i_{\bar{R}}(\bar{M})\big). \tag{16}$$

In particular, $\mathrm{HFitt}_i(M)$ *is independent of the choice of the finite free presentation* $\mathcal{P}$.

Proof. By Definitions 4 and 3,

$$\mathrm{HFitt}_i(M) = \mathrm{Sat}_{\gamma^*}\big(\mathrm{HFitt}_i^{\mathcal{P}}(M)\big) = \pi^{-1}\big(\pi(\mathrm{HFitt}_i^{\mathcal{P}}(M))\big).$$

Now Proposition 1 yields

$$\pi\big(\mathrm{HFitt}_i^{\mathcal{P}}(M)\big) = \mathrm{Fitt}^i_{\bar{R}}(\bar{M}).$$

Therefore (16) follows. Since the right-hand side depends only on $\bar{M}$, the hyperideal $\mathrm{HFitt}_i(M)$ is independent of the chosen presentation $\mathcal{P}$. □

Remark 2. *Theorem 1 shows that the hyper-Fitting ideals are not introduced merely by a formal pullback from the fundamental module. Rather, for Krasner hypermodules admitting finite free presentations, they arise from an intrinsic determinantal construction in the hyperring setting and become presentation-independent after fundamental saturation. Thus the pullback formula* (16) *is the canonical stabilized form of an internal construction.*

3.2 General properties through the fundamental module

From this point on, let M be an arbitrary R-hypermodule such that $\bar{M}$ is finitely presented as an $\bar{R}$-module. We work with the presentation-independent hyper-Fitting ideals

$$\mathrm{HFitt}_i(M) := \pi^{-1}\big(\mathrm{Fitt}^i_{\bar{R}}(\bar{M})\big). \tag{17}$$

Proposition 2. *Let M be an R-hypermodule such that $\bar{M}$ is finitely presented over $\bar{R}$. Then*

(1) for every $i \geq 0$, $\mathrm{HFitt}_i(M)$ is a hyperideal of R;

(2) the family $\{\mathrm{HFitt}_i(M)\}_{i\geq 0}$ is ascending:

$$\mathrm{HFitt}_0(M) \subseteq \mathrm{HFitt}_1(M) \subseteq \mathrm{HFitt}_2(M) \subseteq \cdots ; \tag{18}$$

(3) the ideals $\mathrm{HFitt}_i(M)$ are independent of the choice of a presentation of $\bar{M}$;

(4) for every $i \geq 0$,

$$\mathrm{HFitt}_i(M) = \gamma^*(0) \iff \mathrm{Fitt}^i_{\bar{R}}(\bar{M}) = 0. \tag{19}$$

Proof. Since $\pi : R \to \bar{R}$ is the canonical surjective hyperring homomorphism, the inverse image of an ideal of $\bar{R}$ under π is a hyperideal of R. Hence, by (17), each $\mathrm{HFitt}_i(M)$ is a hyperideal of R, proving (1).

Inverse images preserve inclusions, so from

$$\mathrm{Fitt}^0_{\bar{R}}(\bar{M}) \subseteq \mathrm{Fitt}^1_{\bar{R}}(\bar{M}) \subseteq \mathrm{Fitt}^2_{\bar{R}}(\bar{M}) \subseteq \cdots ,$$

we obtain (18), proving (2).

Since the classical Fitting ideals of $\bar{M}$ are independent of the chosen presentation of $\bar{M}$, the same holds for their inverse images under π, proving (3).

Finally,

$$\mathrm{HFitt}_i(M) = \gamma^*(0) \iff \pi^{-1}\big(\mathrm{Fitt}^i_{\bar{R}}(\bar{M})\big) = \pi^{-1}(0).$$

Because π is surjective, inverse images distinguish ideals of $\bar{R}$. Hence (19) follows, proving (4). □

Since $\bar{M}$ is finitely presented over $\bar{R}$, its classical Fitting ideals are eventually equal to $\bar{R}$. Hence the set

$$\{\, i \geq 0 \mid \mathrm{HFitt}_i(M) \neq \gamma^*(0) \,\}$$

is nonempty, and the following definition is well posed.

Definition 5. *Define*

$$sh(M) := \min\{\, i \geq 0 \mid \mathrm{HFitt}_i(M) \neq \gamma^*(0) \,\}. \tag{20}$$

Equivalently,

$$sh(M) = \min\{\, i \geq 0 \mid \mathrm{Fitt}^i_{\bar{R}}(\bar{M}) \neq 0 \,\}. \tag{21}$$

The first nontrivial hyper-Fitting ideal *of M is*

$$I_h(M) := \mathrm{HFitt}_{sh(M)}(M). \tag{22}$$

Proposition 3 (Compatibility with the classical case)**.** *Let R be an ordinary commutative ring, viewed as a hyperring with singleton addition, and let M be an ordinary R-module. Then*

$$\mathrm{HFitt}_i(M) = \mathrm{Fitt}^i_R(M) \qquad \textit{for all } i \geq 0. \tag{23}$$

In particular,

$$I_h(M) = I(M), \tag{24}$$

where $I(M)$ denotes the first nonzero Fitting ideal of M.

Proof. In the classical situation, the fundamental relations are trivial, so

$$\bar{R} = R, \qquad \bar{M} = M, \qquad \gamma^*(0) = 0.$$

Therefore, by (17),

$$\mathrm{HFitt}_i(M) = \pi^{-1}\big(\mathrm{Fitt}^i_R(M)\big) = \mathrm{Fitt}^i_R(M)$$

for all $i \geq 0$, and then (24) follows from Definition 5. □

Proposition 4. *Let M be an R-hypermodule such that $\bar{M}$ is finitely presented over $\bar{R}$. Then*

$$I_h(M) = \pi^{-1}\big(I(\bar{M})\big), \tag{25}$$

where $I(\bar{M})$ denotes the first nonzero Fitting ideal of the $\bar{R}$-module $\bar{M}$.

Proof. By (17),

$$\mathrm{HFitt}_i(M) = \pi^{-1}\big(\mathrm{Fitt}^i_{\bar{R}}(\bar{M})\big).$$

By Proposition 2, specifically (4),

$$\mathrm{HFitt}_i(M) \neq \gamma^*(0) \iff \mathrm{Fitt}^i_{\bar{R}}(\bar{M}) \neq 0.$$

Therefore, the first index at which the family $\{\mathrm{HFitt}_i(M)\}_{i\geq 0}$ becomes nontrivial is exactly the first index at which the family $\{\mathrm{Fitt}^i_{\bar{R}}(\bar{M})\}_{i\geq 0}$ becomes nonzero.Thus, (25) follows directly from Definition 5. □

Definition 6. *The fundamental annihilator of M is defined by*

$$\mathrm{Ann}_h(M) := \pi^{-1}\big(\mathrm{Ann}_{\bar{R}}(\bar{M})\big). \tag{26}$$

Since $\mathrm{Ann}_{\bar{R}}(\bar{M})$ is an ideal of $\bar{R}$, it follows that $\mathrm{Ann}_h(M)$ is a hyperideal of R.

Lemma 3. *One has*

$$\mathrm{Ann}_h(M) = \{\, r \in R \mid rx \subseteq \varepsilon^*(0) \text{ for all } x \in M \,\}. \tag{27}$$

Proof. Let $r \in R$. Then

$$r \in \mathrm{Ann}_h(M) \iff \pi(r) \in \mathrm{Ann}_{\bar{R}}(\bar{M}) \iff \pi(r)\bar{x} = 0 \quad \text{for all } \bar{x} \in \bar{M}.$$

Now, let $x \in M$. For every $y \in rx$, one has

$$\bar{y} = \pi(r)\bar{x}$$

in $\bar{M}$. Hence,

$$\pi(r)\bar{x} = 0 \iff \bar{y} = 0 \quad \text{for all } y \in rx \iff rx \subseteq \varepsilon^*(0).$$

Therefore (27) follows. □

4 Hypercomultiplication hypermodules

We now introduce the hypermodule analogue of comultiplication modules and prepare the passage to the fundamental module.

Definition 7. *Let I be a hyperideal of R. Define*

$$(0 :_M I) := \{\, x \in M \mid ax \subseteq \varepsilon^*(0) \text{ for all } a \in I \,\}. \tag{28}$$

Lemma 4. *Let I be a hyperideal of R. Then $(0 :_M I)$ is a subhypermodule of M, and*

$$\rho\big((0 :_M I)\big) = (0 :_{\bar{M}} \pi(I)). \tag{29}$$

Proof. We first show that $(0 :_M I)$ is a subhypermodule of M. Clearly $0 \in (0 :_M I)$, since

$$a0 = 0 \subseteq \varepsilon^*(0) \qquad \text{for all } a \in I.$$

Now, let $x, y \in (0 :_M I)$ and let $z \in x + y$. For every $a \in I$, we have

$$az \subseteq a(x+y) \subseteq ax + ay \subseteq \varepsilon^*(0) + \varepsilon^*(0) \subseteq \varepsilon^*(0).$$

Hence $z \in (0 :_M I)$. Also, if $x \in (0 :_M I)$, then for every $a \in I$,

$$a(-x) = -(ax) \subseteq \varepsilon^*(0),$$

so $-x \in (0 :_M I)$. Therefore $(0 :_M I)$ is a subhypermodule of M.

We now prove (29). Let $\bar{x} \in \rho((0 :_M I))$. Then there exists $x \in (0 :_M I)$ such that $\rho(x) = \bar{x}$. For every $a \in I$ and every $y \in ax$, one has

$$\bar{y} = \pi(a)\bar{x}.$$

Since $x \in (0 :_M I)$, we have $ax \subseteq \varepsilon^*(0)$, hence $\bar{y} = 0$. Therefore,

$$\pi(a)\bar{x} = 0 \qquad \text{for all } a \in I,$$

which shows that $\bar{x} \in (0 :_{\bar{M}} \pi(I))$. Thus

$$\rho((0 :_M I)) \subseteq (0 :_{\bar{M}} \pi(I)).$$

Conversely, let $\bar{x} \in (0 :_{\bar{M}} \pi(I))$, and choose $x \in M$ such that $\rho(x) = \bar{x}$. Then for every $a \in I$,

$$\pi(a)\bar{x} = 0.$$

If $y \in ax$, then

$$\bar{y} = \pi(a)\bar{x} = 0,$$

so $y \in \varepsilon^*(0)$. Hence $ax \subseteq \varepsilon^*(0)$ for every $a \in I$, and therefore $x \in (0 :_M I)$. It follows that

$$\bar{x} = \rho(x) \in \rho((0 :_M I)).$$

Thus (29) follows. □

Definition 8. *An R-hypermodule M is called a* hypercomultiplication hypermodule *if for every subhypermodule N of M there exists a hyperideal I of R such that*

$$N = (0 :_M I). \tag{30}$$

Proposition 5. *If M is a hypercomultiplication R-hypermodule, then $\bar{M}$ is a comultiplication $\bar{R}$-module.*

Proof. Let L be a submodule of $\bar{M}$. Set

$$N := \rho^{-1}(L).$$

Then N is a subhypermodule of M. Since M is hypercomultiplication, there exists a hyperideal I of R such that

$$N = (0 :_M I).$$

Applying ρ and using (29), we obtain

$$L = \rho(N) = \rho((0 :_M I)) = (0 :_{\bar{M}} \pi(I)).$$

Therefore every submodule of $\bar{M}$ is of the form $(0 :_{\bar{M}} J)$ for some ideal J of $\bar{R}$, and hence $\bar{M}$ is a comultiplication $\bar{R}$-module. □

5 Krasner hypercomultiplication hypermodules

Throughout this section, assume that M is a Krasner hypercomultiplication R-hypermodule such that $\bar{M}$ is finitely presented as an $\bar{R}$-module. The results of this section may be viewed as a transfer principle from the classical theory of finitely generated comultiplication modules to the setting of Krasner hypercomultiplication hypermodules. The key point is that the fundamental module provides the correct bridge for transporting Fitting-theoretic information back to the hypermodule level via pullback hyperideals.

Definition 9. *We say that M is* fundamentally cyclic *if $\bar{M}$ is a cyclic $\bar{R}$-module. We say that M is* fundamentally simple *if $\bar{M}$ is a simple $\bar{R}$-module. We say that M is* fundamentally semisimple *if $\bar{M}$ is a semisimple $\bar{R}$-module.*

Theorem 2. *Assume that $\bar{R}$ is an integral domain. Then either*

$$I_h(M) = \mathrm{HFitt}_0(M), \tag{31}$$

or

$$\bar{M} \cong \bar{R}. \tag{32}$$

Proof. By Proposition 5, the $\bar{R}$-module $\bar{M}$ is a finitely generated comultiplication module. Since $\bar{R}$ is an integral domain, Theorem 1.3 of [11] yields either

$$I(\bar{M}) = \mathrm{Fitt}^0_{\bar{R}}(\bar{M}),$$

or

$$\bar{M} \cong \bar{R}.$$

In the first case, by (17) and (25), we obtain

$$I_h(M) = \pi^{-1}\big(I(\bar{M})\big) = \pi^{-1}\big(\mathrm{Fitt}^0_{\bar{R}}(\bar{M})\big) = \mathrm{HFitt}_0(M).$$

Hence (31) holds. In the second case we obtain (32). □

Corollary 1. *Assume that $\bar{R}$ is an integral domain and that $\pi(I_h(M))$ is a prime ideal of $\bar{R}$. Then M is fundamentally simple.*

Proof. By (25), we have

$$\pi\big(I_h(M)\big) = I(\bar{M}).$$

Hence $I(\bar{M})$ is a prime ideal of $\bar{R}$. Since $\bar{M}$ is a finitely generated comultiplication $\bar{R}$-module by Proposition 5, Proposition 1.6 of [11] implies that $\bar{M}$ is a simple $\bar{R}$-module. Therefore M is fundamentally simple by Definition 9. □

Proposition 6. *Assume that $\bar{R}$ is a valuation ring. Then M is fundamentally cyclic.*

Proof. By Proposition 5, the $\bar{R}$-module $\bar{M}$ is a finitely generated comultiplication module. Since $\bar{R}$ is a valuation ring, Proposition 1.7 of [11] implies that $\bar{M}$ is cyclic. Therefore, M is fundamentally cyclic by Definition 9. □

Proposition 7. *Assume that $\bar{R}$ is a Dedekind domain. Then M is fundamentally cyclic.*

Proof. By Proposition 5, the $\bar{R}$-module $\bar{M}$ is a finitely generated comultiplication module. Hence Lemma 1.8 of [11] implies that $\bar{M}$ is cyclic. Therefore M is fundamentally cyclic by Definition 9. □

Proposition 8. *The following hold:*

(1) If $\pi(I_h(M))$ is a prime ideal of $\bar{R}$, then

$$\mathrm{Ann}_h(M) \subseteq I_h(M). \tag{33}$$

(2) If

$$\pi(I_h(M)) = \mathfrak{m}_1 \cdots \mathfrak{m}_n, \tag{34}$$

where $\mathfrak{m}_1, \dots, \mathfrak{m}_n$ are distinct maximal ideals of $\bar{R}$, then

$$\mathrm{Ann}_h(M) \subseteq I_h(M). \tag{35}$$

Proof. By Definition 6 and Proposition 4, we have

$$\mathrm{Ann}_h(M) = \pi^{-1}\big(\mathrm{Ann}_{\bar{R}}(\bar{M})\big) \qquad \text{and} \qquad I_h(M) = \pi^{-1}\big(I(\bar{M})\big).$$

For (1), if $\pi(I_h(M))$ is a prime ideal of $\bar{R}$, then $I(\bar{M})$ is a prime ideal of $\bar{R}$. By Theorem 1.12(i) of [11],

$$\mathrm{Ann}_{\bar{R}}(\bar{M}) \subseteq I(\bar{M}).$$

Pulling back through π, we obtain (33).

For (2), if (34) holds, then Theorem 1.12(ii) of [11] yields

$$\mathrm{Ann}_{\bar{R}}(\bar{M}) \subseteq I(\bar{M}).$$

Pulling back through π, we obtain (35). □

Theorem 3. *Assume that*

$$\mathrm{Ann}_h(M) = \pi^{-1}(\mathfrak{m}_1 \cdots \mathfrak{m}_n), \tag{36}$$

where $\mathfrak{m}_1, \ldots, \mathfrak{m}_n$ *are distinct maximal ideals of* $\bar{R}$. *Then*

$$\bar{M} \cong \bar{R}/\mathfrak{m}_1 \oplus \cdots \oplus \bar{R}/\mathfrak{m}_n. \tag{37}$$

In particular, M *is fundamentally semisimple.*

Proof. By Definition 6, (36) is equivalent to

$$\mathrm{Ann}_{\bar{R}}(\bar{M}) = \mathfrak{m}_1 \cdots \mathfrak{m}_n.$$

By Proposition 5, the $\bar{R}$-module $\bar{M}$ is a finitely generated comultiplication module. Therefore Theorem 1.14 of [11] implies (37). The final assertion follows from Definition 9. □

Corollary 2. *Assume that*

$$\mathrm{HFitt}_0(M) = \pi^{-1}(\mathfrak{m}_1 \cdots \mathfrak{m}_n), \tag{38}$$

where $\mathfrak{m}_1, \ldots, \mathfrak{m}_n$ *are distinct maximal ideals of* $\bar{R}$. *Then* M *is fundamentally semisimple.*

Proof. By (17), (38) is equivalent to

$$\mathrm{Fitt}^0_{\bar{R}}(\bar{M}) = \mathfrak{m}_1 \cdots \mathfrak{m}_n.$$

Since $\bar{M}$ is a finitely generated comultiplication module by Proposition 5, Theorem 1.16 of [11] implies that $\bar{M}$ is semisimple. Therefore, M is fundamentally semisimple by Definition 9. □

Proposition 9. *Assume that* $\bar{R}$ *is a Prüfer domain and*

$$\mathrm{HFitt}_0(M) = \pi^{-1}(\mathfrak{q}^n), \tag{39}$$

where $\mathfrak{q}$ *is a maximal ideal of* $\bar{R}$ *and* n *is a positive integer. Then* M *is fundamentally cyclic.*

Proof. By (17), (39) is equivalent to

$$\mathrm{Fitt}^0_{\bar{R}}(\bar{M}) = \mathfrak{q}^n.$$

By Proposition 5, the $\bar{R}$-module $\bar{M}$ is a finitely generated comultiplication module. Since $\bar{R}$ is a Prüfer domain, Theorem 1.21 of [11] implies that $\bar{M}$ is cyclic. Therefore M is fundamentally cyclic by Definition 9. □

Proposition 10. *If* $\bar{R}$ *is not semilocal, then*

$$I_h(M) = \mathrm{HFitt}_0(M). \tag{40}$$

Proof. By Proposition 5, the $\bar{R}$-module $\bar{M}$ is a finitely generated comultiplication module. Since $\bar{R}$ is not semilocal, Corollary 1.23 of [11] yields

$$I(\bar{M}) = \mathrm{Fitt}^0_{\bar{R}}(\bar{M}).$$

Now (40) follows from (17) and (25). □

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