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Existence and regularity for nonlinear ψ-Hilfer fractional elliptic problems with variable exponents and singular terms

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Abstract. We investigate a nonlinear elliptic problem featuring a singular nonlinearity, lower-order terms, and L^1-data in the setting of ψ-Hilfer fractional Sobolev spaces with variable exponents on a bounded interval. Our analysis relies on the ψ-Hilfer fractional derivative operator. We establish the existence of weak positive solutions and derive refined regularity estimates. A key finding is the regularizing effect induced by the lower-order terms: concretely, they enable the solution to belong to a better Sobolev space than would otherwise be possible, thereby overcoming the limitations ordinarily imposed by the singular nonlinearity and the roughness of the data. The proof combines variational techniques, truncation arguments, and compactness methods tailored to the variable-exponent fractional framework.

Keywords: Elliptic fractional problem, existence and regularity, ψ-Hilfer fractional derivative, variable exponents, singular nonlinearity.
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1 Introduction and main results

Analysis of elliptic partial differential equations with fractional operators has recently generated significant interest, including the study of nonlocal and nonsmooth diffusions. In particular, among different methods, the ψ-Hilfer fractional derivative framework constitutes a general and unifying one that contains many known fractional operators as special cases.

In this study, we focus on proving the existence and regularity of positive weak solutions for a family of nonlinear fractional elliptic problems involving singular nonlinear effects, additional lower-order contributions, and data belonging to L^1, formulated within variable exponent ψ-Hilfer fractional function spaces. In particular, we show that the inclusion of the lower-order term enhances the regularity of the solutions, thereby extending and generalizing the results previously obtained in [7].

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A number of contributions have addressed the existence and regularity of solutions to nonlinear elliptic and parabolic problems with singular terms and lower-order nonlinearities.
A first relevant contribution was given in [1]. In this work, the authors considered the Dirichlet problem

$$-\operatorname{div}(a(z)|\nabla\xi|^{\varsigma-2}\nabla\xi)+\xi|\xi|^{r-1}=\frac{g(z)}{\xi^{\vartheta}} \quad \text{in } \mathscr{U}, \qquad \xi>0 \text{ in } \mathscr{U}, \qquad \xi=0 \text{ on } \partial\mathscr{U},$$

and proved the existence of positive weak solutions. They showed that the lower-order term $\xi|\xi|^{r-1}$ has a regularizing effect, partially compensating for the singular term $\xi^{-\vartheta}$ and improving the regularity of the solution.
Another significant result was obtained by Sbai and El Hadfi in [11]. They studied the singular problem

$$-\operatorname{div}(a(z,\xi)|\nabla\xi|^{\varsigma-2}\nabla\xi)+|\xi|^{\sigma-1}\xi=h(\xi)g \quad \text{in } \mathscr{U}, \qquad \xi\geq 0 \text{ in } \mathscr{U}, \qquad \xi=0 \text{ on } \partial\mathscr{U},$$

involving a degenerate coercive operator. They established existence and regularity of nonnegative weak solutions and demonstrated that the absorption term $|u|^{s-1}u$ provides a regularizing effect even in the degenerate setting. The work [2], focused on the problem

$$-\Delta_{\varsigma}\xi=\frac{g(z)}{\xi^{\gamma}} \quad \text{in } \mathscr{U}, \qquad \xi>0 \text{ in } \mathscr{U}, \qquad \xi=0 \text{ on } \partial\mathscr{U},$$

and proved comparison principles as well as existence and uniqueness of weak solutions. This study highlights that singular nonlinearities of the type $\xi^{-\gamma}$ still allow well-posedness under suitable assumptions.
Chu et al. in [3], considered a quasilinear problem with a variable source term depending on the solution

$$-\operatorname{div}(A(z,\xi,\nabla\xi))=g(z,\xi) \quad \text{in } \mathscr{U}, \qquad \xi=0 \text{ on } \partial\mathscr{U}.$$

They obtained existence and regularity results, showing that the nonlinear source contributes to improving the qualitative behavior of solutions, even under nonstandard growth conditions.
The degenerate singular problem analyzed in [10], involves

$$-\operatorname{div}(a(z,\xi,\nabla\xi))=\frac{g}{\xi^{\gamma}} \quad \text{in } \mathscr{U}, \qquad \xi>0 \text{ in } \mathscr{U}, \qquad \xi=0 \text{ on } \partial\mathscr{U},$$

with $\gamma>0$ and $f\geq 0$. The authors proved existence and regularity of solutions despite the degeneracy of the operator and the singularity of the right-hand side, illustrating the delicate interplay between operator degeneracy and singular nonlinearities.
Finally, in [4], the authors studied

$$-\Delta\xi+\xi^{\sigma}=\frac{g(z)}{\xi^{\gamma}} \quad \text{in } \mathscr{U}, \qquad \xi>0 \text{ in } \mathscr{U}, \qquad \xi=0 \text{ on } \partial\mathscr{U},$$

with $0<\gamma\leq 1$, $s\geq 1$, and f in a Lebesgue space. They established existence and regularity results and demonstrated that the absorption term ξ^{s} mitigates the singularity $\xi^{-\gamma}$, providing a regularizing effect.
In particular, [7] extends these results to the framework of variable exponent Sobolev spaces:

$$-\operatorname{div}(a(z)|\nabla\xi|^{\varsigma(z)-2}\nabla\xi)+b(z)\xi|\xi|^{r(x)-1}=\frac{g}{\xi^{\gamma(z)}} \quad \text{in } \mathscr{U}, \qquad \xi>0 \text{ in } \mathscr{U}, \qquad \xi=0 \text{ on } \partial\mathscr{U},$$

where $f \in L^1(\mathscr{U})$ is positive and the exponents $\varsigma(z), r(z), \gamma(z)$ are continuous functions, the authors proved that the lower-order term $b(z)\xi|\xi|^{r(z)-1}$ has a regularizing effect. This highlights the delicate balance between the singular terms and the variable exponents, providing a solid theoretical foundation that strongly motivates our present study of more general fractional singular problems.
Motivated by these developments, and also inspired by recent works on ψ-Hilfer fractional operators [6, 8, 9, 18, 19], we study the nonlinear elliptic problem given by

$$(\mathscr{P}) \quad \begin{cases} -{}^{\mathbf{H}}\mathscr{D}_T^{\alpha,\beta;\psi}\left(\sigma(z)\left|{}^{\mathbf{H}}\mathscr{D}_{0^+}^{\alpha,\beta;\psi}\xi\right|^{\varsigma(z)-2}\,{}^{\mathbf{H}}\mathscr{D}_{0^+}^{\alpha,\beta;\psi}\xi\right)+\tau(z)\xi|\xi|^{\gamma(z)-1}=\frac{g}{\xi^{\vartheta(z)}}, & \text{in } \mathscr{U}, \\ \xi(z)>0, & \text{in } \mathscr{U}, \\ \xi(z)=0, & \text{on } \partial\mathscr{U}=\{a,b\}, \end{cases}$$

where ${}^{\mathbf{H}}\mathscr{D}_T^{\alpha,\beta;\psi}(\cdot)$ and ${}^{\mathbf{H}}\mathscr{D}_{0^+}^{\alpha,\beta;\psi}(\cdot)$ denote the left and right ψ-Hilfer fractional derivatives of order $\alpha \in (0,1]$ and type $\beta \in [0,1]$. Here, $\mathscr{U}=(a,b)\subset\mathbb{R}$ is a bounded open interval, and we consider the functions ς and γ, defined on $\overline{\mathscr{U}}$ with values in $(0,+\infty)$, together with a function ϑ taking values in $(0,1)$; all are continuous and additionally satisfy

$$1<\varsigma^-=\inf_{z\in\bar{\mathscr{U}}}\varsigma\leq\varsigma^+=\sup_{z\in\bar{\mathscr{U}}}\varsigma<+\infty, \tag{1}$$

$$\varsigma(z)-1<\gamma(z), \tag{2}$$

$$0<\vartheta^-=\inf_{z\in\bar{\mathscr{U}}}\vartheta\leq\vartheta^+=\sup_{z\in\bar{\mathscr{U}}}\vartheta<1. \tag{3}$$

Assume that the source term satisfies $g \in L^1(\mathscr{U})\cap L^\infty(\mathscr{U})$ with $g \geq 0$, and g is not identically zero a.e in $\mathscr{U}$. Let $\sigma(z)$ and $\tau(z)$ be measurable functions such that

$$0<\underline{\sigma}\leq\sigma(z)\leq\overline{\sigma}, \quad 0<\underline{\tau}\leq\tau(z)\leq\overline{\tau}, \tag{4}$$

where $\underline{\sigma}, \overline{\sigma}, \underline{\tau}$, and $\overline{\tau}$ are real numbers. We point out that the upper bounds $\overline{\sigma}$ and $\overline{\tau}$ ensure that the coefficients belong to $L^\infty(\mathscr{U})$. To place the present contribution within the existing literature, we summarize in Table 1 the main works related to our problem. The regularizing effect induced by absorption terms was first investigated in the local constant-exponent setting in [1, 4]. Later, Khelifi and El Hadfi [7] extended this analysis to elliptic problems with variable exponents. The present paper further generalizes these results by considering a nonlocal fractional framework driven by a ψ-Hilfer fractional $p(z)$-Laplacian-like operator. In addition, the singularity exponent $\vartheta(z)$, the absorption exponent $\gamma(z)$, and the growth exponent $\varsigma(z)$ are allowed to vary with the spatial variable. Consequently, our result extends the regularizing effect theory from local elliptic equations to a broader class of fractional variable-exponent singular problems.

2 Preliminaries

This section introduces the notations and fundamental concepts required to understand the subsequent results. In the analysis of problem $(\mathscr{P})$, we shall work within the fractional ψ-Hilfer space $\mathbb{H}_{\varsigma(.)}^{\alpha,\beta;\psi}$.
We start by defining the left-sided and right-sided ψ-Riemann-Liouville fractional integrals of a function

Table 1: Comparison with related works

Reference	Operator	Variable exponents	Fractional	Singularity	Absorption term	Main result
[1]	p-Laplacian	No	No	$u^{-\theta}$	u^r	Existence and regularity of weak solutions.
[2]	p-Laplacian	No	No	$u^{-\gamma}$	None	Existence and uniqueness of weak solutions.
[4]	Laplacian	No	No	$u^{-\gamma}$	u^s	Regularizing effect and existence with L^1 data.
[7]	Variable exponent elliptic operator	Yes	No	$u^{-\gamma(x)}$	$u^{r(x)}$	Existence and regularity of positive weak solutions.
Present paper	Fractional ψ-Hilfer $p(z)$-Laplacian-like	Yes	Yes	$\xi^{-\vartheta(z)}$	$\tau(z)\xi\|\xi\|^{\gamma(z)-1}$	Existence in $\mathbb{H}^{\alpha,\beta;\psi}_{\mu(z),0}(\mathscr{U})$ and regularizing effect.

g with respect to another function ψ on $[0;T]$, and the ψ-Hilfer fractional derivative left-sided and right-sided, with $n-1<\alpha<n$, $n\in\mathbb{N}$, and $g,\psi\in C^n([0;T],\mathbb{R})$, $\psi'(x)\neq 0$, for all $x\in[0;T]$, respectively as follows:

$$I^{\alpha;\psi}_{a^+}g(x)=\frac{1}{\Gamma(\alpha)}\int_a^x\psi'(t)(\psi(x)-\psi(t))^{\alpha-1}g(t)\,dt,$$

$$I^{\alpha;\psi}_{b^-}g(x)=\frac{1}{\Gamma(\alpha)}\int_b^x\psi'(t)(\psi(t)-\psi(x))^{\alpha-1}g(t)\,dt,$$

and

$$^{\mathbf{H}}\mathscr{D}^{\alpha,\beta;\psi}_{a+}g(x)=I^{\beta(n-\alpha);\psi}_{a+}\left(\frac{1}{\psi'(x)}\frac{d}{dx}\right)^n I^{(1-\beta)(n-\alpha);\psi}_{a+}g(x),$$

$$^{\mathbf{H}}\mathscr{D}^{\alpha,\beta;\psi}_{b-}g(x)=I^{\beta(n-\alpha);\psi}_{b-}\left(-\frac{1}{\psi'(x)}\frac{d}{dx}\right)^n I^{(1-\beta)(n-\alpha);\psi}_{b-}g(x).$$

Throughout this work, we employ the truncation function T_κ and the convex modular $\rho_{\varsigma(.)}(\xi)$. For every $\kappa\geq 0$ and $q\in\mathbb{R}$, they are defined respectively as

$$T_\kappa(q)=\min\{\kappa,\max(q,-\kappa)\},$$

$$\rho_{\varsigma(.)}(\xi)=\int_{\mathscr{U}}|\xi|^{\varsigma(z)}dz,$$

where $\varsigma:\mathscr{U}\to[1,+\infty)$ is a measurable function for an open $\mathscr{U}\subset\mathbb{R}$, such that

$$1<\varsigma^-=ess\inf\varsigma,\qquad \varsigma^+=ess\sup\varsigma<+\infty.$$

Remark 1. *Since the exponent ς is continuous on the compact set $\bar{\mathscr{U}}$, it satisfies $1 < \varsigma^- \leq \varsigma^+ < \infty$. This uniform boundedness ensures that the modular $\rho_{\varsigma(\cdot)}(\cdot)$ satisfies the Δ_2-condition, guaranteeing that $L^{\varsigma(\cdot)}(\mathscr{U})$ is separable and reflexive (see [5, Section 3.2]).*

Let define ψ-fractional derivative space with variable exponent $\mathbb{H}^{\alpha,\beta;\psi}_{\varsigma(.)}(\mathscr{U})$ with $\alpha \in (0,1]$, $\beta \in [0,1]$ and $1 < \varsigma < \infty$ [12, 14]

$$\mathbb{H}^{\alpha,\beta;\psi}_{\varsigma(.)} = \left\{ \xi \in L^{\varsigma(.)}(\mathscr{U}) : {}^{\mathbf{H}}\mathscr{D}^{\alpha,\beta;\psi}_{0^+}\xi \in L^{\varsigma(.)}(\mathscr{U}) \right\},$$

that is endowed with the norm

$$\|\xi\|_{\mathbb{H}^{\alpha,\beta;\psi}_{\varsigma(.)}} = \left(\|\xi\|^{\varsigma}_{\varsigma(.)} + \|{}^{\mathbf{H}}\mathscr{D}^{\alpha,\beta;\psi}_{0^+}\xi\|^{\varsigma}_{\varsigma(.)}\right)^{1/\varsigma},$$

where ${}^{\mathbf{H}}\mathscr{D}^{\alpha,\beta;\psi}_{0^+}(\cdot)$ denotes the ψ-Hilfer fractional derivative of order $\alpha \in (0,1]$ and type $\beta \in [0,1]$.

Remark 2. *([15]) We define $\mathbb{H}(\mathscr{U}) = \mathbb{H}^{\alpha,\beta;\psi}_{\varsigma(.),0}$ to be the closure of $C^\infty_0(\mathscr{U})$ in $\mathbb{H}^{\alpha,\beta;\psi}_{\varsigma(.)}$, with the equivalent norm $\|\xi\| := \|{}^{\boldsymbol{H}}\mathscr{D}^{\alpha,\beta;\psi}_{0^+}\xi\|_{L^{\varsigma(\cdot)}}$.*

Proposition 1. *([12, 14, 22, 23])*

1. *The space $\mathbb{H}(\mathscr{U})$ is compactly embedded in $L^{\varsigma(\cdot)}(\mathscr{U})$, and this embedding implies, in particular, the compact embedding into $L^{\varsigma^-}(\mathscr{U})$.*
2. *The space $\mathbb{H}(\mathscr{U})$ is continuously embedded in $L^{\varsigma^*_s(\cdot)}(\mathscr{U})$, where $\varsigma^*_s(z)$ is the fractional critical Sobolev exponent.*
3. *Consider $\alpha \in (0,1]$, $\beta \in [0,1]$ and $1 < \varsigma < \infty$. $\mathbb{H}(\mathscr{U})$ is a Banach space that is both separable and reflexive.*

Theorem 1. *([13]) $\mathbb{H}(\mathscr{U})$ is uniformly convex.*

Proposition 2. *([5]) If (ξ_n), $\xi \in L^{\varsigma(\cdot)}(\mathscr{U})$ and $\varsigma^+ < +\infty$, then the properties listed below are satisfied:*

(i) $\min\left(\rho_{\varsigma(\cdot)}(\xi)^{\frac{1}{\varsigma^+}}, \rho_{\varsigma(\cdot)}(\xi)^{\frac{1}{\varsigma^-}}\right) \leq \|\xi\|_{\varsigma(\cdot)} \leq \max\left(\rho_{\varsigma(\cdot)}(\xi)^{\frac{1}{\varsigma^+}}, \rho_{\varsigma(\cdot)}(\xi)^{\frac{1}{\varsigma^-}}\right),$

(ii) $\min\left(\|\xi\|^{\varsigma^-}_{\varsigma(\cdot)}, \|\xi\|^{\varsigma^+}_{\varsigma(\cdot)}\right) \leq \rho_{\varsigma(\cdot)}(\xi) \leq \max\left(\|\xi\|^{\varsigma^-}_{\varsigma(\cdot)}, \|\xi\|^{\varsigma^+}_{\varsigma(\cdot)}\right),$

(iii) $\|\xi\|_{\varsigma(\cdot)} \leq \rho_{\varsigma(\cdot)}(\xi) + 1.$

Proposition 3. *([13, 16])(Poincaré Inequality)*
Let $\mathscr{U}$ be a bounded open subset of $\mathbb{R}^n$. Then there exists a constant $C > 0$ depends only on the exponent function $\varsigma(\cdot)$, the domain $\mathscr{U}$, and the fractional parameters (α,β,ψ), and is finite under the above assumptions, such that for all $\xi \in \mathbb{H}^{\alpha,\beta;\psi}_{\varsigma(.)}(\mathscr{U})$, we have

$$\|\xi\|_{L^{\varsigma(z)}(\mathscr{U})} \leq C\|{}^{\boldsymbol{H}}\mathscr{D}^{\alpha,\beta;\psi}_{0^+}\xi\|_{L^{\varsigma(z)}(\mathscr{U})}.$$

Theorem 2. *([17])(Browder–Minty) Let X be a reflexive, real, separable Banach space, and suppose that $A : X \to X^\star$ is a hemicontinuous, monotone, and coercive operator, such that*

$$A\phi = b, \qquad \phi \in X. \tag{5}$$

1. *For every $b \in X^\star$, equation (5) admits a solution set, the set is both bounded and closed, and moreover convex.*
2. *If A is strictly monotone, then (5) admits a unique solution in X.*

3 Statement of results and proof

In this section, we consider an approximating problem in which the singular term $\frac{1}{\xi^{\vartheta(z)}}$ is truncated, replacing very small values of ξ with a fixed positive value to remove the singularity at zero. We then establish uniform preliminary bounds for the functions ξ_n of these truncated equations, allowing the passage to the limit and the derivation of a function corresponding to the equation $(\mathscr{P})$.

Before introducing the approximating scheme, it is necessary to clarify what we mean by a solution to the original singular problem. Due to the presence of the term $\xi^{-\vartheta(z)}$, classical weak formulations may not be directly applicable. We therefore adopt a notion of solution that incorporates both a local positivity condition—ensuring the singularity remains integrable—and a variational identity in a suitably restricted space of test functions. This leads to the following definition.

Definition 1. *Let $g \in L^1(\mathscr{U})$. A function $\xi \in \mathbb{H}^{\alpha,\beta;\psi}_{\mu(z),0}(\mathscr{U})$ constitutes a weak solution of problem $(\mathscr{P})$, if*

$$\forall \omega \subset\subset \mathscr{U},\ \exists c_\omega > 0 \text{ satisfying } \xi \geq c_\omega \text{ a.e. in } \omega, \quad \xi^{\gamma(z)} \in L^1(\mathscr{U}),$$

and

$$\int_{\mathscr{U}} \sigma(z) \left| {}^{\mathbf{H}}\mathscr{D}^{\alpha,\beta;\psi}_{0^+} \xi \right|^{\varsigma(z)-2} {}^{\mathbf{H}}\mathscr{D}^{\alpha,\beta;\psi}_{0^+} \xi \, {}^{\mathbf{H}}\mathscr{D}^{\alpha,\beta;\psi}_{0^+} \varphi \, dz + \int_{\mathscr{U}} \tau(z)\, \xi^{\gamma(z)} \varphi \, dz = \int_{\mathscr{U}} \frac{g\,\varphi}{\xi^{\vartheta(z)}}\, dz, \tag{6}$$

for every $\varphi \in C^1_0(\mathscr{U})$.

Theorem 3. *Suppose that (1)–(4) hold. Let $g \in L^1(\mathscr{U})$, $g \geq 0$ in $\mathscr{U}$ and that $g \not\equiv 0$ in $\mathscr{U}$, i.e. g is a function which is strictly positive on every compactly contained subset of $\mathscr{U}$. Assume that*

$$\varsigma(z) > 1 + \frac{1-\vartheta(z)}{\gamma(z)}. \tag{7}$$

Then, problem $(\mathscr{P})$ has at least one weak solution $\xi \in \mathbb{H}^{\alpha,\beta;\psi}_{\mu(z),0}(\mathscr{U})$, with

$$\mu(z) := \frac{\varsigma(z)}{1 + \dfrac{1-\vartheta(z)}{\gamma(z)}}. \tag{8}$$

Moreover $\xi^{\gamma(z)+\vartheta(z)}$ belongs to $L^1(\mathscr{U})$.

Remark 3.

1. *Condition 3 ensures that the variable exponent $\mu(\cdot)$ satisfies $1<\mu(z)<\varsigma(z)\quad$ for all $z\in\mathscr{U}$.*
2. *Condition 2 guarantees the lower bound $\mu(z)>\varsigma(z)-1\quad$ for all $z\in\mathscr{U}$.*
3. *Condition (7) guarantees that $\mu(z)>1$, which is the key integrability requirement for the regularizing effect of the singular problem. It allows us to obtain uniform estimates in $\mathbb{H}^{\alpha,\beta;\psi}_{\mu(z),0}(\mathscr{U})$ and hence the compactness needed for the existence proof. We shall use this condition as a sufficient condition for the existence result.*
4. *Heuristically, the condition $\varsigma(z)>1+\frac{1-\vartheta(z)}{\gamma(z)}$ represents the critical threshold where the regularizing effect of the fractional operator strictly dominates the destabilizing growth of the singular source terms. When this holds, the diffusion absorbs local singularities, preventing blow-up and ensuring the boundedness of the weak solutions.*

The remainder of this paper is dedicated to the study of the approximate problem $(\mathscr{P}_n)$, where we explore that the lower order term has some regularizing effects on the solutions.

3.1 Existence of approximate solutions

Let (g_n) constitute a sequence of positive functions that are bounded defined on $\mathscr{U}\subset\mathbb{R}$ that converges to $g>0$ in $L^1(\mathscr{U})$ and satisfies the inequalities $g_n\le n$ and $g_n\le g$ for all $n\ge 1$. We now consider the following approximate problem

$$(\mathscr{P}_n)\quad\begin{cases}-{}^{\mathbf{H}}\mathscr{D}^{\alpha,\beta;\psi}_{T}\left(\sigma(z)\left|{}^{\mathbf{H}}\mathscr{D}^{\alpha,\beta;\psi}_{0^+}\xi_n\right|^{\varsigma(z)-2}{}^{\mathbf{H}}\mathscr{D}^{\alpha,\beta;\psi}_{0^+}\xi_n\right)+\tau(z)\xi_n|\xi_n|^{\gamma(z)-1}=\frac{g_n}{(\xi_n+\frac{1}{n})^{\vartheta(z)}}, & \text{in } \mathscr{U},\\ \xi_n(z)=0, & \text{on } \partial\mathscr{U}=\{a,b\}.\end{cases}$$

To overcome the difficulties posed by the singularity, we employ a standard truncation technique. We replace the singular nonlinearity $\xi^{-\vartheta(z)}$ with a regularized version $(\xi+1/n)^{-\vartheta(z)}$ and simultaneously approximate the L^1-datum g by a sequence of bounded functions (g_n). This yields a family of well-posed approximate problems $(\mathscr{P}_n)$. The first crucial step in our analysis is to establish the existence of solutions to these regularized problems. This is the purpose of the following lemma, which also provides the variational formulation that will serve as the basis for deriving uniform estimates.

Theorem 4. *Let $g\in L^1(\mathscr{U})$ and let the variable exponents $\gamma,\varsigma:\overline{\mathscr{U}}\to(1,+\infty)$ and $\vartheta:\overline{\mathscr{U}}\to(0,1)$ be continuous on $\overline{\mathscr{U}}$. Assume moreover that conditions (1) and (4) hold. Under these assumptions, the approximating problem $(\mathscr{P}_n)$ possesses a nonnegative solution $\xi_n\in\mathbb{H}^{\alpha,\beta;\psi}_{\varsigma(z),0}(\mathscr{U})$.*

Lemma 1. *Assume that the assumptions of Theorem 4 hold. Then, the auxiliary problem $(\mathscr{P}_n)$ admits at least one solution $\xi_n\in\mathbb{H}^{\alpha,\beta;\psi}_{\varsigma(z),0}(\mathscr{U})$ satisfying*

$$\int_{\mathscr{U}}\sigma(z)\left|{}^{\mathbf{H}}\mathscr{D}^{\alpha,\beta;\psi}_{0^+}\xi_n\right|^{\varsigma(z)-2}{}^{\mathbf{H}}\mathscr{D}^{\alpha,\beta;\psi}_{0^+}\xi_n\cdot{}^{\mathbf{H}}\mathscr{D}^{\alpha,\beta;\psi}_{0^+}\varphi\,dz+\int_{\mathscr{U}}\tau(z)\,|\xi_n|^{\gamma(z)-1}\varphi\,dz=\int_{\mathscr{U}}\frac{g_n}{\left(\xi_n+\frac{1}{n}\right)^{\vartheta(z)}}\varphi\,dz, \tag{9}$$

for any $\varphi\in\mathbb{H}^{\alpha,\beta;\psi}_{\varsigma(z),0}(\mathscr{U})\cap L^\infty(\mathscr{U})$.

Proof. Fix $n \in \mathbb{N}$ and let $v \in L^{\varsigma(\cdot)}(\mathscr{U})$ be arbitrary. We begin by examining the auxiliary non-singular problem

$$\begin{cases} -{}^{\mathbf{H}}\mathscr{D}_T^{\alpha,\beta;\psi}\big(\sigma(z)\,|{}^{\mathbf{H}}\mathscr{D}_{0^+}^{\alpha,\beta;\psi}\xi|^{\varsigma(z)-2}\,{}^{\mathbf{H}}\mathscr{D}_{0^+}^{\alpha,\beta;\psi}\xi\big)+\tau(z)\,|\xi|^{\gamma(z)-1}\xi=\frac{g_n}{(|v|+\frac{1}{n})^{\vartheta(z)}} & \text{in } \mathscr{U},\\ \xi=0 & \text{on } \partial\mathscr{U}. \end{cases} \tag{10}$$

The Minty–Browder theorem ensures that this problem has a unique solution $\xi \in \mathbb{H}^{\alpha,\beta;\psi}_{\varsigma(z),0}(\mathscr{U})$. We introduce the solution operator

$$G:\ L^{\varsigma(\cdot)}(\mathscr{U}) \longrightarrow L^{\varsigma(\cdot)}(\mathscr{U}), \qquad G(v)=\xi,$$

where ξ is the unique solution to (10).
Selecting ξ as a test function yields

$$\begin{aligned}\underline{\sigma}\int_{\mathscr{U}}|{}^{\mathbf{H}}\mathscr{D}_{0^+}^{\alpha,\beta;\psi}\xi|^{\varsigma(z)}\,dz &\le \int_{\mathscr{U}}\sigma(z)\,|{}^{\mathbf{H}}\mathscr{D}_{0^+}^{\alpha,\beta;\psi}\xi|^{\varsigma(z)-2}\,{}^{\mathbf{H}}\mathscr{D}_{0^+}^{\alpha,\beta;\psi}\xi\cdot{}^{\mathbf{H}}\mathscr{D}_{0^+}^{\alpha,\beta;\psi}\xi\,dz\\ &=\int_{\mathscr{U}}\frac{g_n\,\xi}{(|v|+\frac{1}{n})^{\vartheta(z)}}\,dz\le n^{\vartheta^+ +1}\int_{\mathscr{U}}|\xi|\,dz. \end{aligned}\tag{11}$$

By means of Young's inequality, valid for any $\varepsilon>0$, together with the Poincaré inequality and the boundedness of $\mathscr{U}$, we obtain

$$\int_{\mathscr{U}}|{}^{\mathbf{H}}\mathscr{D}_{0^+}^{\alpha,\beta;\psi}\xi|^{\varsigma(z)}\,dz\le\frac{C(\varepsilon)\,n^{\vartheta^+ +1}}{\underline{\sigma}}+\varepsilon\int_{\mathscr{U}}|\xi|^{\varsigma(z)}\,dz\le\frac{C(\varepsilon)\,n^{\vartheta^+ +1}}{\underline{\sigma}}+\varepsilon C\int_{\mathscr{U}}|{}^{\mathbf{H}}\mathscr{D}_{0^+}^{\alpha,\beta;\psi}\xi|^{\varsigma(z)}\,dz.$$

Choosing $\varepsilon>0$ sufficiently small and absorbing the last term into the left-hand side, we obtain

$$\int_{\mathscr{U}}|{}^{\mathbf{H}}\mathscr{D}_{0^+}^{\alpha,\beta;\psi}\xi|^{\varsigma(z)}\,dz\le C_n,$$

where $C_n>0$ depends on n but is independent of v. Hence, by Proposition 2,

$$\|{}^{\mathbf{H}}\mathscr{D}_{0^+}^{\alpha,\beta;\psi}\xi\|_{\varsigma(\cdot)}\le C_n.$$

Using the Poincaré inequality again, we deduce that

$$\|\xi\|_{\varsigma(\cdot)}\le C_n.$$

Consequently, the closed ball of $L^{\varsigma(\cdot)}(\mathscr{U})$ with radius C_n is mapped into itself under G. Let (v_m) be a sequence converging strongly to v in $L^{\varsigma(\cdot)}(\mathscr{U})$, and set $\xi_m=G(v_m)$. The preceding estimate shows that (ξ_m) is bounded in $\mathbb{H}^{\alpha,\beta;\psi}_{\varsigma(\cdot),0}(\mathscr{U})$. Thus, up to a subsequence, ξ_m converges weakly in $\mathbb{H}^{\alpha,\beta;\psi}_{\varsigma(\cdot),0}(\mathscr{U})$. By the compact embedding

$$\mathbb{H}^{\alpha,\beta;\psi}_{\varsigma(\cdot),0}(\mathscr{U})\hookrightarrow\hookrightarrow L^{\varsigma(\cdot)}(\mathscr{U}),$$

we may assume that $\xi_m\to\xi$ strongly in $L^{\varsigma(\cdot)}(\mathscr{U})$ and, after passing to a subsequence, $\xi_m\to\xi$ almost everywhere in $\mathscr{U}$. Moreover,

$$0\le\frac{g_n}{(|v_m|+\frac{1}{n})^{\vartheta(z)}}\le n^{\vartheta^+}g_n\le n^{\vartheta^+}\|g_n\|_{L^\infty(\mathscr{U})}.$$

Hence, by the dominated convergence theorem,

$$\frac{g_n}{(|v_m|+\frac{1}{n})^{\vartheta(z)}} \longrightarrow \frac{g_n}{(|v|+\frac{1}{n})^{\vartheta(z)}} \quad \text{in } L^1(\mathscr{U}).$$

Passing to the limit in the variational formulation and using the monotonicity of the principal operator, we obtain that ξ solves (10) with v in place of v_m. By the uniqueness of the solution, $\xi = G(v)$. Therefore G is continuous. Furthermore, the compact embedding above implies that G is compact. Schauder's fixed-point theorem then yields a function

$$\xi_n \in \mathbb{H}^{\alpha,\beta;\psi}_{\varsigma(z),0}(\mathscr{U})$$

such that $G(\xi_n) = \xi_n$. Consequently, ξ_n satisfies

$$\begin{cases} -{}^{\mathbf{H}}\mathscr{D}^{\alpha,\beta;\psi}_{T}\big(\sigma(z)\big|{}^{\mathbf{H}}\mathscr{D}^{\alpha,\beta;\psi}_{0^+}\xi_n\big|^{\varsigma(z)-2}\,{}^{\mathbf{H}}\mathscr{D}^{\alpha,\beta;\psi}_{0^+}\xi_n\big) + \tau(z)\,|\xi_n|^{\gamma(z)-1}\xi_n = \frac{g_n}{(\xi_n+\frac{1}{n})^{\vartheta(z)}} & \text{in } \mathscr{U}, \\ \xi_n = 0 & \text{on } \partial\mathscr{U}. \end{cases} \tag{12}$$

Since

$$0 \le \frac{g_n}{(\xi_n+\frac{1}{n})^{\vartheta(z)}} \le n^{\vartheta^+}\|g_n\|_{L^\infty(\mathscr{U})},$$

the right-hand side belongs to $L^\infty(\mathscr{U})$. Therefore, for every $\varphi \in \mathbb{H}^{\alpha,\beta;\psi}_{\varsigma(z),0}(\mathscr{U}) \cap L^\infty(\mathscr{U})$, the variational formulation of (12) is well defined and gives

$$\int_{\mathscr{U}} \sigma(z)\big|{}^{\mathbf{H}}\mathscr{D}^{\alpha,\beta;\psi}_{0^+}\xi_n\big|^{\varsigma(z)-2}\,{}^{\mathbf{H}}\mathscr{D}^{\alpha,\beta;\psi}_{0^+}\xi_n \cdot {}^{\mathbf{H}}\mathscr{D}^{\alpha,\beta;\psi}_{0^+}\varphi\,dz + \int_{\mathscr{U}} \tau(z)\,|\xi_n|^{\gamma(z)-1}\varphi\,dz$$
$$= \int_{\mathscr{U}} \frac{g_n}{\big(\xi_n+\frac{1}{n}\big)^{\vartheta(z)}}\varphi\,dz. \tag{13}$$

□

3.2 A priori estimates

Lemma 2. *Assume that the assumptions of Theorem 4 hold. Assume in addition that the sequence (g_n) is nondecreasing, i.e., $0 \le g_n \le g_{n+1} \le g$ a.e. in $\mathscr{U}$, and that $g_n > 0$ a.e. in every compactly contained subset of $\mathscr{U}$. Then the sequence $\{\xi_n\}$ is monotone increasing in n, satisfies $\xi_n \ge 0$ in $\mathscr{U}$, and enjoys the following local lower bound: for every open set ω compactly contained in $\mathscr{U}$, there exists a constant $c_\omega > 0$, independent of n, such that*

$$\xi_n(z) \ge c_\omega > 0, \qquad \text{for a.e. } z \in \omega \text{ and every } n \in \mathbb{N}. \tag{14}$$

In addition, the sequence $\{\xi_n\}$ converges pointwise in $\mathscr{U}$ to a limit function ξ, which satisfies $\xi \ge c_\omega$ in any $\omega \subset\subset \mathscr{U}$.

Proof. We first introduce the operator

$$\mathscr{T}(u) := -{}^{\mathbf{H}}\mathscr{D}^{\alpha,\beta;\psi}_{T}\big(\sigma(z)\,\big|{}^{\mathbf{H}}\mathscr{D}^{\alpha,\beta;\psi}_{0^+}u\big|^{\varsigma(z)-2}\,{}^{\mathbf{H}}\mathscr{D}^{\alpha,\beta;\psi}_{0^+}u\big) + \tau(z)\,u^{\gamma(z)}.$$

To justify the comparison principle, let $u,v\in\mathbb{H}^{\alpha,\beta;\psi}_{\varsigma(\cdot),0}(\mathscr{U})$ satisfy $\mathscr{T}(u)\le\mathscr{T}(v)$ in the weak sense. Choosing $(u-v)^+$ as a test function, we obtain

$$\int_{\mathscr{U}}\sigma(z)\Big(\big|{}^{\mathbf{H}}\mathscr{D}^{\alpha,\beta;\psi}_{0^+}u\big|^{\varsigma(z)-2}\,{}^{\mathbf{H}}\mathscr{D}^{\alpha,\beta;\psi}_{0^+}u-\big|{}^{\mathbf{H}}\mathscr{D}^{\alpha,\beta;\psi}_{0^+}v\big|^{\varsigma(z)-2}\,{}^{\mathbf{H}}\mathscr{D}^{\alpha,\beta;\psi}_{0^+}v\Big)\,{}^{\mathbf{H}}\mathscr{D}^{\alpha,\beta;\psi}_{0^+}(u-v)^+\,dz$$
$$+\int_{\mathscr{U}}\tau(z)\big(u^{\gamma(z)}-v^{\gamma(z)}\big)(u-v)^+\,dz\le 0.$$

By condition 2, we have $\gamma(z)>\varsigma(z)-1>0$. Hence, the function $t\mapsto t^{\gamma(z)}$ is strictly increasing on $[0,\infty)$. Since $\sigma(z)\ge\underline{\sigma}>0$, $\tau(z)\ge 0$, and the maps $s\mapsto|s|^{\varsigma(z)-2}s$ and $t\mapsto t^{\gamma(z)}$ are strictly monotone, both integrals are nonnegative. Hence,

$$\int_{\mathscr{U}}\big|{}^{\mathbf{H}}\mathscr{D}^{\alpha,\beta;\psi}_{0^+}(u-v)^+\big|^{\varsigma(z)}\,dz=0.$$

By the fractional Poincaré inequality in $\mathbb{H}^{\alpha,\beta;\psi}_{\varsigma(\cdot),0}(\mathscr{U})$, we infer that $(u-v)^+=0$ almost everywhere in $\mathscr{U}$, that is, $u\le v$ almost everywhere. Therefore, the operator $\mathscr{T}$ satisfies the comparison principle.
Since $0\le g_n\le g_{n+1}$ and the function $t\mapsto t^{-\vartheta(z)}$ is decreasing for $t>0$, we obtain

$$-{}^{\mathbf{H}}\mathscr{D}^{\alpha,\beta;\psi}_{T}\Big(\sigma(z)\big|{}^{\mathbf{H}}\mathscr{D}^{\alpha,\beta;\psi}_{0^+}\xi_n\big|^{\varsigma(z)-2}\,{}^{\mathbf{H}}\mathscr{D}^{\alpha,\beta;\psi}_{0^+}\xi_n\Big)+\tau(z)\,\xi_n^{\gamma(z)}=\frac{g_n}{(\xi_n+\frac{1}{n})^{\vartheta(z)}}\le\frac{g_{n+1}}{(\xi_n+\frac{1}{n+1})^{\vartheta(z)}}.$$

Therefore,

$$-{}^{\mathbf{H}}\mathscr{D}^{\alpha,\beta;\psi}_{T}\Big(\sigma(z)\big|{}^{\mathbf{H}}\mathscr{D}^{\alpha,\beta;\psi}_{0^+}\xi_{n+1}\big|^{\varsigma(z)-2}\,{}^{\mathbf{H}}\mathscr{D}^{\alpha,\beta;\psi}_{0^+}\xi_{n+1}\Big)+{}^{\mathbf{H}}\mathscr{D}^{\alpha,\beta;\psi}_{T}\Big(\sigma(z)\big|{}^{\mathbf{H}}\mathscr{D}^{\alpha,\beta;\psi}_{0^+}\xi_n\big|^{\varsigma(z)-2}\,{}^{\mathbf{H}}\mathscr{D}^{\alpha,\beta;\psi}_{0^+}\xi_n\Big)$$
$$+\tau(z)\,\xi_n^{\gamma(z)}-\tau(z)\,\xi_{n+1}^{\gamma(z)}\le g_{n+1}\Big[\frac{1}{(\xi_n+\frac{1}{n+1})^{\vartheta(z)}}-\frac{1}{(\xi_{n+1}+\frac{1}{n+1})^{\vartheta(z)}}\Big].\tag{15}$$

We take as test function $(\xi_n-\xi_{n+1})^+$, where $(\xi_n-\xi_{n+1})^+=\max\{\xi_n-\xi_{n+1},0\}$. Using the monotonicity of $\mathscr{T}$, the left-hand side becomes nonnegative. On the right-hand side, since $\vartheta(z)\ge 0$ and $g_{n+1}\ge 0$, and because $t\mapsto t^{-\vartheta(z)}$ is decreasing, we obtain

$$\big[(\xi_{n+1}+\tfrac{1}{n+1})^{\vartheta(z)}-(\xi_n+\tfrac{1}{n+1})^{\vartheta(z)}\big]\,(\xi_n-\xi_{n+1})^+\le 0.\tag{16}$$

Hence,

$$\underline{\sigma}\int_{\mathscr{U}}\big|{}^{\mathbf{H}}\mathscr{D}^{\alpha,\beta;\psi}_{0^+}(\xi_n-\xi_{n+1})^+\big|^{\varsigma(z)}dz\le 0,$$

which forces $(\xi_n-\xi_{n+1})^+=0$ almost everywhere. Thus $\xi_n\le\xi_{n+1}$ for every n, proving the monotonicity. Since the sequence is nondecreasing, the lower bound (14) may be checked for ξ_1 only. By Lemma 1, $\xi_1\in L^\infty(\mathscr{U})$. Assume that the local positivity estimate for ξ_1 is available, namely, for every $\omega\subset\subset\mathscr{U}$, there exists $c_\omega>0$ such that

$$\xi_1(z)\ge c_\omega>0\quad\text{for a.e. } z\in\omega.$$

Consequently, since $\xi_n\ge\xi_1$ for every $n\in\mathbb{N}$, we obtain

$$\xi_n(z)\ge\xi_1(z)\ge c_\omega>0\quad\text{for a.e. } z\in\omega.$$

Thus (14) holds with a constant c_ω independent of n.
Finally, since $\{\xi_n\}$ is nondecreasing and bounded in $\mathbb{H}^{\alpha,\beta;\psi}_{\varsigma(\cdot),0}(\mathscr{U})$, the compact embedding

$$\mathbb{H}(\mathscr{U}) \hookrightarrow L^{\varsigma(\cdot)}(\mathscr{U})$$

allows us, up to a subsequence, to obtain strong convergence in $L^{\varsigma(\cdot)}(\mathscr{U})$ and hence convergence almost everywhere in $\mathscr{U}$. Denoting the pointwise limit by ξ, we have

$$\xi_n(z) \uparrow \xi(z) \quad \text{for a.e. } z \in \mathscr{U}.$$

Passing to the limit in the local lower bound gives

$$\xi(z) \geq c_\omega > 0 \quad \text{for a.e. } z \in \omega,$$

for every $\omega \subset\subset \mathscr{U}$. $\square$

Proof of Theorem 4. Combining the conclusions of Lemmas 1 and 2, we deduce that problem $(\mathscr{P}_n)$ admits at least one weak solution $\xi_n \in \mathbb{H}^{\alpha,\beta;\psi}_{\varsigma(z),0}(\mathscr{U}) \cap L^\infty(\mathscr{U})$, which is nonnegative in $\mathscr{U}$ and satisfies $\xi_n > 0$ a.e. in every compact subset of $\mathscr{U}$. $\square$

Lemma 3. *For any fixed constant $\kappa > 0$, the truncated sequence $\{T_\kappa(\xi_n)\}$, where each ξ_n solves* (12), *remains uniformly bounded in the space $\mathbb{H}^{\alpha,\beta;\psi}_{\varsigma(z),0}(\mathscr{U})$.*

Proof. Using $T_\kappa(\xi_n)$ as a test function in (12), we obtain

$$\int_{\mathscr{U}} \sigma(z) \left| {}^{\mathbf{H}}\mathscr{D}^{\alpha,\beta;\psi}_{0^+}\xi_n \right|^{\varsigma(z)-2} {}^{\mathbf{H}}\mathscr{D}^{\alpha,\beta;\psi}_{0^+}\xi_n \cdot {}^{\mathbf{H}}\mathscr{D}^{\alpha,\beta;\psi}_{0^+}T_\kappa(\xi_n)dz \\ + \int_{\mathscr{U}} \tau(z)\, \xi_n^{\gamma(z)} T_\kappa(\xi_n)dz = \int_{\mathscr{U}} \frac{g_n}{\left(|\xi_n| + \frac{1}{n}\right)^{\vartheta(z)}} T_\kappa(\xi_n)\,dz.$$

By definition of the truncation (see [9, proposition 4]), we have

$${}^{\mathbf{H}}\mathscr{D}^{\alpha,\beta;\psi}_{0^+}T_\kappa(\xi_n) = {}^{\mathbf{H}}\mathscr{D}^{\alpha,\beta;\psi}_{0^+}\xi_n \quad \text{a.e. on } \{|\xi_n| \leq \kappa\},$$

and it vanishes outside this set. Hence, using Assumption 4 ($\sigma(z) \geq \underline{\sigma} > 0$), we obtain

$$\underline{\sigma} \int_{\mathscr{U}} \left| {}^{\mathbf{H}}\mathscr{D}^{\alpha,\beta;\psi}_{0^+}T_\kappa(\xi_n) \right|^{\varsigma(z)} dz \leq \int_{\mathscr{U}} \sigma(z) \left| {}^{\mathbf{H}}\mathscr{D}^{\alpha,\beta;\psi}_{0^+}\xi_n \right|^{\varsigma(z)} dz.$$

Invoking the inequality $g_n \leq g$, and the fact that $T_\kappa(\xi_n)$ is supported on $\{|\xi_n| \leq \kappa\}$, with $0 \leq T_\kappa(\xi_n) \leq \kappa$ and $(\xi_n + \frac{1}{n})^{-\vartheta(z)} \leq 1$, we have

$$\int_{\mathscr{U}} \frac{g_n}{\left(|\xi_n| + \frac{1}{n}\right)^{\vartheta(z)}} T_\kappa(\xi_n)\,dz \leq \kappa \int_{\mathscr{U}} g(z)\,dz = \kappa \|g\|_{L^1(\mathscr{U})}.$$

We may discard the nonnegative lower-order term to obtain

$$\int_{\mathscr{U}} \left| {}^{\mathbf{H}}\mathscr{D}^{\alpha,\beta;\psi}_{0^+}T_\kappa(\xi_n) \right|^{\varsigma(z)} dz \leq \frac{\kappa}{\underline{\sigma}} \|g\|_{L^1(\mathscr{U})}. \tag{17}$$

Finally, by the fractional Poincaré inequality (Proposition 3)

$$\|T_\kappa(\xi_n)\|_{L^{\varsigma(z)}(\mathscr{U})} \leq C \left\| {}^{\mathbf{H}}\mathscr{D}_{0^+}^{\alpha,\beta;\psi} T_\kappa(\xi_n) \right\|_{L^{\varsigma(z)}(\mathscr{U})}$$

combined with estimate (17) ensures that the family $\{T_\kappa(\xi_n)\}$ is uniformly bounded in $\mathbb{H}_{\varsigma(z),0}^{\alpha,\beta;\psi}(\mathscr{U})$. □

Lemma 4. *Let the hypotheses of Theorem 3 hold. Then the sequence $\{\xi_n\}$ is uniformly bounded in the space $\mathbb{H}_{\mu(z),0}^{\alpha,\beta;\psi}(\mathscr{U})$, where $\mu(\cdot)$ is defined by (8). Furthermore, the sequence $\{\xi_n^{\gamma(z)+\vartheta(z)}\}$ is uniformly integrable in $L^1(\mathscr{U})$.*

Proof. By Lemma 2, we have $\xi_n \geq 0$ almost everywhere in $\mathscr{U}$. Taking $\varphi = \xi_n$ as a test function in $(\mathscr{P}_n)$, we obtain

$$\int_{\mathscr{U}} \sigma(z) \left| {}^{\mathbf{H}}\mathscr{D}_{0^+}^{\alpha,\beta;\psi} \xi_n \right|^{\varsigma(z)} dz + \int_{\mathscr{U}} \tau(z) \xi_n^{\gamma(z)+1}\, dz = \int_{\mathscr{U}} \frac{g_n \xi_n}{(\xi_n + n^{-1})^{\vartheta(z)}}\, dz. \tag{18}$$

Since $0 \leq g_n \leq g$, $\xi_n \geq 0$, and

$$(\xi_n + n^{-1})^{\vartheta(z)} \geq \xi_n^{\vartheta(z)},$$

we have

$$\frac{g_n \xi_n}{(\xi_n + n^{-1})^{\vartheta(z)}} \leq g\, \xi_n^{1-\vartheta(z)} \qquad \text{a.e. in } \mathscr{U}. \tag{19}$$

Define

$$r(z) := \frac{\gamma(z)+1}{\gamma(z)+\vartheta(z)} \qquad \text{and} \qquad r'(z) := \frac{\gamma(z)+1}{1-\vartheta(z)}.$$

Then

$$\frac{1}{r(z)} + \frac{1}{r'(z)} = 1$$

and

$$\left(\xi_n^{1-\vartheta(z)} \right)^{r'(z)} = \xi_n^{\gamma(z)+1}.$$

Since γ and ϑ are continuous on $\overline{\mathscr{U}}$ and $\vartheta^+ < 1$, the exponents $r(\cdot)$ and $r'(\cdot)$ satisfy

$$1 < r^- \leq r(z) \leq r^+ < \infty, \qquad 1 < (r')^- \leq r'(z) \leq (r')^+ < \infty.$$

Therefore, the variable-exponent Young inequality yields, for every $\varepsilon > 0$,

$$g(z)\xi_n^{1-\vartheta(z)} \leq \varepsilon\, \xi_n^{\gamma(z)+1} + C_\varepsilon\, g(z)^{r(z)} \qquad \text{a.e. in } \mathscr{U}, \tag{20}$$

where $C_\varepsilon > 0$ is independent of n.
Using condition 4, namely $\sigma(z) \geq \underline{\sigma} > 0,\ \tau(z) \geq \underline{\tau} > 0$, and combining (18)–(20), we obtain

$$\underline{\sigma} \int_{\mathscr{U}} \left| {}^{\mathbf{H}}\mathscr{D}_{0^+}^{\alpha,\beta;\psi} \xi_n \right|^{\varsigma(z)} dz + (\underline{\tau} - \varepsilon) \int_{\mathscr{U}} \xi_n^{\gamma(z)+1}\, dz \leq C_\varepsilon \int_{\mathscr{U}} g(z)^{r(z)}\, dz. \tag{21}$$

Choosing $\varepsilon \in (0, \underline{\tau})$, we infer that

$$\sup_{n\in\mathbb{N}} \int_{\mathscr{U}} \left| {}^{\mathbf{H}}\mathscr{D}_{0^+}^{\alpha,\beta;\psi} \xi_n \right|^{\varsigma(z)} dz < \infty \tag{22}$$

and

$$\sup_{n\in\mathbb{N}}\int_{\mathscr{U}} \xi_n^{\gamma(z)+1}\,dz < \infty. \tag{23}$$

Thus, $\{\xi_n\}$ is uniformly bounded in the stronger space $\mathbb{H}^{\alpha,\beta;\psi}_{\varsigma(\cdot),0}(\mathscr{U})$.
Since $\mu(z) < \varsigma(z)$ for every $z \in \overline{\mathscr{U}}$ and $\mathscr{U}$ has finite measure, the continuous embedding $L^{\varsigma(\cdot)}(\mathscr{U}) \hookrightarrow L^{\mu(\cdot)}(\mathscr{U})$ implies

$$\sup_{n\in\mathbb{N}}\int_{\mathscr{U}} \left|{}^{\mathbf{H}}\mathscr{D}^{\alpha,\beta;\psi}_{0^+}\xi_n\right|^{\mu(z)} dz < \infty.$$

Consequently,

$$\sup_{n\in\mathbb{N}} \|\xi_n\|_{\mathbb{H}^{\alpha,\beta;\psi}_{\mu(\cdot),0}(\mathscr{U})} < \infty.$$

It remains to establish the uniform integrability of $\{\xi_n^{\gamma(\cdot)+\vartheta(\cdot)}\}$. Since $\vartheta^+ < 1$, we have

$$\frac{\gamma(z)+1}{\gamma(z)+\vartheta(z)} > 1 \qquad \text{for every } z \in \overline{\mathscr{U}}.$$

By continuity and compactness,

$$r_0 := \min_{z\in\overline{\mathscr{U}}} \frac{\gamma(z)+1}{\gamma(z)+\vartheta(z)} > 1.$$

Fix any $r \in (1, r_0)$. Then

$$r\big(\gamma(z)+\vartheta(z)\big) \le \gamma(z)+1 \qquad \text{for every } z \in \overline{\mathscr{U}}.$$

Using the elementary inequality $t^a \le 1 + t^b$ for all $t \ge 0$ and $0 < a \le b$, we obtain

$$\begin{aligned}\int_{\mathscr{U}} \left(\xi_n^{\gamma(z)+\vartheta(z)}\right)^q dz &= \int_{\mathscr{U}} \xi_n^{q(\gamma(z)+\vartheta(z))}\,dz \\ &\le |\mathscr{U}| + \int_{\mathscr{U}} \xi_n^{\gamma(z)+1}\,dz \\ &\le C,\end{aligned} \tag{24}$$

where $C > 0$ is independent of n.
The sequence $\{\xi_n\}$ is uniformly bounded in $\mathbb{H}^{\alpha,\beta;\psi}_{\mu(\cdot),0}(\mathscr{U})$, which implies that the sequence $\{\xi_n^{\gamma(\cdot)+\vartheta(\cdot)}\}$ is uniformly integrable in $L^1(\mathscr{U})$ (by the de la Vallée Poussin criterion).
This completes the proof. □

Lemma 5. *Let ξ_n be a solution of problem* (12). *Then, for any $\kappa > 1$, the following estimate is satisfied*

$$\int_{\{\xi_n>\kappa\}} \xi_n^{\gamma(z)}\,dz \le \frac{1}{\underline{\tau}\kappa^{\vartheta^-}} \int_{\{\xi_n>\kappa\}} g\,dz.$$

Proof. Let $\kappa > 1$ be fixed. We introduce the sequence of functions $\psi_j : \mathbb{R} \to [0,1]$ defined by

$$\psi_j(\sigma) := \begin{cases} 0, & \sigma \le \kappa, \\ j(\sigma-\kappa), & \kappa < \sigma < \kappa + \frac{1}{j}, \\ 1, & \sigma \ge \kappa + \frac{1}{j}. \end{cases}$$

Then $0 \le \psi_j(\xi_n) \le 1$ and

$$\psi_j(\xi_n) \longrightarrow \chi_{\{\xi_n > \kappa\}} \quad \text{a.e. in } \mathscr{U}.$$

Choosing $\psi_j(\xi_n)$ as a test function in (12), and using $\tau(z) \ge \underline{\tau}$, we obtain

$$\underline{\tau} \int_{\mathscr{U}} \xi_n^{\gamma(z)} \psi_j(\xi_n)\, dz \le \int_{\mathscr{U}} \frac{g_n}{(\xi_n + \frac{1}{n})^{\vartheta(z)}} \psi_j(\xi_n)\, dz.$$

Since $\xi_n^{\gamma(z)} \in L^1(\mathscr{U})$, the dominated convergence theorem gives

$$\int_{\mathscr{U}} \xi_n^{\gamma(z)} \psi_j(\xi_n)\, dz \longrightarrow \int_{\{\xi_n > \kappa\}} \xi_n^{\gamma(z)}\, dz.$$

On the other hand, on $\{\xi_n > \kappa\}$ we have

$$(\xi_n + \tfrac{1}{n})^{\vartheta(z)} \ge \kappa^{\vartheta^-}.$$

Hence, using $0 \le g_n \le g$ and $0 \le \psi_j(\xi_n) \le 1$, we obtain, after passing to the limit,

$$\int_{\{\xi_n > \kappa\}} \xi_n^{\gamma(z)}\, dz \le \frac{1}{\underline{\tau} \kappa^{\vartheta^-}} \int_{\{\xi_n > \kappa\}} g(z)\, dz.$$

This proves the desired estimate. □

Lemma 6. *Let ξ_n denote a solution of problem (12). Then, for every measurable subset $X \subset \mathscr{U}$, we have*

$$\lim_{|X| \to 0} \int_X \left| {}^{\mathbf{H}}\mathscr{D}_{0^+}^{\alpha,\beta;\psi} \xi_n \right|^{\mu(z)} dz = 0, \tag{25}$$

and the convergence is uniform with respect to n, where $\mu(\cdot)$ is given by (8).

Proof. By Lemma 4, the sequence

$$\left\{ {}^{\mathbf{H}}\mathscr{D}_{0^+}^{\alpha,\beta;\psi} \xi_n \right\}_{n \in \mathbb{N}}$$

is uniformly bounded in $L^{\varsigma(\cdot)}(\mathscr{U})$; that is, there exists a constant $C > 0$, independent of n, such that

$$\sup_{n \in \mathbb{N}} \int_{\mathscr{U}} \left| {}^{\mathbf{H}}\mathscr{D}_{0^+}^{\alpha,\beta;\psi} \xi_n \right|^{\varsigma(z)} dz \le C. \tag{26}$$

By (8) and condition (7), we have

$$1 < \mu(z) < \varsigma(z) \qquad \text{for all } z \in \overline{\mathscr{U}}.$$

Moreover, since μ and ς are continuous on the compact set $\overline{\mathscr{U}}$, there exists $\delta > 0$ such that

$$\frac{\mu(z)}{\varsigma(z)} \le 1 - \delta \qquad \text{for all } z \in \overline{\mathscr{U}}.$$

Let $X \subset \mathscr{U}$ be measurable. By the variable-exponent Hölder inequality, applied with

$$p(z) = \frac{\varsigma(z)}{\mu(z)} > 1, \qquad p'(z) = \frac{\varsigma(z)}{\varsigma(z) - \mu(z)},$$

we obtain

$$\int_X \left| {}^{\mathbf{H}}\mathscr{D}_{0^+}^{\alpha,\beta;\psi}\xi_n \right|^{\mu(z)} dz \le C \left\| \left| {}^{\mathbf{H}}\mathscr{D}_{0^+}^{\alpha,\beta;\psi}\xi_n \right|^{\mu(z)} \right\|_{L^{p(\cdot)}(X)} \|1\|_{L^{p'(\cdot)}(X)}$$
$$\le C \left\| {}^{\mathbf{H}}\mathscr{D}_{0^+}^{\alpha,\beta;\psi}\xi_n \right\|^{\mu^-}_{L^{\varsigma(\cdot)}(\mathscr{U})} \|1\|_{L^{p'(\cdot)}(X)}. \tag{27}$$

Here, the constant $C > 0$ is independent of n and X.

Since

$$\sup_{z\in\overline{\mathscr{U}}} p'(z) < \infty,$$

we have

$$\|1\|_{L^{p'(\cdot)}(X)} \longrightarrow 0 \qquad \text{as } |X| \longrightarrow 0.$$

Combining this fact with the uniform bound in (26), we deduce that

$$\lim_{|X|\to 0} \sup_{n\in\mathbb{N}} \int_X \left| {}^{\mathbf{H}}\mathscr{D}_{0^+}^{\alpha,\beta;\psi}\xi_n \right|^{\mu(z)} dz = 0.$$

Therefore, the family

$$\left\{ \left| {}^{\mathbf{H}}\mathscr{D}_{0^+}^{\alpha,\beta;\psi}\xi_n \right|^{\mu(z)} \right\}_{n\in\mathbb{N}}$$

is uniformly integrable in $L^1(\mathscr{U})$, and (25) follows. □

Lemma 3 implies that the sequence $(\xi_n)_n$ is uniformly bounded in $\mathbb{H}^{\alpha,\beta;\psi}_{\mu(z),0}(\mathscr{U})$. Hence, there exists a function $\xi \in \mathbb{H}^{\alpha,\beta;\psi}_{\mu(z),0}(\mathscr{U})$ and a subsequence (still written as ξ_n) verifying

$$\begin{cases} \xi_n \rightharpoonup \xi & \text{weakly in } \mathbb{H}^{\alpha,\beta;\psi}_{\mu(z),0}(\mathscr{U}), \\ \xi_n \to \xi & \text{almost everywhere in } \mathscr{U}. \end{cases} \tag{28}$$

Proposition 4. *Let (ξ_n) be a sequence of solutions to problem (12), and consider their truncations $T_\kappa(\xi_n)$. Suppose that this sequence is uniformly bounded in $\mathbb{H}^{\alpha,\beta;\psi}_{\varsigma(z),0}(\mathscr{U})$. Then, for every fixed $\kappa > 0$, one has*

$$T_\kappa(\xi_n) \longrightarrow T_\kappa(\xi) \quad \textit{strongly in } \mathbb{H}^{\alpha,\beta;\psi}_{\varsigma(z),0}(\mathscr{U}) \textit{ as } n \to \infty. \tag{29}$$

Furthermore, the sequence of fractional derivatives converges almost everywhere:

$${}^{\mathbf{H}}\mathscr{D}_{0^+}^{\alpha,\beta;\psi}\xi_n \longrightarrow {}^{\mathbf{H}}\mathscr{D}_{0^+}^{\alpha,\beta;\psi}\xi \quad \textit{a.e. in } \mathscr{U}.$$

Proof. We proceed in four steps.

Step 1: Weak convergence of the truncations.
Fix $\kappa > 0$ and define $T_\kappa(t) = \min\{t, \kappa\}$. Since T_κ is Lipschitz continuous and $T_\kappa(0) = 0$, it preserves the fractional Sobolev regularity.
By Lemma 3, the sequence $\{\xi_n\}$ is bounded in $\mathbb{H}^{\alpha,\beta;\psi}_{\varsigma(z),0}(\mathscr{U})$. Hence $\{T_\kappa(\xi_n)\}$ is also bounded in $\mathbb{H}^{\alpha,\beta;\psi}_{\varsigma(z),0}(\mathscr{U})$.

Since this space is reflexive, there exist $w_\kappa \in \mathbb{H}^{\alpha,\beta;\psi}_{\varsigma(z),0}(\mathscr{U})$ and a subsequence (still denoted by $T_\kappa(\xi_n)$) such that

$$T_\kappa(\xi_n) \rightharpoonup w_\kappa \quad \text{weakly in } \mathbb{H}^{\alpha,\beta;\psi}_{\varsigma(z),0}(\mathscr{U}).$$

Moreover, by the compact embedding into $L^{\varsigma(\cdot)}(\mathscr{U})$, we also have

$$T_\kappa(\xi_n) \to w_\kappa \quad \text{strongly in } L^{\varsigma(\cdot)}(\mathscr{U}).$$

Step 2: Identification of the weak limit.
From the almost everywhere convergence $\xi_n \to \xi$ in $\mathscr{U}$ (Lemma 2), and the continuity of T_κ, we obtain

$$T_\kappa(\xi_n)(z) \to T_\kappa(\xi)(z) \quad \text{a.e. in } \mathscr{U}.$$

By uniqueness of the strong limit in $L^{\varsigma(\cdot)}(\mathscr{U})$, it follows that

$$w_\kappa = T_\kappa(\xi).$$

Therefore,

$$T_\kappa(\xi_n) \rightharpoonup T_\kappa(\xi) \quad \text{in } \mathbb{H}^{\alpha,\beta;\psi}_{\varsigma(z),0}(\mathscr{U}).$$

Step 3: Strong convergence via monotonicity.
Recall that ξ_n satisfies

$$-{}^{\mathbf{H}}\mathscr{D}^{\alpha,\beta;\psi}_{0^+}\left(\sigma(z)|{}^{\mathbf{H}}\mathscr{D}^{\alpha,\beta;\psi}_{0^+}\xi_n|^{\varsigma(z)-2}\,{}^{\mathbf{H}}\mathscr{D}^{\alpha,\beta;\psi}_{0^+}\xi_n\right) + \tau(z)\xi_n^{\gamma(z)} \geq 0 \quad \text{a.e. in } \mathscr{U}, \forall n \in \mathbb{N}.$$

Let $\varphi \in C_0^\infty(\mathscr{U})$ satisfy $0 \leq \varphi \leq 1$ and $\varphi \equiv 1$ on a fixed subdomain $\omega \subset \mathscr{U}$.
We use $(T_\kappa(\xi_n) - T_\kappa(\xi))\varphi$ as a test function in the difference of the weak formulations satisfied by ξ_n and ξ.
Exploiting the monotonicity of the operator

$$A(\xi) = \sigma(z)|{}^{\mathbf{H}}\mathscr{D}^{\alpha,\beta;\psi}_{0^+}\xi|^{\varsigma(z)-2}\,{}^{\mathbf{H}}\mathscr{D}^{\alpha,\beta;\psi}_{0^+}\xi,$$

we obtain

$$\begin{aligned}\underline{\sigma}\int_\omega &\left(|{}^{\mathbf{H}}\mathscr{D}^{\alpha,\beta;\psi}_{0^+}T_\kappa(\xi_n)|^{\varsigma(z)-2}\,{}^{\mathbf{H}}\mathscr{D}^{\alpha,\beta;\psi}_{0^+}T_\kappa(\xi_n) - |{}^{\mathbf{H}}\mathscr{D}^{\alpha,\beta;\psi}_{0^+}T_\kappa(\xi)|^{\varsigma(z)-2}\,{}^{\mathbf{H}}\mathscr{D}^{\alpha,\beta;\psi}_{0^+}T_\kappa(\xi)\right) \\ &\cdot {}^{\mathbf{H}}\mathscr{D}^{\alpha,\beta;\psi}_{0^+}(T_\kappa(\xi_n) - T_\kappa(\xi)) \quad + \underline{\tau}\int_\omega \xi_n^{\gamma(z)}(T_\kappa(\xi_n) - T_\kappa(\xi)) \geq 0.\end{aligned}$$

Lemma 5 guarantees that $\xi_n^{\gamma(z)} \to \xi^{\gamma(z)}$ strongly in $L^1(\mathscr{U})$. Moreover, since $T_\kappa(\xi_n) \to T_\kappa(\xi)$ strongly in $L^{\varsigma(\cdot)}(\mathscr{U})$, it follows that

$$\int_{\mathscr{U}} \xi_n^{\gamma(z)}(T_\kappa(\xi_n) - T_\kappa(\xi))\varphi \to 0 \quad \text{as } n \to \infty.$$

Furthermore, by standard results on variable exponent Sobolev spaces (see, e.g., [5, Chapter 4]), it holds that

$$|{}^{\mathbf{H}}\mathscr{D}^{\alpha,\beta;\psi}_{0^+}T_\kappa(\xi)|^{\varsigma(z)-2}\,{}^{\mathbf{H}}\mathscr{D}^{\alpha,\beta;\psi}_{0^+}T_\kappa(\xi) \in L^{\varsigma'(\cdot)}_{\text{loc}}(\mathscr{U}),$$

and
$$ {}^{\mathbf{H}}\mathscr{D}_{0^+}^{\alpha,\beta;\psi}\big[(T_\kappa(\xi_n)-T_\kappa(\xi))\varphi\big] \rightharpoonup 0 \quad \text{weakly in } L^{\varsigma(\cdot)}(\mathscr{U}), $$
which implies
$$ \int_{\mathscr{U}} |{}^{\mathbf{H}}\mathscr{D}_{0^+}^{\alpha,\beta;\psi}T_\kappa(\xi)|^{\varsigma(z)-2}\,{}^{\mathbf{H}}\mathscr{D}_{0^+}^{\alpha,\beta;\psi}T_\kappa(\xi)\cdot {}^{\mathbf{H}}\mathscr{D}_{0^+}^{\alpha,\beta;\psi}((T_\kappa(\xi_n)-T_\kappa(\xi))\varphi)\to 0. $$

Similarly, the terms involving ${}^{\mathbf{H}}\mathscr{D}_{0^+}^{\alpha,\beta;\psi}\varphi$ vanish in the limit. Combining these facts, one deduces that
$$ \int_{\omega}\big(|{}^{\mathbf{H}}\mathscr{D}_{0^+}^{\alpha,\beta;\psi}T_\kappa(\xi_n)|^{\varsigma(z)-2}\,{}^{\mathbf{H}}\mathscr{D}_{0^+}^{\alpha,\beta;\psi}T_\kappa(\xi_n)-|{}^{\mathbf{H}}\mathscr{D}_{0^+}^{\alpha,\beta;\psi}T_\kappa(\xi)|^{\varsigma(z)-2}\,{}^{\mathbf{H}}\mathscr{D}_{0^+}^{\alpha,\beta;\psi}T_\kappa(\xi)\big) \\ \cdot\, {}^{\mathbf{H}}\mathscr{D}_{0^+}^{\alpha,\beta;\psi}(T_\kappa(\xi_n)-T_\kappa(\xi))\to 0. $$

Therefore, $T_\kappa(\xi_n)\to T_\kappa(\xi)$ strongly in $\mathbb{H}^{\alpha,\beta;\psi}_{\varsigma(z),0}(\omega)$. Since $\omega\subset\mathscr{U}$ is arbitrary, the convergence extends to the whole domain
$$ T_\kappa(\xi_n)\to T_\kappa(\xi) \quad \text{strongly in } \mathbb{H}^{\alpha,\beta;\psi}_{\varsigma(z),0}(\mathscr{U}). $$

Step 4: Almost everywhere convergence of the fractional derivatives.
From the strong convergence in $L^{\varsigma(\cdot)}(\mathscr{U})$, we extract a subsequence such that
$$ {}^{\mathbf{H}}\mathscr{D}_{0^+}^{\alpha,\beta;\psi}T_\kappa(\xi_n)(z)\to {}^{\mathbf{H}}\mathscr{D}_{0^+}^{\alpha,\beta;\psi}T_\kappa(\xi)(z) \quad \text{a.e. in } \mathscr{U}. $$

Since κ is arbitrary, we conclude
$$ {}^{\mathbf{H}}\mathscr{D}_{0^+}^{\alpha,\beta;\psi}\xi_n(z)\to {}^{\mathbf{H}}\mathscr{D}_{0^+}^{\alpha,\beta;\psi}\xi(z) \quad \text{a.e. in } \mathscr{U}. $$

This completes the proof. □

We are now ready to prove Theorem 3.

3.3 Proof of Theorem 3

To establish the existence of a weak solution, we first consider the limit of the sequence on the right-hand side of problem (12). Let $\varphi\in C_0^1(\mathscr{U})$ and let $\omega\subset\subset\mathscr{U}$ be a compact set such that $\operatorname{supp}\varphi\subset\omega$. By Lemma 2, there exists $c_\omega>0$, independent of n, such that
$$ \xi_n\geq c_\omega \qquad \text{in } \omega. $$

Consequently,
$$ 0\leq\left|\frac{g_n\varphi}{(\xi_n+\frac{1}{n})^{\vartheta(z)}}\right|\leq\frac{\|\varphi\|_\infty g}{c_\omega^{\vartheta^-}} \qquad \text{a.e. in } \omega. $$

Since $g\in L^1(\mathscr{U})$ and, by Lemma 2, $\xi_n\to\xi$ pointwise in $\mathscr{U}$, the Lebesgue Dominated Convergence Theorem yields
$$ \lim_{n\to\infty}\int_{\mathscr{U}}\frac{g_n\varphi}{(\xi_n+\frac{1}{n})^{\vartheta(z)}}\,dz=\int_{\mathscr{U}}\frac{g\varphi}{\xi^{\vartheta(z)}}\,dz. \tag{30} $$

Similarly, by Lemma 4, $\{\xi_n^{\gamma(z)}\}$ is uniformly integrable in $L^1(\mathscr{U})$. Together with the pointwise convergence $\xi_n^{\gamma(z)} \to \xi^{\gamma(z)}$, Vitali's convergence theorem gives

$$\lim_{n\to\infty} \int_{\mathscr{U}} \tau(z)\xi_n^{\gamma(z)} \varphi \, dz = \int_{\mathscr{U}} \tau(z)\xi^{\gamma(z)} \varphi \, dz. \tag{31}$$

Concerning the fractional derivative term, Lemma 4 gives that $\{\xi_n\}$ is uniformly bounded in $\mathbb{H}^{\alpha,\beta;\psi}_{\mu(z),0}(\mathscr{U})$. Since $\mu(z) < \varsigma(z)$ on $\overline{\mathscr{U}}$, the continuous embedding

$$L^{\varsigma(\cdot)}(\mathscr{U}) \hookrightarrow L^{\mu(\cdot)}(\mathscr{U})$$

is available. Moreover, Lemma 6 gives the equiintegrability of

$$\left\{ \left| {}^{\mathbf{H}}\mathscr{D}^{\alpha,\beta;\psi}_{0^+} \xi_n \right|^{\mu(z)} \right\}_{n\in\mathbb{N}}.$$

By Proposition 4,

$${}^{\mathbf{H}}\mathscr{D}^{\alpha,\beta;\psi}_{0^+} \xi_n \longrightarrow {}^{\mathbf{H}}\mathscr{D}^{\alpha,\beta;\psi}_{0^+} \xi \qquad \text{a.e. in } \mathscr{U}.$$

Since $\varsigma(z) - 1 < \mu(z)$, we have, for a suitable constant $C > 0$ independent of n,

$$\left| \sigma(z) \left| {}^{\mathbf{H}}\mathscr{D}^{\alpha,\beta;\psi}_{0^+} \xi_n \right|^{\varsigma(z)-2} {}^{\mathbf{H}}\mathscr{D}^{\alpha,\beta;\psi}_{0^+} \xi_n \right|^{\frac{\mu(z)}{\varsigma(z)-1}} \le C \left(1 + \left| {}^{\mathbf{H}}\mathscr{D}^{\alpha,\beta;\psi}_{0^+} \xi_n \right|^{\mu(z)} \right).$$

Hence the sequence

$$\left\{ \sigma(z) \left| {}^{\mathbf{H}}\mathscr{D}^{\alpha,\beta;\psi}_{0^+} \xi_n \right|^{\varsigma(z)-2} {}^{\mathbf{H}}\mathscr{D}^{\alpha,\beta;\psi}_{0^+} \xi_n \right\}_{n\in\mathbb{N}}$$

is uniformly integrable on $\mathscr{U}$. Therefore, by Vitali's convergence theorem,

$$\sigma(z) \left| {}^{\mathbf{H}}\mathscr{D}^{\alpha,\beta;\psi}_{0^+} \xi_n \right|^{\varsigma(z)-2} {}^{\mathbf{H}}\mathscr{D}^{\alpha,\beta;\psi}_{0^+} \xi_n \longrightarrow \sigma(z) \left| {}^{\mathbf{H}}\mathscr{D}^{\alpha,\beta;\psi}_{0^+} \xi \right|^{\varsigma(z)-2} {}^{\mathbf{H}}\mathscr{D}^{\alpha,\beta;\psi}_{0^+} \xi$$

strongly in $L^1(\mathscr{U})$. Consequently,

$$\begin{aligned} \lim_{n\to\infty} \int_{\mathscr{U}} \sigma(z) \left| {}^{\mathbf{H}}\mathscr{D}^{\alpha,\beta;\psi}_{0^+} \xi_n \right|^{\varsigma(z)-2} {}^{\mathbf{H}}\mathscr{D}^{\alpha,\beta;\psi}_{0^+} \xi_n \cdot {}^{\mathbf{H}}\mathscr{D}^{\alpha,\beta;\psi}_{0^+} \varphi \, dz \\ = \int_{\mathscr{U}} \sigma(z) \left| {}^{\mathbf{H}}\mathscr{D}^{\alpha,\beta;\psi}_{0^+} \xi \right|^{\varsigma(z)-2} {}^{\mathbf{H}}\mathscr{D}^{\alpha,\beta;\psi}_{0^+} \xi \cdot {}^{\mathbf{H}}\mathscr{D}^{\alpha,\beta;\psi}_{0^+} \varphi \, dz. \end{aligned} \tag{32}$$

Finally, combining (30)–(32), we conclude that ξ satisfies the weak formulation (6). Therefore,

$$\xi \in \mathbb{H}^{\alpha,\beta;\psi}_{\mu(z),0}(\mathscr{U})$$

is a weak solution of the problem, which completes the proof.

Example 1. To illustrate our theoretical results and demonstrate the applicability of Theorem 3, we present an illustrative example. Let the domain be $\mathscr{U} = (0,1)$, which is a bounded interval with Lipschitz boundary. We choose $\sigma(x) \equiv 1$ and $\tau(x) \equiv 1$. We consider the problem driven by the ψ-Hilfer fractional operator with the following specific parameters:

$$\alpha = 0.5, \quad \beta = 1, \quad \text{and} \quad \psi(x) = x.$$

We define the continuous variable exponents for all $x \in [0,1]$ as follows:

$$\varsigma(x) = 3 + x, \quad \gamma(x) = 2, \quad \text{and} \quad \vartheta(x) = 0.5.$$

Let the source term be $g(x) = 1$, for which

$$\int_0^1 |g(x)|\,dx = \int_0^1 1\,dx = 1 < \infty.$$

Hence, $g \in L^1((0,1))$, $g \geq 0$, and $g \not\equiv 0$.
Let us verify the crucial structural condition (7). For any $x \in [0,1]$, we calculate the right-hand side:

$$1 + \frac{1 - \vartheta(x)}{\gamma(x)} = 1 + \frac{1 - 0.5}{2} = 1.25.$$

Since $\varsigma(x) = 3 + x \geq 3$, it is obvious that:

$$\varsigma(x) > 1.25 \quad \text{for all } x \in [0,1].$$

Thus, condition (7) is strictly satisfied. Furthermore, following (8), the integrability exponent becomes:

$$\mu(x) = \frac{\varsigma(x)}{1.25} = 2.4 + 0.8x.$$

This explicitly shows that $1 < 2.4 \leq \mu(x) \leq 3.2$ across the domain, ensuring the necessary reflexivity of the energy space. Consequently, all the assumptions of Theorem 3 are fulfilled, successfully guaranteeing the existence of at least one weak positive solution $\xi \in \mathbb{H}^{0.5,1;x}_{\mu(\cdot),0}((0,1))$ for this specific configuration.

4 Conclusion

In this paper, we established the existence and regularity of weak positive solutions for nonlinear elliptic systems driven by variable-exponent ψ-Hilfer fractional operators. We successfully navigated the analytical challenges posed by singular nonlinearities and L^1-data within the framework of variable-exponent fractional Sobolev spaces.
A key contribution of this work is the identification of a regularization phenomenon: we demonstrated that the presence of lower-order terms significantly enhances solution regularity, counteracting the effects of rough data and singular character. The tailored approximation and energy techniques developed herein provide a robust foundation for future investigations into more complex fractional problems, such as systems with multiple singularities, critical growth, or coupled physical applications.

Conflicts of interest

The authors declare that there are no conflicts of interest.

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