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Performance assessment under negative data and uncertainty: The robust data envelopment analysis approach

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Abstract. This paper introduces a novel robust Data Envelopment Analysis (DEA) framework for assessing and ranking Decision-Making Units (DMUs) in the presence of negative data and deep uncertainty. Its methodological contribution is the derivation, via robust-optimization duality, of tractable linear robust counterparts of the Range Directional Measure (RDM) and Variant of Radial Measure (VRM) models under a composite Box–Polyhedral (budgeted) uncertainty set. Requiring only interval information on input/output data, the framework makes no assumptions about their probability distributions. By integrating the RDM and VRM models, which inherently accommodate negative data, with Robust Optimization (RO), the approach generates robust efficiency scores protected against simultaneous perturbations of all inputs and outputs within prescribed uncertainty bounds. Its efficacy is validated through a stock market application using financial indicators that may assume negative values, such as net profit and return on assets. The case study evaluates 20 stocks in the Banks and Credit Institutions industry of the Tehran Stock Exchange across 24 robustness–perturbation configurations. Comparative sensitivity analysis shows that the two robust models involve different trade-offs between the stability of efficiency-score levels and stock rankings under uncertainty, helping managers and investors identify stocks most sensitive to data imprecision. The results confirm that the proposed Robust DEA (RDEA) approach balances robustness and discriminatory power, providing a decision-support tool for financial analysis and other domains characterized by complex and uncertain data environments.

Keywords: Data envelopment analysis; directional distance function; negative data; range directional measure; variant of radial measure; deep uncertainty; robust optimization; stock market.
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1 Introduction

Performance evaluation, as a systematic process for measuring the success of operational entities in converting resources into outputs, plays a fundamental role in strategic decision-making. This concept holds critical importance irrespective of any particular application domain, in any organization, institution, industry, or system where resource allocation and productivity are of concern. From healthcare and educational systems to economic enterprises, financial institutions, transportation networks, agricultural units, governmental organizations, and even the assessment of technologies and energy systems, all require accurate and reliable tools for measuring efficiency [17, 30, 39, 54, 74]. In essence, wherever a set of homogeneous units consume specified inputs to produce outputs, the question of their relative performance and the identification of efficient and inefficient units naturally arises. Addressing this question constitutes a prerequisite for any improvement initiative, optimal resource allocation, and evidence-based policymaking.

Among the various approaches to performance evaluation, Data Envelopment Analysis (DEA) has attained a distinguished position as a nonparametric technique grounded in mathematical programming [13, 34, 43]. Originally introduced by Charnes et al. [11] building upon Farrell's [18] concept of efficiency, DEA computes the relative efficiency of Decision-Making Units (DMUs) through comparison with an empirical efficiency frontier, without requiring a pre-specified production function or a predetermined weighting structure. The inherent flexibility of DEA in handling multiple inputs and outputs measured on different scales, its independence from preassigned prices or weights, and its capacity to identify the sources of inefficiency have collectively rendered it a widely adopted tool across a broad spectrum of scientific and practical domains [44]. Following the introduction of the original CCR model, numerous theoretical extensions were developed, among the most notable of which are the BCC model incorporating variable returns to scale [3], the Slacks-Based Measure (SBM) models, Directional Distance Function (DDF) models, and super-efficiency models. While each of these extensions has addressed certain limitations of its predecessors, fundamental challenges persist in the practical application of DEA that continue to attract scholarly attention.

One of the most significant of these challenges is the inability of classical DEA models to accommodate negative data. The mathematical structure of the original DEA models is predicated on the assumption that all inputs and outputs are non-negative. Although this assumption is mathematically convenient and computationally desirable, it is violated in a wide range of real-world problems. In financial analysis, indicators such as net profit, return on assets, return on equity, and operating cash flow can naturally assume negative values. In environmental performance assessment, indicators such as net changes in pollutant emissions may be negative. In banking, indicators like operating profit and loss, and in economics, GDP growth rates during recessionary periods, represent further instances of potentially negative data. Consequently, the development of models capable of evaluating efficiency without restrictions on the sign of data constitutes an undeniable necessity.

Within the DEA literature, numerous efforts have been undertaken to overcome the limitation posed by negative data. The earliest approaches relied on data transformations and the property of translation invariance. Researchers such as Lovell [31], Lovell & Pastor [32], Pastor [40], and Seiford & Zhu [61] applied suitable translations to the original data, converting negative values into non-negative ones before applying standard DEA models. Although this approach proves effective under certain conditions, data transformation can alter the original informational structure of the problem and influence the efficiency evaluation results. This limitation motivated the development of models capable of handling negative values directly, without any translation or transformation of the original data.

Along this trajectory, Portela et al. [51] proposed the Range Directional Measure (RDM) model based on the directional distance approach. By defining a directional vector proportional to the observed range of variation in the data, this model enables efficiency evaluation without any sign restriction. Sharp et al. [64] introduced the Modified Slacks-Based Measure (MSBM) for measuring the efficiency of decision-making units in the presence of negative values in both inputs and outputs. Emrouznejad et al. [15] presented the Semi-Oriented Radial Measure (SORM) model, employing a partitioning approach to deal with negative values in DEA. Cheng et al. [12] proposed the Variant of Radial Measure (VRM) models for performance assessment of DMUs where negative data are present. By redefining the radial measure in a manner that eliminates the structural dependence on data positivity, the VRM offers a flexible evaluation framework. Kazemi Matin et al. [23] further proposed a modified semi-oriented radial measure model to address issues related to target setting within the SORM framework. This line of research has continued to expand along several complementary directions. Emrouznejad et al. [14] established boundedness properties of the SORM model with negative data, while Portela and Thanassoulis [52] extended Malmquist-type productivity indices to settings with negative data. Sahoo et al. [58] and Allahyar and Rostamy-Malkhalifeh [1] examined returns to scale and most productive scale size under negative data. Wei et al. [71] and Tone et al. [70] extended the Slacks-Based Measure to accommodate negative data, respectively via a modified SBM-based ranking method and a general treatment of negative data within SBM models, while Lin and Chen [29] developed a directional-distance-based super-efficiency model for negative data. On the inverse-DEA side, Amin and Al-Muharrami [2] and Soltanifar et al. [65] proposed inverse DEA and inverse DEA-R models for merger analysis with negative data, Ghiyasi and Zhu [20] proposed an inverse semi-oriented radial measure, and, most recently, Srivastava et al. [67] proposed a non-homogeneous DEA approach for decision-making under negative data. This body of research collectively reflects the sustained effort of the scholarly community to resolve the negative data limitation in DEA. A critical observation, however, is that the majority of these models have been developed under deterministic conditions, assuming that the observed data are precise and free from uncertainty.

The second fundamental challenge affecting the practical application of DEA models is the uncertainty inherent in the data. In real-world decision-making environments, the data employed in performance evaluation are rarely exact or deterministic. Measurement errors, environmental fluctuations, temporal variations [46], incomplete information, and the inherently uncertain nature of certain indicators all contribute to varying degrees of uncertainty in model inputs. Disregarding this uncertainty and relying solely on observed point values can lead to unreliable efficiency scores, unstable rankings, and ultimately flawed decisions. This concern assumes heightened significance in domains such as financial markets, where severe volatility and deep uncertainty are intrinsic characteristics.

Within the DEA literature, multiple approaches from the domain of optimization under uncertainty have been integrated to manage data uncertainty in the performance evaluation process. Stochastic programming, fuzzy optimization, interval programming, uncertainty theory, and Z-number theory are among the most prominent of these approaches, each addressing the problem of uncertainty in DEA through a distinct philosophy and mathematical apparatus [16, 36, 45, 72]. Stochastic DEA models uncertainty by assuming specific probability distributions for the uncertain parameters, yet this requires precise statistical information about data distributions that is frequently unavailable in practical settings. Fuzzy DEA leverages the concepts of fuzzy sets to model uncertainty through membership functions, though the specification of these functions can itself be accompanied by ambiguity and subjectivity.

Interval-based approaches, while intuitive and straightforward, may yield excessively wide efficiency intervals that lack discriminatory power.

Among the various approaches to managing uncertainty, Robust Optimization (RO) is recognized as one of the most powerful and practically viable tools [5, 7, 19, 21]. The fundamental advantage of robust optimization lies in its ability to produce solutions that remain feasible and near-optimal across all possible realizations of the parameters within a defined uncertainty set, without requiring the assumption of any specific probability distribution for the uncertain parameters [4, 6, 66]. This characteristic renders robust optimization particularly well-suited for conditions of deep uncertainty (i.e., conditions in which precise probabilistic information is unavailable). Moreover, robust models are typically reformulated into tractable optimization problems with reasonable computational complexity, offering a considerable practical advantage over certain alternative uncertainty management approaches. Additionally, robust optimization provides the capability to control the level of conservatism of the solution through the calibration of the uncertainty set parameters, thereby enabling the decision-maker to establish a desirable trade-off between performance and resilience against uncertainty [25–27].

The integration of robust optimization with data envelopment analysis has attracted growing research interest in recent years, and a number of studies have been conducted in this area [42]. Beyond the foundational review of [42], several recent contributions have advanced robust DEA along complementary directions. Toloo et al. [69] established a rigorous duality relationship between the multiplier and envelopment forms of robust DEA under a budgeted uncertainty set, showing that the worst-case multiplier problem is equivalent to the best-case dual envelopment problem. Peykani et al. [47] extended robust optimization to generalized leader-follower network DEA structures under uncertain data. In parallel, a distinct methodological thread has embedded robust optimization within the common-weights formulation of DEA [59] and within Russell and enhanced Russell efficiency measures [60], while Shakouri et al. [62] developed a stochastic p-robust DEA approach. More recently, Pourmahmoud and Arami [53] evaluated the cost efficiency of decision-making units in an uncertain environment, whereas Shakouri [63] integrated genetic algorithms and machine learning into a p-robust network cost-efficiency framework. Notwithstanding this growing body of work, the existing literature on Robust DEA still confronts significant research gaps. The most prominent of these gaps is the absence of a unified framework that simultaneously manages both the challenge of negative data and deep uncertainty within the efficiency evaluation process. A review of the literature reveals that prior studies have predominantly addressed each of these two issues independently and in isolation from the other. Models designed for negative data have generally disregarded uncertainty, while robust DEA models have largely been built upon the assumption of data non-negativity. Yet in many real-world applications, particularly in the domain of financial analysis and capital markets, both challenges coexist simultaneously. Financial indicators can assume negative values and, due to market fluctuations and economic conditions, are accompanied by considerable uncertainty. Neglecting either of these two dimensions severely undermines the validity and reliability of the evaluation results. To the best of the authors' knowledge, no prior study has integrated robust optimization with the RDM or VRM models so as to jointly accommodate negative data and deep uncertainty within a single DEA framework. A methodologically distinct approach to a closely related question was recently proposed by the authors [50], who combined the RDM model with uncertainty theory, rather than robust optimization, to handle negative and uncertain financial data; that approach, however, requires specifying an uncertainty distribution for every uncertain parameter, whereas the robust-optimization framework developed here requires only interval (deep-uncertainty) information and makes no distributional assumption, which is a materially different and, for many practical settings, a less restrictive modeling requirement.

The present research aims to fill this gap by proposing a novel framework based on Robust Data Envelopment Analysis (RDEA) that simultaneously addresses both of the aforementioned challenges. The proposed model integrates the analytical capabilities of the RDM and VRM models, which inherently possess the ability to handle negative data, with the principles of robust optimization. This integration is carried out such that the computed efficiency scores are immunized against simultaneous perturbations of all inputs and outputs within prescribed uncertainty bounds. In other words, the proposed framework seeks to evaluate the efficiency of decision-making units not merely on the basis of observed data values, but by accounting for the plausible range of their variations. Such an approach ensures that the model's results are more stable in the face of data fluctuations and provide greater reliability for decision-makers.

To examine the practical efficacy of the proposed framework, an empirical study in the domain of stock market analysis is conducted [41]. In this application, a set of firms is considered as decision-making units, and their performance is evaluated and ranked using several financial indicators. The selection of the stock market as the empirical setting for this research is noteworthy from multiple perspectives. First, many of the financial indicators used in firm performance analysis, such as net profit or return on assets, can assume negative values. Second, financial markets are inherently characterized by high levels of uncertainty and volatility. These features render the stock market a suitable testbed for examining the efficacy of the proposed model under conditions where both the challenges of negative data and uncertainty are simultaneously present. The results of the empirical analysis demonstrate that the proposed RDEA approach is capable of establishing an appropriate balance between model robustness against uncertainty and discriminatory power among decision-making units, thereby providing a reliable decision-support tool for complex and uncertain data environments.

The remainder of this paper is organized as follows. Section 2 addresses the methodological examination and review of foundational models for confronting the challenge of negative data. Section 3 defines the theoretical framework of deep uncertainty and presents in detail the robustification process of the proposed model based on uncertainty sets. Section 4 is devoted to practical implementation and a case study in the stock market domain, wherein the efficiency and robust ranking capability of the presented framework are evaluated. Finally, Section 5 concludes by synthesizing key research findings, articulating limitations, and offering recommendations for future research directions.

2 Modeling negative data in RDM and VRM

Suppose that there are L homogenous decision making units $DMU_k(k = 1,...,L)$ that convert M inputs $\alpha_{ik}(i = 1,...,M)$ into N outputs $\beta_{jk}(j = 1,...,N)$ and where the DMU_p is an under evaluation DMU. The non-negative intensity variables λ_k $(k = 1,...,L)$ determine the weight assigned to each observed DMU_k in constructing the composite reference unit against which DMU_p is evaluated. The Generic Directional Distance (GDD) model under variable returns to scale assumption is as follows [9,10]:

$$
\begin{aligned}
&\max\ \theta_p^{GDD} && && (1)\\
&\text{s.}t.\ \sum_{k=1}^{L}\alpha_{ik}\lambda_k \leq \alpha_{ip} - \psi_{\alpha i}\theta_p^{GDD}, && \forall i\\
&\qquad \sum_{k=1}^{L}\beta_{jk}\lambda_k \geq \beta_{jp} + \psi_{\beta j}\theta_p^{GDD}, && \forall j
\end{aligned}
$$

$$
\begin{aligned}
&\sum_{k=1}^{L} \lambda_k = 1, \\
&\lambda_k \geq 0,\ \forall k,\ \theta_p^{GDD} \geq 0.
\end{aligned}
$$

Portela et al. [51] proposed the RDM model for dealing with negative inputs and/or outputs by applying a modified version of the GDD model. They modified GDD model using an ideal point (α_1, β_1) that is presented in Equation (2) to identify direction vectors $(\psi_{\alpha i}, \psi_{\beta j})$. It should be explained that this ideal point can be one of $DMUs$, but this point usually is out of feasibility domain.

$$
\text{Ideal point} = (\alpha_1, \beta_1), \quad \text{where} \quad (\alpha_1)_i = \min_k \{\alpha_{ik}\},\ i = 1, \ldots, M, \quad \text{and} \quad (\beta_1)_j = \max_k \{\beta_{jk}\},\ j = 1, \ldots, N \tag{2}
$$

Then, Portela et al. [51] defined the vectors δ_{ip}^{-} and δ_{jp}^{+} as Equations (3) and (4) and named these vectors, the range of possible improvement for the decision making unit that is under evaluation.

$$
\delta_{ip}^{-} = \alpha_{ip} - \min_k \{\alpha_{ik}\}, \qquad \forall i \tag{3}
$$

$$
\delta_{jp}^{+} = \max_k \{\beta_{jk}\} - \beta_{jp}, \qquad \forall j \tag{4}
$$

The direction from DMU_p to the ideal point I is $(\psi_{\alpha i}, \psi_{\beta j}) = (\delta_{ip}^{-}, \delta_{jp}^{+})$. Based on the notion of the range of possible improvements, the RDM model is defined as follows:

$$
\begin{aligned}
\max\ & \theta_p^{RDM} \\
\text{s.t.}\ & \sum_{k=1}^{L} \alpha_{ik}\lambda_k \leq \alpha_{ip} - \delta_{ip}^{-}\theta_p^{RDM}, \qquad \forall i \\
& \sum_{k=1}^{L} \beta_{jk}\lambda_k \geq \beta_{jp} + \delta_{jp}^{+}\theta_p^{RDM}, \qquad \forall j \\
& \sum_{k=1}^{L} \lambda_k = 1, \\
& \lambda_k \geq 0,\ \forall k,\ \theta_p^{RDM} \geq 0.
\end{aligned} \tag{5}
$$

With respect to this notion that the range of possible improvement is always non-negative, RDM model can be used in the presence of negative values. Taking the dual of Model (5) introduces two sets of non-negative dual variables, namely γ_i $(i = 1, \ldots, M)$ and μ_j $(j = 1, \ldots, N)$, associated respectively with the M input constraints and the N output constraints of the envelopment form; these dual variables act as the multiplier weights of Model (6) below, and should not be confused with the intensity variables λ_k of Model (5). It should be noted that the dual form of RDM is as Model (6). Notably, Models (5) and (6) are the envelopment form and the multiplier form of RDM model, respectively.

$$
\begin{aligned}
\min\ & \sum_{i=1}^{M} \alpha_{ip}\gamma_i - \sum_{j=1}^{N} \beta_{jp}\mu_j + \omega_p \\
\text{s.t.}\ & \sum_{i=1}^{M} \alpha_{ik}\gamma_i - \sum_{j=1}^{N} \beta_{jk}\mu_j + \omega_p \geq 0, \qquad \forall k
\end{aligned} \tag{6}
$$

$$\sum_{i=1}^{M} \delta_{ip}^{-}\gamma_i + \sum_{j=1}^{N} \delta_{jp}^{+}\mu_j \geq 1$$
$$\gamma_i, \mu_j \geq 0, \qquad \forall i, j$$

In the following, Cheng et al. [12] introduced a VRM model for performance measurement of DMUs where negative values are present. They applied $|\alpha_{ip}|$ and $|\beta_{jp}|$ instead of δ_{ip}^{-} and δ_{jp}^{+} vectors that are the range of possible improvements for DMU_p and defined the envelopment form VRM model as follows:

$$\begin{aligned}
&\max\ \theta_p^{VRM} \\
&\text{s.}t.\ \sum_{k=1}^{L} \alpha_{ik}\lambda_k \leq \alpha_{ip} - |\alpha_{ip}|\theta_p^{VRM}, \qquad \forall i \\
&\sum_{k=1}^{L} \beta_{jk}\lambda_k \geq \beta_{jp} + |\beta_{jp}|\theta_p^{VRM}, \qquad \forall j \\
&\sum_{k=1}^{L} \lambda_k = 1, \\
&\lambda_k \geq 0,\ \forall k,\ \theta_p^{VRM} \geq 0.
\end{aligned} \tag{7}$$

Also, the dual formulation of the VRM model, which provides the basis for deriving its multiplier form, is presented as follows:

$$\begin{aligned}
&\min\ \sum_{i=1}^{M} \alpha_{ip}\gamma_i - \sum_{j=1}^{N} \beta_{jp}\mu_j + \omega_p \\
&\text{s.}t.\ \sum_{i=1}^{M} \alpha_{ik}\gamma_i - \sum_{j=1}^{N} \beta_{jk}\mu_j + \omega_p \geq 0, \qquad \forall k \\
&\sum_{i=1}^{M} |\alpha_{ip}|\gamma_i + \sum_{j=1}^{N} |\beta_{jp}|\mu_j \geq 1, \\
&\gamma_i, \mu_j \geq 0,\ \forall i, j.
\end{aligned} \tag{8}$$

It should be noted that, in the RDM and VRM models, $\theta_p^{\text{RDM}*}$ and $\theta_p^{\text{VRM}*}$ represent the optimal inefficiency scores of DMU$_p$, obtained from Models (5) and (7), respectively. Because the RDM and VRM models define their inefficiency scores differently, model-specific transformations are used to obtain the corresponding efficiency scores. Specifically, the efficiency scores are calculated as $1-\theta_p^{\text{RDM}*}$ for the RDM model and $1/(1+\theta_p^{\text{VRM}*})$ for the VRM model. Both transformations assign a score of 1 to a fully efficient DMU and yield progressively lower efficiency scores as inefficiency increases. This convention is used consistently in Tables 3-5 of Section 4.

At the conclusion of this section, it is important to highlight that the multiplier forms of the RDM and VRM models, presented as Models (6) and (8), serve as the foundational frameworks of the study and form the basis for developing the robust methodological approach introduced in the following section.

3 The proposed robust DEA approach

Throughout this paper, the term *deep uncertainty* is used, following the decision-making-under-uncertainty literature, to denote a setting in which the analyst is unable, or unwilling, to specify a probability distribution over the possible realizations of an uncertain parameter and possesses only interval (bounded) information on the range within which the parameter may fall; this stands in contrast to *risk*, where a full probability distribution is assumed known. Robust optimization is the natural analytical tool for deep uncertainty because, unlike stochastic or fuzzy programming, it requires no distributional assumption whatsoever: the decision-maker specifies only an uncertainty set, here the Box-Polyhedral (budgeted) uncertainty set of Bertsimas and Sim [6], together with a tunable budget of robustness Γ, and the resulting model is immunized against every realization of the uncertain parameters within that set. This is precisely the sense in which the uncertainty sets in Equations (11)-(16) below are constructed: each uncertain coefficient is known only to lie within a symmetric interval around its nominal (observed) value, with no assumption on the likelihood of any particular value within that interval. In this section, the robust counterparts of the RDM and VRM models are developed, and the step-by-step procedure for formulating them within the framework of robust optimization is described. It is important to emphasize that all parameters in the uncertain versions of the RDM and VRM models, presented as Models (9) and (10), are characterized by deep uncertainty for which only interval information consisting of lower and upper bounds is available.

$$
\begin{aligned}
\min\ & \sum_{i=1}^{M} \tilde{\alpha}_{ip}\gamma_i - \sum_{j=1}^{N} \tilde{\beta}_{jp}\mu_j + \omega_p \\
\text{s.}t.\ & \sum_{i=1}^{M} \tilde{\alpha}_{ik}\gamma_i - \sum_{j=1}^{N} \tilde{\beta}_{jk}\mu_j + \omega_p \geq 0, \qquad \forall k \\
& \sum_{i=1}^{M} \tilde{\delta}_{ip}^{-}\gamma_i + \sum_{j=1}^{N} \tilde{\delta}_{jp}^{+}\mu_j \geq 1 \\
& \gamma_i, \mu_j \geq 0, \qquad \forall i, j
\end{aligned} \tag{9}
$$

$$
\begin{aligned}
\min\ & \sum_{i=1}^{M} \tilde{\alpha}_{ip}\gamma_i - \sum_{j=1}^{N} \tilde{\beta}_{jp}\mu_j + \omega_p \\
\text{s.}t.\ & \sum_{i=1}^{M} \tilde{\alpha}_{ik}\gamma_i - \sum_{j=1}^{N} \tilde{\beta}_{jk}\mu_j + \omega_p \geq 0, \qquad \forall k \\
& \sum_{i=1}^{M} |\tilde{\alpha}_{ip}|\gamma_i + \sum_{j=1}^{N} |\tilde{\beta}_{jp}|\mu_j \geq 1 \\
& \gamma_i, \mu_j \geq 0, \qquad \forall i, j
\end{aligned} \tag{10}
$$

It is important to note that, for deriving the robust counterparts of the uncertain constraints in the proposed models, a composite uncertainty set constructed as the intersection of the Box and Polyhedral sets is employed. This integrated uncertainty structure, originally introduced by Bertsimas and Sim [6] and subsequently generalized by Li et al. [25], is among the most efficient and widely utilized formulations across various applications, as it allows the level of conservatism to be adjusted through a tuning parameter while ensuring that the robust counterpart of each uncertain constraint remains linear [6, 25, 42]. Furthermore, all uncertain parameters, denoted by the "˜" symbol, are assumed to be independent, symmetric, and bounded random variables whose values follow Relations (11) to (16). In these relations, the "ˆ" symbol represents the nominal value of each uncertain parameter, while ξ and ϖ denote the percentage of perturbation and random variables taking values within the interval $[-1,1]$, respectively.

$$\left\{ \tilde{\alpha}_{ik} \,|\, \tilde{\alpha}_{ik} = \hat{\alpha}_{ik} + \xi\, \hat{\alpha}_{ik} \varpi^{\alpha}_{ik};\ |\varpi^{\alpha}_{ik}| \leq 1;\ \sum_{i=1}^{M} |\varpi^{\alpha}_{ik}| \leq \Gamma^{\alpha},\ \forall i,k \right\}, \tag{11}$$

$$\left\{ \tilde{\beta}_{jk} \,|\, \tilde{\beta}_{jk} = \hat{\beta}_{jk} + \xi\, \hat{\beta}_{jk} \varpi^{\beta}_{jk};\ |\varpi^{\beta}_{jk}| \leq 1;\ \sum_{j=1}^{N} |\varpi^{\beta}_{jk}| \leq \Gamma^{\beta},\ \forall j,k \right\}, \tag{12}$$

$$\left\{ \tilde{\delta}^{-}_{ik} \,|\, \tilde{\delta}^{-}_{ik} = \hat{\delta}^{-}_{ik} + \xi\, \hat{\delta}^{-}_{ik} \varpi^{\delta^{-}}_{ik};\ |\varpi^{\delta^{-}}_{ik}| \leq 1;\ \sum_{i=1}^{M} |\varpi^{\delta^{-}}_{ik}| \leq \Gamma^{\delta^{-}},\ \forall i,k \right\}, \tag{13}$$

$$\left\{ \tilde{\delta}^{+}_{jk} \,|\, \tilde{\delta}^{+}_{jk} = \hat{\delta}^{+}_{jk} + \xi\, \hat{\delta}^{+}_{jk} \varpi^{\delta^{+}}_{jk};\ |\varpi^{\delta^{+}}_{jk}| \leq 1;\ \sum_{j=1}^{N} |\varpi^{\delta^{+}}_{jk}| \leq \Gamma^{\delta^{+}},\ \forall j,k \right\}, \tag{14}$$

$$\left\{ |\tilde{\alpha}_{ik}| \,|\, |\tilde{\alpha}_{ik}| = |\hat{\alpha}_{ik}| + \xi\, |\hat{\alpha}_{ik}| \varpi^{\alpha}_{ik};\ |\varpi^{\alpha}_{ik}| \leq 1;\ \sum_{i=1}^{M} |\varpi^{\alpha}_{ik}| \leq \Gamma^{\alpha},\ \forall i,k \right\}, \tag{15}$$

$$\left\{ |\tilde{\beta}_{jk}| \,|\, |\tilde{\beta}_{jk}| = |\hat{\beta}_{jk}| + \xi\, |\hat{\beta}_{jk}| \varpi^{\beta}_{jk};\ |\varpi^{\beta}_{jk}| \leq 1;\ \sum_{j=1}^{N} |\varpi^{\beta}_{jk}| \leq \Gamma^{\beta},\ \forall j,k \right\}. \tag{16}$$

Each of the uncertainty sets in Equations (11)-(16) plays a distinct role in the robustification of the RDM and VRM models. Equations (11) and (12) describe the uncertainty affecting the raw input and output coefficients α_{ik} and β_{jk} that enter the technology (envelopment) constraints of Models (9) and (10); their robust counterparts protect the feasibility of these constraints against simultaneous adverse perturbations of up to Γ^{α} inputs and Γ^{β} outputs. Equations (13) and (14) describe the uncertainty affecting the RDM-specific range-of-improvement coefficients δ^{-}_{ip} and δ^{+}_{jp}, which enter only the normalization constraint of Model (9); since $\delta^{-}_{ip} = \alpha_{ip} - \min_k\{\alpha_{ik}\}$ is itself a function of uncertain data (Equation (3)), its uncertainty is specified directly, in the same additive-perturbation form, rather than propagated analytically from the uncertainty of α_{ip} and $\min_k\{\alpha_{ik}\}$ taken separately, which would otherwise require characterizing the uncertainty of a minimum over several uncertain quantities, a considerably more complex problem. Equations (15) and (16) play the analogous role for the VRM-specific coefficients $|\alpha_{ip}|$ and $|\beta_{jp}|$, which enter the normalization constraint of Model (10).

A modeling caveat regarding the uncertainty sets in Equations (15)-(16) deserves explicit discussion. Because $|\tilde{\alpha}_{ik}|$ and $|\tilde{\beta}_{jk}|$ are perturbed directly at the level of their absolute value, the sign of the underlying uncertain parameter is not tracked by the model: the perturbation is applied to the magnitude $|\hat{\alpha}_{ik}|$ rather than to the signed value $\hat{\alpha}_{ik}$ itself, so a coefficient with negative nominal value remains associated with a perturbed magnitude without a reconstructed sign. This treatment is consistent with, and directly mirrors, the VRM direction vector, which by construction uses only $|\alpha_{ip}|$ and $|\beta_{jp}|$ rather than the signed values as the direction of improvement (Model (7)), so that the robust counterpart in Equation (18) perturbs precisely the same magnitude-based quantities that already define the nominal VRM model. This design implies that the proposed uncertainty set is not intended to, and does not, capture scenarios in which a realization could plausibly cross zero and change sign during the observation period (e.g., a firm's net profit shifting from a small loss to a small gain); the sign of every observation is treated as fixed and known, and only its magnitude is uncertain. We regard this as a deliberate simplification, consistent with the underlying VRM formulation, rather than an oversight, and we return to the design of a sign-aware uncertainty set as an explicit avenue for future research in Section 5.

The robust counterparts of the Uncertain Range Directional Measure (URDM) and Uncertain Variant of Radial Measure (UVRM) models are formulated as Models (17) and (18), respectively. Notably, $\Phi(\gamma_i,\mu_j,\Gamma_p^{\alpha\beta})$, $\Phi(\gamma_i,\mu_j,\Gamma_k^{\alpha\beta})$, $\Phi(\gamma_i,\mu_j,\Gamma_p^{\delta^-\delta^+})$ and $\Phi(\gamma_i,\mu_j,\Gamma_p^{|\alpha||\beta|})$ are the protection functions corresponding to the uncertain data:

$$\begin{aligned}
&\min \Upsilon^{URDM} \\
&\text{s.t.} \sum_{i=1}^{M}\alpha_{ip}\gamma_i-\sum_{j=1}^{N}\beta_{jp}\mu_j+\omega_p+\Phi(\gamma_i,\mu_j,\Gamma_p^{\alpha\beta})\le\Upsilon^{URDM}, \\
&\quad \sum_{i=1}^{M}\alpha_{ik}\gamma_i-\sum_{j=1}^{N}\beta_{jk}\mu_j+\omega_p-\Phi(\gamma_i,\mu_j,\Gamma_k^{\alpha\beta})\ge 0,\ \forall k \\
&\quad \sum_{i=1}^{M}\delta_{ip}^{-}\gamma_i+\sum_{j=1}^{N}\delta_{jp}^{+}\mu_j-\Phi(\gamma_i,\mu_j,\Gamma_p^{\delta^-\delta^+})\ge 1, \\
&\quad \gamma_i,\ \mu_j\ge 0, \qquad \forall i,j
\end{aligned} \tag{17}$$

$$\begin{aligned}
&\min \Upsilon^{UVRM} \\
&\text{s.t.} \sum_{i=1}^{M}\alpha_{ip}\gamma_i-\sum_{j=1}^{N}\beta_{jp}\mu_j+\omega_p+\Phi(\gamma_i,\mu_j,\Gamma_p^{\alpha\beta})\le\Upsilon^{UVRM}, \\
&\quad \sum_{i=1}^{M}\alpha_{ik}\gamma_i-\sum_{j=1}^{N}\beta_{jk}\mu_j+\omega_p-\Phi(\gamma_i,\mu_j,\Gamma_k^{\alpha\beta})\ge 0,\ \forall k \\
&\quad \sum_{i=1}^{M}|\alpha_{ip}|\gamma_i+\sum_{j=1}^{N}|\beta_{jp}|\mu_j-\Phi(\gamma_i,\mu_j,\Gamma_p^{|\alpha||\beta|})\ge 1, \\
&\quad \gamma_i,\ \mu_j\ge 0, \qquad \forall i,j
\end{aligned} \tag{18}$$

Before formally defining these sets, we first fix the notation used in Equations (19)-(24): $\Lambda^{\alpha}=\{1,\ldots,M\}$, $\Lambda^{\beta}=\{1,\ldots,N\}$, $\Lambda^{\delta^-}=\{1,\ldots,M\}$, $\Lambda^{\delta^+}=\{1,\ldots,N\}$, $\Lambda^{|\alpha|}=\{1,\ldots,M\}$, and $\Lambda^{|\beta|}=\{1,\ldots,N\}$ denote, respectively, the full index sets of the uncertain input, output, range-of-improvement, and absolute-value coefficients that are allowed to deviate from their nominal values. For each uncertainty set, $\Psi^{(\cdot)}$ denotes a subset of $\lfloor\Gamma^{(\cdot)}\rfloor$ indices whose associated coefficients are permitted to attain their extreme (worst-case) perturbation simultaneously, and $\Omega^{(\cdot)}$ denotes one additional index, not already contained in $\Psi^{(\cdot)}$, whose coefficient is allowed to deviate by only the residual fractional share $\Gamma^{(\cdot)}-\lfloor\Gamma^{(\cdot)}\rfloor$; the set $\dagger^{(\cdot)}=\Psi^{(\cdot)}\cup\{\Omega^{(\cdot)}\}$ thus collects the indices that jointly attain their worst case for a given realization of the budget of robustness $\Gamma^{(\cdot)}$. Subsequently, the sets $\dagger^{\alpha}$, $\dagger^{\beta}$, $\dagger^{\delta^-}$, $\dagger^{\delta^+}$, $\dagger^{|\alpha|}$, and $\dagger^{|\beta|}$ are formally defined by Equations (19) to (24), respectively:

$$\dagger^{\alpha}=\{\Psi^{\alpha}\cup\{\Omega^{\alpha}\}\,|\,\Psi^{\alpha}\subseteq\Lambda^{\alpha},\ |\Psi^{\alpha}|=\lfloor\Gamma^{\alpha}\rfloor,\ \Omega^{\alpha}\in\Lambda^{\alpha}\setminus\Psi^{\alpha}\}, \tag{19}$$

$$\dagger^{\beta} = \{\Psi^{\beta} \cup \{\Omega^{\beta}\} \,|\, \Psi^{\beta} \subseteq \Lambda^{\beta},\ |\Psi^{\beta}| = \lfloor \Gamma^{\beta} \rfloor,\ \Omega^{\beta} \in \Lambda^{\beta} \setminus \Psi^{\beta}\}, \tag{20}$$

$$\dagger^{\delta^-} = \{\Psi^{\delta^-} \cup \{\Omega^{\delta^-}\} \,|\, \Psi^{\delta^-} \subseteq \Lambda^{\delta^-},\ |\Psi^{\delta^-}| = \lfloor \Gamma^{\delta^-} \rfloor,\ \Omega^{\delta^-} \in \Lambda^{\delta^-} \setminus \Psi^{\delta^-}\}, \tag{21}$$

$$\dagger^{\delta^+} = \{\Psi^{\delta^+} \cup \{\Omega^{\delta^+}\} \,|\, \Psi^{\delta^+} \subseteq \Lambda^{\delta^+},\ |\Psi^{\delta^+}| = \lfloor \Gamma^{\delta^+} \rfloor,\ \Omega^{\delta^+} \in \Lambda^{\delta^+} \setminus \Psi^{\delta^+}\}, \tag{22}$$

$$\dagger^{|\alpha|} = \{\Psi^{|\alpha|} \cup \{\Omega^{|\alpha|}\} \,|\, \Psi^{|\alpha|} \subseteq \Lambda^{|\alpha|},\ |\Psi^{|\alpha|}| = \lfloor \Gamma^{|\alpha|} \rfloor,\ \Omega^{|\alpha|} \in \Lambda^{|\alpha|} \setminus \Psi^{|\alpha|}\}, \tag{23}$$

$$\dagger^{|\beta|} = \{\Psi^{|\beta|} \cup \{\Omega^{|\beta|}\} \,|\, \Psi^{|\beta|} \subseteq \Lambda^{|\beta|},\ |\Psi^{|\beta|}| = \lfloor \Gamma^{|\beta|} \rfloor,\ \Omega^{|\beta|} \in \Lambda^{|\beta|} \setminus \Psi^{|\beta|}\}. \tag{24}$$

Consequently, the protection functions $\Phi(\gamma_i, \mu_j, \Gamma_p^{\alpha\beta})$, $\Phi(\gamma_i, \mu_j, \Gamma_k^{\alpha\beta})$, $\Phi(\gamma_i, \mu_j, \Gamma_p^{\delta^-\delta^+})$ and $\Phi(\gamma_i, \mu_j, \Gamma_p^{|\alpha||\beta|})$ are defined in Relations (25) to (28), respectively:

$$\Phi(\gamma_i^*, \mu_j^*, \Gamma_p^{\alpha\beta}) = \max_{\dagger^{\alpha}, \dagger^{\beta}} \left\{ \begin{array}{l} \sum_{i \in \Psi^{\alpha}} \xi \hat{\alpha}_{ip} \gamma_i + (\Gamma^{\alpha} - \lfloor \Gamma^{\alpha} \rfloor) \xi \hat{\alpha}_{\Omega^{\alpha} p} \gamma_{\Omega^{\alpha}} + \\ \sum_{j \in \Psi^{\beta}} \xi \hat{\beta}_{jp} \mu_j + (\Gamma^{\beta} - \lfloor \Gamma^{\beta} \rfloor) \xi \hat{\beta}_{\Omega^{\beta} p} \mu_{\Omega^{\beta}} \end{array} \right\}, \tag{25}$$

$$\Phi(\gamma_i^*, \mu_j^*, \Gamma_k^{\alpha\beta}) = \max_{\dagger^{\alpha}, \dagger^{\beta}} \left\{ \begin{array}{l} \sum_{i \in \Psi^{\alpha}} \xi \hat{\alpha}_{ik} \gamma_i + (\Gamma^{\alpha} - \lfloor \Gamma^{\alpha} \rfloor) \xi \hat{\alpha}_{\Omega^{\alpha} k} \gamma_{\Omega^{\alpha}} + \\ \sum_{j \in \Psi^{\beta}} \xi \hat{\beta}_{jk} \mu_j + (\Gamma^{\beta} - \lfloor \Gamma^{\beta} \rfloor) \xi \hat{\beta}_{\Omega^{\beta} k} \mu_{\Omega^{\beta}} \end{array} \right\}, \tag{26}$$

$$\Phi(\gamma_i, \mu_j, \Gamma_p^{\delta^-\delta^+}) = \max_{\dagger^{\delta^-}, \dagger^{\delta^+}} \left\{ \begin{array}{l} \sum_{i \in \Psi^{\delta^-}} \xi \hat{\delta}_{ip}^- \gamma_i + (\Gamma^{\delta^-} - \lfloor \Gamma^{\delta^-} \rfloor) \xi \hat{\delta}_{\Omega^{\delta^-} p}^- \gamma_{\Omega^{\delta^-}} + \\ \sum_{j \in \Psi^{\delta^+}} \xi \hat{\delta}_{jp}^+ \mu_j + (\Gamma^{\delta^+} - \lfloor \Gamma^{\delta^+} \rfloor) \xi \hat{\delta}_{\Omega^{\delta^+} p}^+ \mu_{\Omega^{\delta^+}} \end{array} \right\}, \tag{27}$$

$$\Phi(\gamma_i, \mu_j, \Gamma_p^{|\alpha||\beta|}) = \max_{\dagger^{|\alpha|}, \dagger^{|\beta|}} \left\{ \begin{array}{l} \sum_{i \in \Psi^{|\alpha|}} \xi |\hat{\alpha}_{ip}| \gamma_i + (\Gamma^{|\alpha|} - \lfloor \Gamma^{|\alpha|} \rfloor) \xi |\hat{\alpha}_{\Omega^{|\alpha|} p}| \gamma_{\Omega^{|\alpha|}} + \\ \sum_{j \in \Psi^{|\beta|}} \xi |\hat{\beta}_{jp}| \mu_j + (\Gamma^{|\beta|} - \lfloor \Gamma^{|\beta|} \rfloor) \xi |\hat{\beta}_{\Omega^{|\beta|} p}| \mu_{\Omega^{|\beta|}} \end{array} \right\}. \tag{28}$$

In Equations (25)–(28), the summations over $\Psi^{(\cdot)}$ collect the $\lfloor \Gamma^{(\cdot)} \rfloor$ largest worst-case coefficient contributions, while the term indexed by $\Omega^{(\cdot)}$ represents the fractional contribution of one additional coefficient. For example, when $\Gamma^{\alpha} = 2.5$, the protection function selects the two largest input-side contributions in full and adds one-half of the next-largest contribution.

Before stating the final linear robust counterparts, it is instructive to show explicitly how the protection functions defined in Equations (25)-(28), each formulated as a combinatorial maximization problem over the index sets $\dagger^{\alpha}, \dagger^{\beta}, \dagger^{\delta^-}, \dagger^{\delta^+}, \dagger^{|\alpha|}, \dagger^{|\beta|}$, are transformed into an equivalent system of linear constraints; this step, omitted from the original submission, follows the classical result of Bertsimas and Sim [6] on budgeted (Box-Polyhedral) uncertainty sets, as further clarified in the duality treatment of Toloo et al. [69]. Consider, without loss of generality, the protection function $\Phi(\gamma_i^*, \mu_j^*, \Gamma_p^{\alpha\beta})$ of Equation (25). Because the combinatorial maximization there selects, among all uncertain coefficients indexed by $\Lambda^{\alpha} \cup \Lambda^{\beta}$, the $\lfloor \Gamma_p^{\alpha\beta} \rfloor$ largest absolute contributions plus a fractional share of one additional coefficient, it is equivalent, by the argument of Bertsimas and Sim [6], to the following bounded linear program in auxiliary variables $z_i, z_j \in [0,1]$:

$$\begin{aligned}
\Phi(\gamma_i^*, \mu_j^*, \Gamma_p^{\alpha\beta}) = \max \ & \sum_{i=1}^{M} \xi \hat{\alpha}_{ip} \gamma_i z_i + \sum_{j=1}^{N} \xi \hat{\beta}_{jp} \mu_j z_j \\
\text{s.}t. \ & \sum_{i=1}^{M} z_i + \sum_{j=1}^{N} z_j \leq \Gamma_p^{\alpha\beta}, \\
& 0 \leq z_i \leq 1, \ 0 \leq z_j \leq 1.
\end{aligned} \tag{29}$$

Because Model (29) is a bounded linear program for any fixed (γ_i, μ_j), strong LP duality applies. Taking the dual of Model (29), with $\Theta_p^{\alpha\beta} \geq 0$ the dual variable associated with the budget constraint $\sum_i z_i + \sum_j z_j \leq \Gamma_p^{\alpha\beta}$, and $\Delta_{ip}^{\alpha\beta}, \nabla_{jp}^{\alpha\beta} \geq 0$ the dual variables associated with the upper bounds $z_i \leq 1$ and $z_j \leq 1$, respectively, yields

$$\begin{aligned}
\Phi(\gamma_i^*, \mu_j^*, \Gamma_p^{\alpha\beta}) = \min \ & \Theta_p^{\alpha\beta} \Gamma_p^{\alpha\beta} + \sum_{i=1}^{M} \Delta_{ip}^{\alpha\beta} + \sum_{j=1}^{N} \nabla_{jp}^{\alpha\beta} \\
\text{s.}t. \ & \Theta_p^{\alpha\beta} + \Delta_{ip}^{\alpha\beta} \geq \xi \hat{\alpha}_{ip} \gamma_i, \quad \forall i \\
& \Theta_p^{\alpha\beta} + \nabla_{jp}^{\alpha\beta} \geq \xi \hat{\beta}_{jp} \mu_j, \quad \forall j \\
& \Theta_p^{\alpha\beta}, \Delta_{ip}^{\alpha\beta}, \nabla_{jp}^{\alpha\beta} \geq 0.
\end{aligned} \tag{30}$$

Substituting the dual (minimization) representation (30) of the protection function directly in place of $\Phi(\gamma_i, \mu_j, \Gamma_p^{\alpha\beta})$ in the first constraint of Model (17) removes the embedded maximization operator: since the outer problem already minimizes Υ^{URDM} subject to that constraint holding for the worst case, replacing $\Phi(\cdot)$ by its dual minimization problem (30), together with the associated dual feasibility constraints, preserves equivalence while yielding a single-level linear program. The remaining protection functions $\Phi(\gamma_i, \mu_j, \Gamma_k^{\alpha\beta})$, $\Phi(\gamma_i, \mu_j, \Gamma_p^{\delta^-\delta^+})$, and $\Phi(\gamma_i, \mu_j, \Gamma_p^{|\alpha||\beta|})$ of Equations (26)-(28) are linearized in exactly the same manner, each introducing its own triple of dual variables (Θ, Δ, ∇). Applying this substitution to all four protection functions in Models (17) and (18) yields precisely the linear robust counterparts stated in Models (31) and (32) below. In these models, $\Theta_p^{\alpha\beta}$ and $\Theta_k^{\alpha\beta}$ denote the dual budget variables associated with the protection functions evaluated at the perturbed data of DMU_p and of each comparator DMU_k, respectively; $\Delta_{i(\cdot)}^{(\cdot)}$ denotes the dual bound variable associated with the input-side (α-type) uncertain coefficients, while $\nabla_{j(\cdot)}^{(\cdot)}$ denotes the dual bound variable associated with the output-side (β-type) uncertain coefficients; and $\Theta_p^{\delta^-\delta^+}$ and $\Theta_p^{|\alpha||\beta|}$ play the analogous role for the third (normalization) constraints of Models (31) and (32), respectively.

Finally, utilizing the Box-Polyhedral uncertainty set, the Robust Range Directional Measure (RRDM) and Robust Variant of Radial Measure (RVRM) models, which are suitable for settings involving negative data and deep uncertainty, are formulated as Models (31) and (32), respectively, where Υ^{RRDM} and Υ^{RVRM} denote the worst-case (robust) objective values of the RRDM and RVRM models, i.e., the counterparts of Υ^{URDM} and Υ^{UVRM} once every protection function has been linearized via the duality argument above:

$$\min \Upsilon^{RRDM} \tag{31}$$

$$
\begin{aligned}
\text{s.}t.\ & \sum_{i=1}^{M} \alpha_{ip}\gamma_i - \sum_{j=1}^{N} \beta_{jp}\mu_j + \omega_p + \Theta_p^{\alpha\beta}\Gamma_p^{\alpha\beta} + \sum_{i=1}^{M} \Delta_{ip}^{\alpha\beta} + \sum_{j=1}^{N} \nabla_{jp}^{\alpha\beta} \leq \Upsilon^{RRDM}, \\
& \sum_{i=1}^{M} \alpha_{ik}\gamma_i - \sum_{j=1}^{N} \beta_{jk}\mu_j + \omega_p - \Theta_k^{\alpha\beta}\Gamma_k^{\alpha\beta} - \sum_{i=1}^{M} \Delta_{ik}^{\alpha\beta} - \sum_{j=1}^{N} \nabla_{jk}^{\alpha\beta} \geq 0,\ \forall k \\
& \sum_{i=1}^{M} \delta_{ip}^{-}\gamma_i + \sum_{j=1}^{N} \delta_{jp}^{+}\mu_j - \Theta_p^{\delta^-\delta^+}\Gamma_p^{\delta^-\delta^+} - \sum_{i=1}^{M} \Delta_{ip}^{\delta^-\delta^+} - \sum_{j=1}^{N} \nabla_{jp}^{\delta^-\delta^+} \geq 1, \\
& \Theta_p^{\alpha\beta} + \Delta_{ip}^{\alpha\beta} \geq \xi\hat{\alpha}_{ip}\gamma_i, && \forall i \in \Lambda^{\alpha}, \\
& \Theta_p^{\alpha\beta} + \nabla_{jp}^{\alpha\beta} \geq \xi\hat{\beta}_{jp}\mu_j, && \forall j \in \Lambda^{\beta} \\
& \Theta_k^{\alpha\beta} + \Delta_{ik}^{\alpha\beta} \geq \xi\hat{\alpha}_{ik}\gamma_i, && \forall k, i \in \Lambda^{\alpha} \\
& \Theta_k^{\alpha\beta} + \nabla_{jk}^{\alpha\beta} \geq \xi\hat{\beta}_{jk}\mu_j, && \forall k, j \in \Lambda^{\beta} \\
& \Theta_p^{\delta^-\delta^+} + \Delta_{ip}^{\delta^-\delta^+} \geq \xi\hat{\delta}_{ip}^{-}\gamma_i, && \forall i \in \Lambda^{\delta^-} \\
& \Theta_p^{\delta^-\delta^+} + \nabla_{jp}^{\delta^-\delta^+} \geq \xi\hat{\delta}_{jp}^{+}\mu_j, && \forall j \in \Lambda^{\delta^+} \\
& \Theta_p^{\alpha\beta}, \Theta_k^{\alpha\beta}, \Theta_p^{\delta^-\delta^+} \geq 0, && \forall k \\
& \Delta_{ip}^{\alpha\beta}, \nabla_{jp}^{\alpha\beta}, \Delta_{ik}^{\alpha\beta}, \nabla_{jk}^{\alpha\beta}, \Delta_{ip}^{\delta^-\delta^+}, \nabla_{jp}^{\delta^-\delta^+} \geq 0, \forall i, j, k \\
& \gamma_i, \mu_j \geq 0, && \forall i, j
\end{aligned}
$$

$$
\begin{aligned}
& \min \Upsilon^{RVRM} \qquad\qquad (32) \\
\text{s.}t.\ & \sum_{i=1}^{M} \alpha_{ip}\gamma_i - \sum_{j=1}^{N} \beta_{jp}\mu_j + \omega_p + \Theta_p^{\alpha\beta}\Gamma_p^{\alpha\beta} + \sum_{i=1}^{M} \Delta_{ip}^{\alpha\beta} + \sum_{j=1}^{N} \nabla_{jp}^{\alpha\beta} \leq \Upsilon^{RVRM}, \\
& \sum_{i=1}^{M} \alpha_{ik}\gamma_i - \sum_{j=1}^{N} \beta_{jk}\mu_j + \omega_p - \Theta_k^{\alpha\beta}\Gamma_k^{\alpha\beta} - \sum_{i=1}^{M} \Delta_{ik}^{\alpha\beta} - \sum_{j=1}^{N} \nabla_{jk}^{\alpha\beta} \geq 0,\ \forall k \\
& \sum_{i=1}^{M} |\alpha_{ip}|\gamma_i + \sum_{j=1}^{N} |\beta_{jp}|\mu_j - \Theta_p^{|\alpha||\beta|}\Gamma_p^{|\alpha||\beta|} - \sum_{i=1}^{M} \Delta_{ip}^{|\alpha||\beta|} - \sum_{j=1}^{N} \nabla_{jp}^{|\alpha||\beta|} \geq 1, \\
& \Theta_p^{\alpha\beta} + \Delta_{ip}^{\alpha\beta} \geq \xi\hat{\alpha}_{ip}\gamma_i, && \forall i \in \Lambda^{\alpha} \\
& \Theta_p^{\alpha\beta} + \nabla_{jp}^{\alpha\beta} \geq \xi\hat{\beta}_{jp}\mu_j, && \forall j \in \Lambda^{\beta} \\
& \Theta_k^{\alpha\beta} + \Delta_{ik}^{\alpha\beta} \geq \xi\hat{\alpha}_{ik}\gamma_i, && \forall k, i \in \Lambda^{\alpha} \\
& \Theta_k^{\alpha\beta} + \nabla_{jk}^{\alpha\beta} \geq \xi\hat{\beta}_{jk}\mu_j, && \forall k, j \in \Lambda^{\beta} \\
& \Theta_p^{|\alpha||\beta|} + \Delta_{ip}^{|\alpha||\beta|} \geq \xi|\hat{\alpha}_{ip}|\gamma_i, && \forall i \in \Lambda^{|\alpha|} \\
& \Theta_p^{|\alpha||\beta|} + \nabla_{jp}^{|\alpha||\beta|} \geq \xi|\hat{\beta}_{jp}|\mu_j, && \forall j \in \Lambda^{|\beta|} \\
& \Theta_p^{\alpha\beta}, \Theta_k^{\alpha\beta}, \Theta_p^{|\alpha||\beta|} \geq 0, && \forall k \\
& \Delta_{ip}^{\alpha\beta}, \nabla_{jp}^{\alpha\beta}, \Delta_{ik}^{\alpha\beta}, \nabla_{jk}^{\alpha\beta}, \Delta_{ip}^{|\alpha||\beta|}, \nabla_{jp}^{|\alpha||\beta|} \geq 0, \forall i, j, k \\
& \gamma_i, \mu_j \geq 0, && \forall i, j
\end{aligned}
$$

An important issue in implementing the developed RRDM and RVRM models is the selection of an appropriate robustness level. In these models, the adjustable parameter Γ controls the trade-off between robustness and the degree of conservatism imposed on the solution, where $0 \leq \Gamma \leq s$, and s denotes the number of uncertain coefficients in each constraint. In the empirical application, $s = 6$. When $\Gamma = 0$, no protection against perturbations in the uncertain parameters is provided, and the resulting formulation reduces to its nominal counterpart. As Γ increases, the model becomes protected against a larger number of simultaneous adverse perturbations. At the upper bound, $\Gamma = s$, the model provides full protection against all admissible perturbations within the specified uncertainty set.

In the empirical application that follows, the robustness budget is reported both as an absolute value, Γ, and as a percentage of the maximum allowable budget s. This dual representation is adopted to facilitate interpretation: the percentage value indicates the fraction of the maximum budget being used, while the absolute value is the actual parameter employed in the computational implementation. The same definition is applied consistently across all relevant constraints of both robust models.

The value of Γ can also be selected according to a prescribed probabilistic guarantee, rather than being assigned arbitrarily or swept over an illustrative range alone. Under the standard Gaussian approximation to the binomial tail bound of Bertsimas and Sim [6], if the probability of violating constraint r is required to be no greater than τ_r, a sufficient value of Γ_r can be determined using the following relationship, where $\Phi(\cdot)$ denotes the cumulative distribution function of the standard Gaussian distribution, $\Phi^{-1}(\cdot)$ denotes its inverse, and s represents the number of uncertain parameters in constraint r:

$$1 - \tau_r = \Phi\left(\frac{\Gamma_r - 1}{\sqrt{s}}\right) \quad \Longleftrightarrow \quad \Gamma_r = 1 + \Phi^{-1}(1 - \tau_r)\sqrt{s}. \tag{33}$$

Equation (33) therefore provides a systematic mechanism for selecting the robustness parameter according to the desired confidence level $1 - \tau_r$, rather than assigning it arbitrarily. A higher confidence level generally requires a larger value of Γ_r and consequently leads to a more conservative, but better-protected, solution. In the present application, each of the first-stage constraints of Models (31) and (32) involves $s = 6$ uncertain coefficients, corresponding to the $M = 4$ inputs and $N = 2$ outputs listed in Table 1. For a desired confidence level of 90% (i.e., $\tau = 0.10$), Equation (33) yields $\Gamma = 1 + \Phi^{-1}(0.90)\sqrt{6} \approx 1 + 1.2816 \times 2.4495 \approx 4.14$. This calibrated value is used, together with three illustrative perturbation levels $\xi \in \{1\%, 5\%, 10\%\}$, to produce the results reported in Table 3 below, which therefore reflects a targeted probabilistic guarantee rather than an arbitrary scenario choice. This calibrated operating point complements the broader sensitivity analysis over $\Gamma \in \{0, 1.5, 3, 6\}$, corresponding to $\{0\%, 25\%, 50\%, 100\%\}$ of the maximum allowable budget, reported subsequently in Tables 4–5. This analysis illustrates the trade-off between optimality and robustness across representative discrete robustness budgets rather than at a single calibrated setting.

4 Case study and experimental results

To validate the practical applicability and demonstrate the computational efficacy of the proposed robust DEA framework, an empirical study is conducted within a real-world financial setting. The Tehran Stock Exchange (TSE), with a history spanning nearly half a century, stands as one of the most prominent and attractive financial markets in the Middle East region. The inherent characteristics of this market, including the simultaneous presence of negative valued financial indicators and considerable data uncertainty

Table 1: Inputs and outputs of the DEA models for stock evaluation

Type	Symbol	Financial Criteria	Perspective
Inputs	I_1	QuickRatio	Liquidity
	I_2	SolvencyRatio-II	Leverage
	I_3	ReceivableTurnoverRatio	Asset Utilization
	I_4	Price to Earnings Ratio	Valuation
Outputs	O_1	Return on Equity	Profitability
	O_2	EPSGrowthRate	Growth

driven by macroeconomic fluctuations, regulatory changes, and market volatility, render it a particularly suitable testbed for examining the proposed models under conditions where both fundamental challenges addressed in this research coexist.

The objective of this section is to measure the relative efficiency of stocks within the Banks and Credit Institutions industry using the proposed robust DEA models. To this end, the inputs and outputs of the DEA models are carefully selected to capture multiple critical dimensions of stock performance. Specifically, the evaluation criteria are drawn from six fundamental categories of financial analysis: liquidity, leverage, asset utilization, valuation, profitability, and growth. This multidimensional selection ensures that the efficiency assessment reflects a comprehensive perspective on firm performance rather than being confined to a single financial aspect. The selected inputs and outputs, along with their definitions, are presented in Table 1.

As shown in Table 1, four input indicators and two output indicators are employed in the evaluation framework. On the input side, the Quick Ratio (I_1) captures the liquidity position of each firm by measuring the ratio of total current assets minus inventory to total current liabilities. The Solvency Ratio (I_2) reflects the leverage structure by computing total liabilities relative to shareholders' equity. The Receivable Turnover Ratio (I_3) serves as a measure of asset utilization efficiency, defined as net receivable sales divided by average net receivables. The Price to Earnings Ratio (I_4) represents the valuation dimension, calculated as the stock price divided by net income per share. Although the P/E ratio is not a resource that is physically consumed in production, it is included among the inputs consistent with common practice in DEA-based stock evaluation, since a lower P/E ratio is preferred, all else equal: a high price paid per unit of earnings signals that investors are paying a comparatively higher cost to obtain a given level of earnings. Accordingly, in this application, inputs are more precisely interpreted as indicators for which lower values are desirable and outputs as indicators for which higher values are desirable, rather than strictly as physically consumed resources versus produced goods; this orientation-based treatment of valuation and risk-type ratios as DEA inputs is a widely adopted convention in the financial DEA literature.

The RDM and VRM models are particularly well suited to this financial application because both define an economically interpretable direction of improvement without requiring any transformation of the raw indicators. In RDM, the direction of improvement $(\delta_{ip}^{-}, \delta_{jp}^{+})$ for a given stock is the observed range of possible improvement relative to its peers, i.e., the gap between the stock's own indicator values and the best (minimum-input, maximum-output) values attained by any peer in the sample; a stock is deemed more, respectively less, efficient the smaller, respectively larger, the fraction of that peer-relative gap it would need to close to reach the frontier. In VRM, the direction of improvement is instead set proportional to the absolute value of the stock's own indicator, $(|\alpha_{ip}|, |\beta_{jp}|)$, so improvement is measured relative to the stock's own scale rather than to the cross-sectional range of the peer group. In the context of financial ratio analysis, the RDM direction therefore has the natural interpretation of *peer-relative* improvement potential (how far a stock lags the best-performing peers on each indicator, in the original units of that indicator), while the VRM direction has the natural interpretation of *self-relative* improvement potential (how large an adjustment, proportional to the stock's own current position, would be required to remove inefficiency). Both notions are economically meaningful for indicators that can be negative, such as Return on Equity or the EPS Growth Rate: the RDM direction anchors improvement to the best peer's value, while the VRM direction anchors improvement to the magnitude of the stock's own value, so that a small negative outcome and a small positive outcome are treated as requiring comparably small adjustments, whereas a large negative outcome requires a comparably large adjustment.

On the output side, Return on Equity (O_1) captures the profitability dimension by measuring the net income generated per unit of common shareholders' equity, while the Earnings per Share Growth Rate (O_2) reflects the growth dimension, computed as the ratio of the current quarter's EPS to the previous quarter's EPS minus one. It is worth noting that among these indicators, Return on Equity and EPS Growth Rate can naturally assume negative values, which underscores the necessity of employing DEA models equipped to handle negative data in this empirical context. The Banks and Credit Institutions industry comprising 20 listed stocks is considered as the set of decision-making units in this empirical study. The summary of the collected financial data is presented in Table 2. Following data collection, all nominal and robust RDM/VRM models developed in this paper, namely Models (5)–(8) and Models (31)–(32), were coded and solved using the LINGO optimization software. Since Models (31) and (32) are linear programs for any fixed combination of Γ and ξ, each instance was solved to global optimality. Using the Bertsimas–Sim-calibrated conservatism level $\Gamma \approx 4.14$ derived in Section 3, and considering three illustrative perturbation levels, $\xi \in \{1\%, 5\%, 10\%\}$, the resulting efficiency scores obtained from the RRDM and RVRM models are reported in Table 3. To further examine the robustness of the results, a sensitivity analysis is conducted for different combinations of Γ and ξ. The sensitivity analysis results obtained from the RRDM and RVRM models are presented in Tables 4 and 5, respectively, while the corresponding efficiency trends are illustrated in Figures 1 and 2, respectively.
Throughout the sensitivity analysis reported in Tables 4 and 5, the robustness budget Γ is expressed as a percentage of the maximum allowable budget s, where $s = 6$ corresponds to the number of uncertain coefficients in each constraint. Thus, the percentage values 0%, 25%, 50%, and 100% correspond to the absolute budgets $\Gamma = 0$, $\Gamma = 1.5$, $\Gamma = 3$, and $\Gamma = 6$, respectively. The absolute values are those used in the computational implementation, whereas the percentages are normalized labels adopted to facilitate interpretation of the trade-off between robustness and optimality across the range from the nominal case ($\Gamma = 0$) to the fully conservative case ($\Gamma = s = 6$).

It should be noted that the calibrated value $\Gamma \approx 4.14$ (69%) used in Table 3 is derived from the probabilistic guarantee of Equation (33) and represents a single operating point associated with a 90% confidence level [57]. By contrast, the broader sensitivity analysis in Tables 4 and 5 spans $\Gamma \in \{0, 1.5, 3, 6\}$ (i.e., 0%, 25%, 50%, and 100% of the maximum budget) and complements the calibrated value by tracing the trade-off between optimality and robustness across a representative range of discrete robustness budgets from the fully nominal case to the fully conservative case. The perturbation level is varied over $\xi \in \{1\%, 10\%\}$ in Tables 4–5. Grounding ξ in the observed historical volatility of the financial indicators, rather than in the illustrative scenario values used here, remains an important avenue for future

Table 2: Financial data of banks and credit institutions industry stocks on the STE

Stocks	Inputs				Outputs	
	I_1	I_2	I_3	I_4	O_1	O_2
Stock01	16.132	19.859	28.651	3886.540	1.092	90.000
Stock02	0.893	10.926	6.396	271.517	15.210	7.568
Stock03	3.616	21.409	31.882	107.354	12.738	-99.451
Stock04	0.880	13.116	7.243	445.596	18.202	12.540
Stock05	0.830	44.782	20.680	4378.995	-25.630	39.766
Stock06	0.920	16.306	22.451	614.028	7.041	-73.214
Stock07	1.081	5.901	3.555	326.829	14.930	21.645
Stock08	0.909	17.588	2.511	374.217	10.873	-170.455
Stock09	0.816	6.246	8.032	122.958	6.893	85.185
Stock10	5.498	6.180	6.515	176.830	23.612	30.882
Stock11	0.805	22.207	5.105	2224.847	18.416	5.769
Stock12	0.878	10.928	6.742	37.943	16.002	16.300
Stock13	0.932	14.555	3.941	1128.656	-22.442	9.118
Stock14	21.388	64.009	4.491	232.747	4.969	-8.185
Stock15	0.969	25.389	8.308	250.208	18.530	-58.871
Stock16	0.955	30.619	9.459	3212.074	8.462	-22.558
Stock17	3.148	7.122	11.041	76.889	17.610	-20.382
Stock18	0.887	27.115	63.500	274.822	0.548	566.667
Stock19	12.142	59.780	8.890	748.457	12.134	-44.861
Stock20	3.342	23.119	3.210	336.715	4.687	-8.632
Mean	3.851	22.358	13.130	961.411	8.194	18.942
SD	5.673	16.182	14.229	1306.964	12.357	138.517
Max	21.388	64.009	63.500	4378.995	23.612	566.667
Min	0.805	5.901	2.511	37.943	-25.630	-170.455

research (see Section 5).

Note that, because a budget of robustness $\Gamma = 0$ (0%) forces all deviation (perturbation) variables to zero irrespective of the assumed perturbation level ξ, the $\Gamma = 0$ (0%) scenario coincides with the nominal (non-robust) efficiency score; only a single reference column ($\xi = 0\%$) is therefore reported for it in Tables 4 and 5, and no distinct $\Gamma = 0$ (0%) entries exist for $\xi = 1\%$ or $\xi = 10\%$. As reported in Tables 4 and 5, the proposed robust RDM and robust VRM models under the multiplier form are executed across the various uncertainty configurations described in the preceding paragraph, namely absolute robustness budgets of $\Gamma = 0$, 1.5, 3, and 6, corresponding to 0%, 25%, 50%, and 100% of the maximum allowable budget, respectively, each combined with perturbation levels of 1% and 10%. These tables provide a detailed, stock-by-stock comparison of efficiency scores, enabling a thorough examination of how different degrees of uncertainty affect the evaluation outcomes under both models.

Moreover, as visually depicted in Figures 1 and 2, and entirely consistent with theoretical expectations, the results reveal two clear and intuitive patterns. First, as the budget of robustness Γ increases

Table 3: Stock evaluation results under the robust RDM and robust VRM models

	Robust RDM			Robust VRM		
Stocks	$\xi = 1\%$	$\xi = 5\%$	$\xi = 10\%$	$\xi = 1\%$	$\xi = 5\%$	$\xi = 10\%$
Stock 01	0.744573	0.690796	0.613849	0.650560	0.612434	0.567362
Stock 02	0.966787	0.931612	0.887092	0.975480	0.900867	0.815152
Stock 03	0.718879	0.665110	0.589435	0.711764	0.661700	0.603981
Stock 04	0.992004	0.957629	0.908519	0.981885	0.912736	0.833103
Stock 05	0.763236	0.719606	0.656378	0.957404	0.885297	0.802222
Stock 06	0.680920	0.629495	0.561328	0.880564	0.819204	0.747576
Stock 07	0.997363	0.985485	0.967124	0.985012	0.926660	0.857113
Stock 08	0.988564	0.935907	0.847342	0.990000	0.950000	0.900000
Stock 09	0.994046	0.968045	0.929884	0.985671	0.930133	0.864532
Stock 10	0.994848	0.972433	0.939736	0.984289	0.923718	0.852867
Stock 11	0.994313	0.969348	0.932350	0.981629	0.911382	0.830353
Stock 12	0.996860	0.982592	0.960248	0.985051	0.926956	0.858049
Stock 13	0.992177	0.956279	0.932645	0.980817	0.907476	0.822797
Stock 14	0.969558	0.926114	0.885084	0.971757	0.898156	0.813411
Stock 15	0.973787	0.868339	0.799178	0.981333	0.910216	0.828577
Stock 16	0.830204	0.802418	0.762959	0.849423	0.792185	0.725011
Stock 17	0.989765	0.945524	0.881610	0.982590	0.915889	0.838564
Stock 18	0.982974	0.910669	0.809747	0.982743	0.916904	0.841133
Stock 19	0.814487	0.786696	0.746571	0.664024	0.623818	0.576530
Stock 20	0.990671	0.948861	0.926128	0.980701	0.906964	0.821914

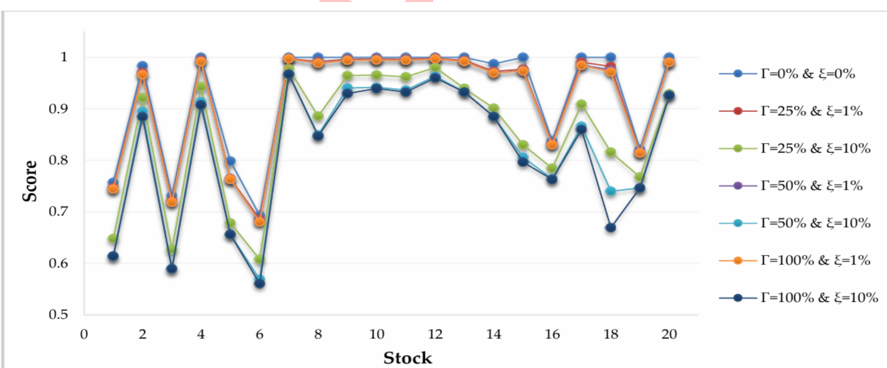

Figure 1: Efficiency sensitivity trends under the RRDM model for different Γ and ξ values

from 0 (0%) to 6 (100%), thereby permitting a progressively larger subset of uncertain parameters to simultaneously realize their worst-case values, the objective function deteriorates monotonically. This degradation reflects the fundamental price of robustness: achieving greater immunization against data uncertainty necessarily sacrifices optimality, pushing efficiency scores toward more conservative esti-

Table 4: Sensitivity analysis of the robust RDM model for different values of Γ and ξ

	$\Gamma = 0$	$\Gamma = 1.5$		$\Gamma = 3$		$\Gamma = 6$	
Stocks	$\xi = 0\%$	$\xi = 1\%$	$\xi = 10\%$	$\xi = 1\%$	$\xi = 10\%$	$\xi = 1\%$	$\xi = 10\%$
Stock01	0.757057	0.747252	0.647771	0.744573	0.613849	0.744573	0.613849
Stock02	0.983348	0.970818	0.921451	0.967238	0.895141	0.966690	0.884562
Stock03	0.731489	0.721861	0.626921	0.718879	0.589435	0.718879	0.589435
Stock04	1.000000	0.994962	0.942657	0.992416	0.913889	0.991928	0.907576
Stock05	0.798461	0.765104	0.678565	0.763236	0.656378	0.763236	0.656378
Stock06	0.693126	0.685568	0.607794	0.682052	0.568927	0.680472	0.560788
Stock07	1.000000	0.998177	0.978110	0.997363	0.967124	0.997363	0.967124
Stock08	1.000000	0.991395	0.885769	0.988642	0.849018	0.988564	0.847342
Stock09	1.000000	0.996712	0.964393	0.994782	0.940595	0.994046	0.929884
Stock10	1.000000	0.997008	0.965801	0.995101	0.942091	0.994848	0.939736
Stock11	1.000000	0.996688	0.962027	0.994523	0.936380	0.994313	0.932350
Stock12	1.000000	0.998368	0.980420	0.997097	0.963170	0.996860	0.960248
Stock13	1.000000	0.994042	0.939807	0.992177	0.932645	0.992177	0.932645
Stock14	0.987425	0.972775	0.901174	0.969558	0.885084	0.969558	0.885084
Stock15	1.000000	0.977362	0.830208	0.974882	0.806064	0.973787	0.796464
Stock16	0.837643	0.832941	0.784833	0.830884	0.764292	0.830204	0.762959
Stock17	1.000000	0.990536	0.908725	0.985267	0.866091	0.984737	0.859717
Stock18	1.000000	0.982307	0.816004	0.975531	0.740008	0.971787	0.669396
Stock19	0.820912	0.816224	0.767529	0.814487	0.746571	0.814487	0.746571
Stock20	1.000000	0.990872	0.929496	0.990671	0.926128	0.990671	0.926128

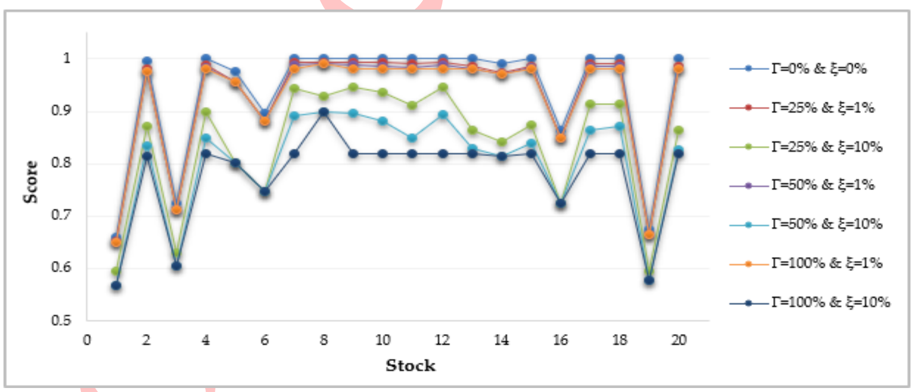

Figure 2: Efficiency sensitivity trends under the RVRM model for different Γ and ξ values

mates. Second, as the perturbation level ξ grows from 0.01 to 0.1, the objective function worsens relative to the nominal problem, signifying that wider uncertainty intervals around the nominal data amplify the conservatism of the robust solution. Taken together, these two observations not only align with the well-established theoretical properties of robust optimization but also empirically validate the correct-

Table 5: Sensitivity analysis of the robust VRM model for different values of Γ and ξ

	$\Gamma = 0$	$\Gamma = 1.5$		$\Gamma = 3$		$\Gamma = 6$	
Stocks	$\xi = 0\%$	$\xi = 1\%$	$\xi = 10\%$	$\xi = 1\%$	$\xi = 10\%$	$\xi = 1\%$	$\xi = 10\%$
Stock 01	0.660408	0.652186	0.595398	0.650560	0.567362	0.650560	0.567362
Stock 02	0.995055	0.981202	0.871285	0.977187	0.834297	0.975480	0.815151
Stock 03	0.724876	0.713415	0.629668	0.711764	0.604319	0.711764	0.603981
Stock 04	1.000000	0.989441	0.898708	0.983662	0.849488	0.980198	0.818182
Stock 05	0.976290	0.957466	0.802681	0.957404	0.802222	0.957404	0.802222
Stock 06	0.896512	0.880591	0.748248	0.880564	0.747576	0.880564	0.747576
Stock 07	1.000000	0.994389	0.944862	0.988808	0.892608	0.980198	0.818182
Stock 08	1.000000	0.993137	0.929018	0.990000	0.900000	0.990000	0.900000
Stock 09	1.000000	0.994643	0.947153	0.989311	0.896813	0.980198	0.818182
Stock 10	1.000000	0.993514	0.936322	0.987338	0.880980	0.980198	0.818182
Stock 11	1.000000	0.990964	0.912427	0.984014	0.849802	0.980198	0.818182
Stock 12	1.000000	0.994561	0.946439	0.989146	0.895160	0.980198	0.818182
Stock 13	1.000000	0.985663	0.864422	0.981468	0.828356	0.980198	0.818182
Stock 14	0.991059	0.974915	0.840815	0.971666	0.813411	0.971666	0.813411
Stock 15	1.000000	0.986278	0.874028	0.982601	0.840053	0.980198	0.818182
Stock 16	0.864253	0.849423	0.725379	0.849423	0.725011	0.849423	0.725011
Stock 17	1.000000	0.991237	0.912899	0.985752	0.864853	0.980198	0.818182
Stock 18	1.000000	0.990875	0.913900	0.986117	0.871627	0.980198	0.818182
Stock 19	0.674439	0.665866	0.596263	0.664024	0.576530	0.664024	0.576530
Stock 20	1.000000	0.985679	0.865158	0.981143	0.826159	0.980198	0.818182

ness and coherence of the proposed models in systematically quantifying the impact of data uncertainty on efficiency measurement.

To determine which of the two proposed robust models is more robust against uncertainty, we compare the sensitivity of the RRDM and RVRM efficiency scores reported in Tables 4 and 5 between the nominal scenario ($\Gamma = 0$ (0%), $\xi = 0\%$) and the most conservative scenario ($\Gamma = 6$ (100%), $\xi = 10\%$). On average across the 20 stocks, the RRDM efficiency score declines by 12.34% (SD 7.15%) between these two extremes, whereas the RVRM efficiency score declines by a smaller and markedly more uniform 17.09% (SD 2.03%). In terms of the magnitude of the efficiency score, RRDM is therefore more sensitive, on average, to worst-case perturbations of the input and output data, but its sensitivity also varies far more widely across stocks than that of RVRM, whose degradation is compressed into a narrow 10.0%–18.2% band regardless of the stock under evaluation. Robustness in score level is not, however, the only relevant criterion for a decision-support tool; a second, complementary criterion is the stability of the induced ranking of stocks. Because both efficiency-score transformations introduced in Section 2 are strictly monotonic in the underlying inefficiency measure θ_p, the rank-based statistics below are invariant to which of the two transformations is used. Measured by the Spearman rank correlation between the nominal ranking and the worst-case ranking, RVRM exhibits substantially higher rank stability ($\rho = 0.970$, Kendall $\tau = 0.958$, mean absolute rank shift of 1.10 positions) than RRDM

($\rho = 0.760$, Kendall $\tau = 0.678$, mean absolute rank shift of 3.30 positions, with Stock18 moving by as many as 14 positions). The two models thus offer a genuine trade-off: RRDM exhibits a wider spread of score degradation that is informative for identifying which stocks are most fragile to uncertainty (see below), while RVRM produces a more uniform, and thus more predictable, level of score degradation and better preserves the relative ordering of the DMUs. Decision-makers primarily interested in a stable ranking of alternatives, e.g., for portfolio screening purposes, should favor RVRM, whereas those primarily interested in discriminating which specific stocks are most exposed to data uncertainty should favor RRDM.

From the perspective of discriminatory power, both models classify 12 of the 20 stocks as fully efficient (score of 1) under the nominal scenario, and both lose all of these efficient classifications once any positive budget of robustness is combined with a positive perturbation level, since no stock's efficiency score remains exactly at unity once uncertainty is admitted; this confirms that the robustness of the classification of *efficient* DMUs, as opposed to the ranking of inefficient ones, is itself highly sensitive to uncertainty under both models. Comparing the top-5 ranked stocks under the nominal and the worst-case scenario, only 2 of the top-5 stocks under RRDM remain in the top-5 once uncertainty is introduced, compared to 4 of the top-5 under RVRM, corroborating the higher ranking stability of RVRM noted above. From a managerial standpoint, and using the more discriminating RRDM sensitivity profile, Stock18, Stock15, and Stock03 emerge as the stocks whose efficiency assessment is most sensitive to uncertainty (with RRDM efficiency-score reductions of 33.1%, 20.3%, and 19.4%, respectively, at the most conservative setting), while Stock07, Stock12, and Stock10 are the least sensitive, with RRDM score reductions below 7%. Since the inputs and outputs used in this study are drawn from liquidity, leverage, asset-utilization, valuation, profitability, and growth indicators, a high sensitivity to the perturbation of these indicators signals that a stock's apparent efficiency is comparatively fragile with respect to measurement or estimation error in its financial statements, e.g., due to volatile earnings, which directly affects Return on Equity and the EPS Growth Rate, the two negative-data-prone outputs in this study. Investors and portfolio managers seeking downside protection should therefore treat the nominal efficiency ranking of such highly sensitive stocks with additional caution and may prefer to base allocation decisions on the conservative (high-Γ, high-ξ) efficiency scores, or on the calibrated Table 3 scores, rather than on the nominal scores alone; conversely, stocks whose efficiency scores decline the least in the worst case represent comparatively uncertainty-robust investment candidates whose efficiency ranking can be relied upon with greater confidence even under adverse data-quality conditions.

5 Conclusions and future research directions

Data Envelopment Analysis has long served as a powerful nonparametric tool for evaluating the relative efficiency of decision-making units. However, two persistent challenges have limited its applicability in real-world settings: the inability of classical DEA models to handle negative data values, and the vulnerability of efficiency scores to data uncertainty. While the literature has addressed each of these issues in isolation through semi-oriented models such as RDM and VRM on one hand, and robust optimization techniques on the other, no unified framework has been proposed to tackle both challenges simultaneously. This paper bridged that gap by developing robust counterparts of the RDM and VRM models under the multiplier form, leveraging the budget of robustness framework to provide decision-makers with explicit control over the trade-off between efficiency optimality and protection against uncertain data.

The proposed models were formulated by introducing uncertainty sets governed by two key parameters: the budget of robustness, which controls the proportion of uncertain parameters allowed to deviate simultaneously, and the perturbation level, which defines the magnitude of deviation around nominal values. This dual-parameter structure offers a flexible and interpretable mechanism for adjusting the degree of conservatism in efficiency evaluation, ranging from the fully nominal case to the most conservative worst-case scenario.

The empirical application on 20 stocks listed in the Banks and Credit Institutions industry of the Tehran Stock Exchange demonstrated the practical relevance of the proposed framework. The selected financial indicators naturally exhibited negative values in profitability and growth metrics, making this case study a genuine testbed for semi-oriented robust models rather than an artificial illustration. The computational results confirmed that efficiency scores degrade monotonically as either the budget of robustness or the perturbation level increases, which is fully consistent with the theoretical properties of robust optimization. More importantly, the results revealed that the ranking of stocks is not invariant to uncertainty settings, implying that efficiency assessments derived from nominal data alone may lead to misleading conclusions in environments characterized by data imprecision.

From a methodological standpoint, this study contributes to the DEA literature in several ways. It establishes that robust optimization and semi-oriented radial measures can be integrated without sacrificing the computational tractability of the resulting models. It also demonstrates that the multiplier form, rather than the envelopment form, provides a natural and effective platform for incorporating uncertainty into models that accommodate negative data. Furthermore, the framework preserves the unit invariance and translation invariance properties that are essential for meaningful efficiency measurement in the presence of mixed-sign data.

Several promising avenues remain open for future investigation. First, the proposed robust semi-oriented framework could be extended to accommodate more complex production structures where decision-making units operate through multiple interconnected stages or divisions. In such network configurations, intermediate measures linking consecutive stages may exhibit negative values, particularly in financial institutions where internal fund transfers, risk-adjusted returns, or divisional profit contributions can be negative. Two-stage and network DEA models with negative data have already been explored to some extent in non-robust settings, e.g., via a two-stage range directional measure for sustainable supply chains [22], a dynamic range directional measure for two-stage networks [68], a multi-objective network DEA model with negative data and undesirable outputs for insurance companies [37], and two-stage models with negative system outputs for evaluating government financial policies [33], while robust network DEA under uncertainty (without negative data) has been developed for leader-follower structures [47], mutual funds [48], and, via a Z-number approach with shared resources and undesirable outputs, for airline efficiency [73]. Developing robust network DEA models that simultaneously handle negative intermediate products and data uncertainty would provide a more realistic representation of organizational efficiency in hierarchical or multi-divisional settings. Second, incorporating distributionally robust optimization, where the probability distribution of uncertain parameters is itself ambiguous, could provide a more refined alternative to the budget of robustness approach, especially when limited historical data makes it difficult to specify precise uncertainty sets. Data-driven robust optimization [8, 35] and distributionally robust optimization [24, 28], which construct uncertainty sets or ambiguity sets directly from data rather than from predefined bounds, have already been applied within DEA in a small number of studies, e.g., to the operational efficiency of provincial endowment insurance systems [55] and to provincial carbon emissions efficiency [56] in China via data-driven robust DEA, and, most recently, to DEA under hybrid distributional and fuzzy uncertainty [38]; extending the negative-data robust framework proposed in this paper along either of these two directions constitutes a promising direction for future research. Finally, integrating machine learning techniques for data-driven construction of uncertainty sets, rather than relying on predefined perturbation levels, could enhance the practical calibration of the robustness parameters based on observed data variability and historical volatility patterns [49]. In a related vein, the uncertainty sets for the absolute-value coefficients $|\alpha_{ip}|$ and $|\beta_{jp}|$ used in the VRM-based robust models perturb the magnitude of each coefficient directly and, as discussed in Section 3, do not capture the possibility that a coefficient's underlying sign could itself change under perturbation; designing a sign-aware uncertainty set, e.g., by perturbing the signed coefficient $\hat{\alpha}_{ik}$ directly and propagating this perturbation analytically to $|\tilde{\alpha}_{ik}|$, or by explicitly modeling the probability of a sign reversal, is an important and non-trivial extension that we leave for future work. Likewise, the budgets of robustness and perturbation levels used in the empirical application were varied systematically as a sensitivity-analysis device rather than calibrated to the historical volatility, estimation-error patterns, or data-revision history of the specific financial indicators employed; grounding the uncertainty sets directly in such empirical evidence, for instance by estimating ξ from the historical standard deviation of each indicator or from realized forecast/restatement errors, would strengthen the empirical link between the assumed uncertainty and the actual imprecision of financial-statement data, and constitutes an important direction for extending the present sensitivity-based design toward a fully calibrated robust DEA framework.

Conflicts of Interest

The authors declare no conflict of interest.

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References

[1] M. Allahyar, M. Rostamy-Malkhalifeh, *Negative data in data envelopment analysis: Efficiency analysis and estimating returns to scale*, Comput. Ind. Eng. **82** (2015) 78–81.

[2] G.R. Amin, S. Al-Muharrami, *A new inverse data envelopment analysis model for mergers with negative data*, IMA J. Manag. Math. **29 (2)** (2018) 137–149.

[3] R.D. Banker, A. Charnes, W.W. Cooper, *Some models for estimating technical and scale inefficiencies in data envelopment analysis*, Manag. Sci. **30 (9)** (1984) 1078–1092.

[4] A. Ben-Tal, A. Nemirovski, *Robust solutions of linear programming problems contaminated with uncertain data*, Math. Program. **88 (3)** (2000) 411–424.

[5] A. Ben-Tal, A. Nemirovski, L. El Ghaoui, *Robust Optimization*, Princeton University Press, Princeton, NJ, 2009.

[6] D. Bertsimas, M. Sim, *The price of robustness*, Oper. Res. **52 (1)** (2004) 35–53.

[7] D. Bertsimas, D.B. Brown, C. Caramanis, *Theory and applications of robust optimization*, SIAM Rev. **53 (3)** (2011) 464–501.

[8] D. Bertsimas, V. Gupta, N. Kallus, *Data-driven robust optimization*, Math. Program. **167 (2)** (2018) 235–292.

[9] R.G. Chambers, Y. Chung, R. Färe, *Benefit and distance functions*, J. Econ. Theory **70 (2)** (1996) 407–419.

[10] R.G. Chambers, Y. Chung, R. Färe, *Profit, directional distance functions, and Nerlovian efficiency*, J. Optim. Theory Appl. **98 (2)** (1998) 351–364.

[11] A. Charnes, W.W. Cooper, E. Rhodes, *Measuring the efficiency of decision making units*, Eur. J. Oper. Res. **2 (6)** (1978) 429–444.

[12] G. Cheng, P. Zervopoulos, Z. Qian, *A variant of radial measure capable of dealing with negative inputs and outputs in data envelopment analysis*, Eur. J. Oper. Res. **225 (1)** (2013) 100–105.

[13] W.W. Cooper, L.M. Seiford, J. Zhu, *Handbook on Data Envelopment Analysis*, Springer New York, NY, 2011.

[14] A. Emrouznejad, G.R. Amin, E. Thanassoulis, A.L. Anouze, *On the boundedness of the SORM DEA models with negative data*, Eur. J. Oper. Res. **206 (1)** (2010) 265–268.

[15] A. Emrouznejad, A.L. Anouze, E. Thanassoulis, *A semi-oriented radial measure for measuring the efficiency of decision making units with negative data, using DEA*, Eur. J. Oper. Res. **200** (2010) 297–304.

[16] A. Emrouznejad, M. Tavana, *Performance Measurement with Fuzzy Data Envelopment Analysis*, Springer Berlin, Heidelberg, 2014.

[17] A. Emrouznejad, G.L. Yang, *A survey and analysis of the first 40 years of scholarly literature in DEA: 1978–2016*, Socio-Econ. Plan. Sci. **61** (2018) 4–8.

[18] M.J. Farrell, *The measurement of productive efficiency*, J. R. Stat. Soc. Ser. A **120 (3)** (1957) 253–281.

[19] V. Gabrel, C. Murat, A. Thiele, *Recent advances in robust optimization: An overview*, Eur. J. Oper. Res. **235 (3)** (2014) 471–483.

[20] M. Ghiyasi, N. Zhu, *An inverse semi-oriented radial data envelopment analysis measure for dealing with negative data*, IMA J. Manag. Math. **31 (4)** (2020) 505–516.

[21] B.L. Gorissen, D. Den Hertog, *A practical guide to robust optimization*, Omega **53** (2015) 124–137.

[22] M. Izadikhah, R. Farzipoor Saen, *Evaluating sustainability of supply chains by two-stage range directional measure in the presence of negative data*, Transp. Res. Part D Transp. Environ. **49** (2016) 110–126.

[23] R. Kazemi Matin, G.R. Amin, A. Emrouznejad, *A modified semi-oriented radial measure for target setting with negative data*, Measurement **54** (2014) 152–158.

[24] D. Kuhn, S. Shafiee, W. Wiesemann, *Distributionally robust optimization*, Acta Numer. **34** (2025) 579–804.

[25] Z. Li, R. Ding, C.A. Floudas, *A comparative theoretical and computational study on robust counterpart optimization: I. Robust linear optimization and robust mixed integer linear optimization*, Ind. Eng. Chem. Res. **50 (18)** (2011) 10567–10603.

[26] Z. Li, Q. Tang, C.A. Floudas, *A comparative theoretical and computational study on robust counterpart optimization: II. Probabilistic guarantees on constraint satisfaction*, Ind. Eng. Chem. Res. **51 (19)** (2012) 6769–6788.

[27] Z. Li, C.A. Floudas, *A comparative theoretical and computational study on robust counterpart optimization: III. Improving the quality of robust solutions*, Ind. Eng. Chem. Res. **53 (33)** (2014) 13112–13124.

[28] F. Lin, X. Fang, Z. Gao, *Distributionally robust optimization: A review on theory and applications*, Numer. Algebra Control Optim. **12 (1)** (2022) 159–212.

[29] R. Lin, Z. Chen, *A directional distance based super-efficiency DEA model handling negative data*, J. Oper. Res. Soc. **68 (11)** (2017) 1312–1322.

[30] J.S. Liu, L.Y. Lu, W.M. Lu, B.J. Lin, *A survey of DEA applications*, Omega **41 (5)** (2013) 893–902.

[31] C.K. Lovell, *Measuring the macroeconomic performance of the Taiwanese economy*, Int. J. Prod. Econ. **39 (1–2)** (1995) 165–178.

[32] C.K. Lovell, J.T. Pastor, *Units invariant and translation invariant DEA models*, Oper. Res. Lett. **18 (3)** (1995) 147–151.

[33] A.V. Lychev, S.V. Ratner, V.E. Krivonozhko, *Two-stage data envelopment analysis models with negative system outputs for the efficiency evaluation of government financial policies*, Mathematics **11 (24)** (2023) 4873.

[34] A. Mergoni, A. Emrouznejad, K.D. Witte, *Fifty years of data envelopment analysis*, Eur. J. Oper. Res. **326 (3)** (2025) 389–412.

[35] M. Namakshenas, M.S. Pishvaee, *Data-driven robust optimization*, in: Robust and Constrained Optimization: Methods and Applications, 2019, pp. 1–40.

[36] O.B. Olesen, N.C. Petersen, *Stochastic data envelopment analysis—A review*, Eur. J. Oper. Res. **251 (1)** (2016) 2–21.

[37] H. Omrani, A. Emrouznejad, M. Shamsi, P. Fahimi, *Evaluation of insurance companies considering uncertainty: A multi-objective network data envelopment analysis model with negative data and undesirable outputs*, Socio-Econ. Plan. Sci. **82** (2022) 101306.

[38] H. Omrani, R. Imanirad, *Data envelopment analysis under hybrid uncertainty: A distributionally robust optimization and intuitionistic fuzzy modeling approach*, Ann. Oper. Res. (2026) 1–40.

[39] A. Panwar, M. Olfati, M. Pant, V. Snasel, *A review on the 40 years of existence of data envelopment analysis models: Historic development and current trends*, Arch. Comput. Methods Eng. **29 (7)** (2022) 5397–5426.

[40] J.T. Pastor, *Translation invariance in data envelopment analysis: A generalization*, Ann. Oper. Res. **66 (2)** (1996) 91–102.

[41] P. Peykani, E. Mohammadi, A. Jabbarzadeh, M. Rostamy-Malkhalifeh, M.S. Pishvaee, *A novel two-phase robust portfolio selection and optimization approach under uncertainty: A case study of Tehran stock exchange*, PLoS ONE **15 (10)** (2020) e0239810.

[42] P. Peykani, E. Mohammadi, R. Farzipoor Saen, S.J. Sadjadi, M. Rostamy-Malkhalifeh, *Data envelopment analysis and robust optimization: A review*, Expert Syst. **37 (4)** (2020) e12534.

[43] P. Peykani, R. Farzipoor Saen, F.S. Seyed Esmaeili, J. Gheidar-Kheljani, *Window data envelopment analysis approach: A review and bibliometric analysis*, Expert Syst. **38 (7)** (2021) e12721.

[44] P. Peykani, F.S. Seyed Esmaeili, M. Mirmozaffari, A. Jabbarzadeh, M. Khamechian, *Input/output variables selection in data envelopment analysis: A Shannon entropy approach*, Mach. Learn. Knowl. Extr. **4 (3)** (2022) 688–699.

[45] P. Peykani, F. Hosseinzadeh Lotfi, S.J. Sadjadi, A. Ebrahimnejad, E. Mohammadi, *Fuzzy chance-constrained data envelopment analysis: a structured literature review, current trends, and future directions*, Fuzzy Optim. Decis. Mak. **21 (2)** (2022) 197–261.

[46] P. Peykani, J. Gheidar-Kheljani, R. Farzipoor Saen, E. Mohammadi, *Generalized robust window data envelopment analysis approach for dynamic performance measurement under uncertain panel data*, Oper. Res. **22 (5)** (2022) 5529–5567.

[47] P. Peykani, A. Emrouznejad, A. Mahmoodirad, M. Nouri, N. Arabjazi, *A robust framework for generalized leader-follower network data envelopment analysis under uncertain data*, in: Advances in the Theory and Practice of Data Envelopment Analysis, Cham: Springer Nature Switzerland, 2024, pp. 10–26.

[48] P. Peykani, A. Emrouznejad, E. Mohammadi, J. Gheidar-Kheljani, *A novel robust network data envelopment analysis approach for performance assessment of mutual funds under uncertainty*, Ann. Oper. Res. **339 (3)** (2024) 1149–1175.

[49] P. Peykani, R. Soltani, C. Tanasescu, S.E. Shojaie, A. Jandaghian, *The Robust Malmquist productivity index: A framework for measuring productivity changes over time under uncertainty*, Mathematics **13 (11)** (2025) 1727.

[50] P. Peykani, S.J. Sadjadi, S.E. Shojaie, C. Tanasescu, Z. Abbasi, *Uncertain data envelopment analysis approach for handling negative values: An application to stock market*, Int. J. Optim. Control Theor. Appl., in press (2026).

[51] M.S. Portela, E. Thanassoulis, G. Simpson, *Negative data in DEA: A directional distance approach applied to bank branches*, J. Oper. Res. Soc. **55 (10)** (2004) 1111–1121.

[52] M.C. Portela, E. Thanassoulis, *Malmquist-type indices in the presence of negative data: An application to bank branches*, J. Bank. Finance **34 (7)** (2010) 1472–1483.

[53] J. Pourmahmoud, S. Arami, *Evaluating cost efficiency of decision-making units in an uncertain environment*, J. Math. Model. **12 (3)** (2024) 565–582.

[54] D. Pramanik, S. Mondal, A. Halder, *Framework for performance analysis of healthcare system: A strategic expert decision making approach under uncertainty*, Healthcare Eng. **2 (3)** (2026) 163–180.

[55] S. Qu, C. Feng, S. Jiang, J. Wei, Y. Xu, *Data-driven robust DEA models for measuring operational efficiency of endowment insurance system of different provinces in China*, Sustainability **14 (16)** (2022) 9954.

[56] S. Qu, Y. Xu, Y. Ji, C. Feng, J. Wei, S. Jiang, *Data-driven robust data envelopment analysis for evaluating the carbon emissions efficiency of provinces in China*, Sustainability **14 (20)** (2022) 13318.

[57] S.J. Sadjadi, H. Omrani, *Data envelopment analysis with uncertain data: An application for Iranian electricity distribution companies*, Energy Policy **36 (11)** (2008) 4247–4254.

[58] B.K. Sahoo, M. Khoveyni, R. Eslami, P. Chaudhury, *Returns to scale and most productive scale size in DEA with negative data*, Eur. J. Oper. Res. **255 (2)** (2016) 545–558.

[59] M. Salahi, N. Torabi, A. Amiri, *An optimistic robust optimization approach to common set of weights in DEA*, Measurement **93** (2016) 67–73.

[60] M. Salahi, M. Toloo, Z. Hesabirad, *Robust Russell and enhanced Russell measures in DEA*, J. Oper. Res. Soc. **70 (8)** (2019) 1275–1283.

[61] L.M. Seiford, J. Zhu, *Modeling undesirable factors in efficiency evaluation*, Eur. J. Oper. Res. **142 (1)** (2002) 16–20.

[62] R. Shakouri, M. Salahi, S. Kordrostami, *Stochastic p-robust DEA efficiency scores approach to banking sector*, J. Model. Manag. **15 (3)** (2020) 893–917.

[63] R. Shakouri, *p-robust network cost efficiency with genetic algorithms and machine learning*, J. Math. Model. **14 (1)** (2026) 231–251.

[64] J.A. Sharp, W. Meng, W. Liu, *A modified slacks-based measure model for data envelopment analysis with 'natural'negative outputs and inputs*, J. Oper. Res. Soc. **58 (12)** (2007) 1672–1677.

[65] M. Soltanifar, M. Ghiyasi, H. Sharafi, *Inverse DEA-R models for merger analysis with negative data*, IMA J. Manag. Math. **34 (3)** (2023) 491–510.

[66] A.L. Soyster, *Convex programming with set-inclusive constraints and applications to inexact linear programming*, Oper. Res. **21 (5)** (1973) 1154–1157.

[67] S. Srivastava, A. Aggarwal, P. Bansal, *Non-homogeneous DEA approach in the presence of negative data: A promising prospective approach to enhance decision-making*, Comput. Appl. Math. **43 (4)** (2024) 1.

[68] M. Tavana, M. Izadikhah, D. Di Caprio, R. Farzipoor Saen, *A new dynamic range directional measure for two-stage data envelopment analysis models with negative data*, Comput. Ind. Eng. **115** (2018) 427–448.

[69] M. Toloo, E.K. Mensah, M. Salahi, *Robust optimization and its duality in data envelopment analysis*, Omega **108** (2022) 102583.

[70] K. Tone, T.S. Chang, C.H. Wu, *Handling negative data in slacks-based measure data envelopment analysis models*, Eur. J. Oper. Res. **282 (3)** (2020) 926–935.

[71] F. Wei, J. Song, C. Jiao, F. Yang, *A modified slacks-based ranking method handling negative data in data envelopment analysis*, Expert Syst. **36 (1)** (2019) e12329.

[72] M. Wen, *Uncertain Data Envelopment Analysis*, Springer Berlin, Heidelberg, 2015.

[73] Z. Yang, H. Omrani, R. Imanirad, *Assessing airline efficiency with a network DEA model: A Z-number approach with shared resources, undesirable outputs, and negative data*, Socio-Econ. Plan. Sci. **96** (2024) 102080.

[74] Z. Zarinkia, *Comparative analysis of hospital efficiency in Iran: A multi-methodological study using DEA and TOPSIS techniques*, Healthcare Eng. **2 (2)** (2026) 78–100.