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A spectral collocation method using orthogonal polynomials for Volterra integro-differential equations of the second kind with discontinuous kernels

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Abstract. This paper presents a spectral collocation approach using shifted Legendre polynomials on the interval $I = [0, T]$ for solving a class of linear Volterra integro-differential equations with discontinuous kernels. The unknown solution and the kernels are approximated by truncated orthogonal polynomial expansions, and suitable operational matrices for integration, differentiation, and product are constructed. The collocation scheme transforms the original problem into a system of linear algebraic equations for unknown coefficients. An explicit L^2-error bound for the proposed approximation is derived under appropriate regularity assumptions. Numerical examples are provided to demonstrate efficiency and accuracy of the method. The method is developed under the assumption that the discontinuity curves are affine functions; extensions to general curves are discussed as a future direction.

Keywords: Volterra integro-differential equations, Discontinuous kernels, Orthogonal polynomials, Spectral collocation method, Error analysis.
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1 Introduction

The theoretical foundations of integral equations trace their origins to concepts introduced by Fourier [1], while Volterra [2] made seminal contributions to their advancement and practical applications. The year 1895 marked a pivotal milestone in this field. Over the past 15-20 years, there has been resurgence of interest in Volterra integral equations (VIEs), propelled by their expanding domains of application and an enhanced understanding of their properties. VIEs are fundamental mathematical tools for modeling dynamical systems with hereditary influences in physics, engineering, economics, chemistry, and populations growth models [3–7]. Among these, Volterra integral equations of the second kind are of greater

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practical importance in numerical solution due to their well-posedness and inherent stability compared to the first kind [8]. Recently, research focus has shifted to a specific class of Volterra integro-differential equations (VIDEs) characterized by discontinuous kernels [8–13]. As a classical problem in mathematics, Volterra equations have been studied for decades. In contrast, the analysis of such equations has received increasing attention in recent years, with contributions from various research groups. This research focuses on the VIDEs of the second kind with discontinuous kernels, represented as follows

$$g'(x) = f(x) + \sum_{l=1}^{n} \lambda_l \int_{\xi_{l-1}(x)}^{\xi_l(x)} k_l(x,t)g(t)\mathrm{d}t, \quad x,t \in I = [0,T], \tag{1.1}$$

with the initial condition $g(0) = 0$. Here, for each $l = 1,2,\ldots,n$, we define the subdomains $I_l := [\xi_{l-1}(x), \xi_l(x)]$. Then $f(x) \in L^2(I)$ and $k_l(x,t) \in L^2(I_l \times I_l)$ are known functions, $g(x)$ is the unknown function to be determined, and $\{\lambda_l\}_{l=1}^n$ are real numbers.

The kernels $\{k_l(x,t)\}_{l=1}^n$ are discontinuous along the continuous curves $\xi_l(x)$, which satisfy $0 = \xi_0(x) < \xi_1(x) < \cdots < \xi_n(x) = x \leq T$. On each subdomain I_l, the kernels are assumed to be sufficiently smooth for the purpose of error analysis.

The method itself does not require global smoothness; the discontinuities are handled through the piecewise-defined integration limits. In this work, we further assume that these discontinuity curves are affine functions of x, i.e., $\xi_l(x) = a_l x$ with $0 = a_0 < a_1 < \cdots < a_n = 1$. This assumption is essential for the operational matrix formulation and will be discussed in detail in Section 3. This specific class of equations, was first introduced by Sidorov and Lorenzi, and subsequently developed by several researchers. From the analytical viewpoint, Evans [14] was among the first to study integral equations of the second kind with discontinuous kernels, while Anselone [15] established the foundation of the uniform approximation theory for such problems. Later, Denisov and Lorenzi [16] derived sufficient conditions for the existence of continuous solutions to linear VIEs with jump discontinuities, and Sidorov and co-authors [17–19] extended these results by providing both existence and uniqueness criteria for continuous solutions to system of VIEs of the first kind with discontinuous kernels along finitely many continuous curves. From an applied perspective, Volterra models are distinguished by their ability to incorporate memory effects of dynamical systems, where past states influence current behavior. This fundamental feature has led to the widespread use of VIEs in various domains such as macroeconomic modeling (the Vintage Capital model), ecology and population dynamics [20–23], and particularly in the mathematical modeling of power and energy systems [24]. In the latter context, the Volterra approach has been effectively applied to load forecasting, energy storage analysis, and optimal power flow problems involving renewable generation and battery systems, where iterative numerical methods based on the modified Newton-Kantorovich procedure have proven to be efficient and convergent. Solving Volterra integral equations with discontinuous kernels is often challenging, and obtaining exact solutions is usually difficult or even impossible. Therefore, it is essential to employ numerical approaches to achieve approximate solutions. Various numerical techniques have been developed for this purpose, such as the Adomian decomposition method [25], homotopy perturbation method [12], matrix method based on Genocchi polynomials [26], Taylor-collocation method [27], regularization method [28], a novel iterative method based on FPA/CESTAC/CADNA [29], Lagrange-collocation method combined with the CESTAC/CADNA approach [**?**], and the method based on triangular orthogonal functions for two-dimensional Volterra integral equations [30]. In this paper, a spectral collocation method based on orthogonal polynomials is employed. The approximate solution is constructed using these basis functions and collocated at the roots of the corresponding polynomials. The main novelty of this work lies

in the following aspects. First, unlike previous Legendre-based collocation methods that assume global smoothness of the kernel, our approach directly handles piecewise integration limits without requiring kernel regularization. Second, we provide an explicit construction of the product operational matrix for shifted Legendre polynomials, which enables efficient treatment of the integral terms without expensive numerical quadrature. Third, we derive rigorous L^2-error bounds that confirm rapid convergence under appropriate regularity assumptions. These features distinguish our method from existing approaches and establish it as a viable tool for problems with discontinuous kernels. The main features that motivate the use of this method can be summarized as follows

- Polynomial-based computations
- Banded structure of the integration operational matrix
- Moderate spectral resolution
- Transformation of the original problem into a system of algebraic equations
- Simple and straightforward implementation

Consequently, the proposed approach provides an accurate, stable, and computationally efficient numerical framework for solving this class of equations. The remainder of this paper is organized as follows. Section 2 presents some basic properties of Legendre polynomials and their shifted forms together with several useful operational matrices. In Section 3, the proposed spectral collocation method based on orthogonal polynomials is developed for solving the considered integro-differential equation with discontinuous kernels. In Section 4, an explicit L^2-error bound for the proposed approximation is derived under suitable regularity assumptions. Section 5 provides two numerical examples to illustrate the accuracy and efficiency of the proposed method. Finally, concluding remarks are given in the Section 6.

2 Preliminaries and requirements

2.1 Classical Legendre polynomials (LPs)

The classical Legendre polynomials, denoted by $L_n(x)$, are typically defined over the interval $[-1,1]$. They form an important orthogonal basis, crucial for spectral methods applied to the solution of differential and integral equations. These polynomials satisfy the recurrence relation

$$(n+1)L_{n+1}(x) = (2n+1)xL_n(x) - nL_{n-1}(x), \quad n \geq 1,$$

with $L_0(x) = 1$ and $L_1(x) = x$. The orthogonality property of these polynomials with respect to a weight function of unity over the interval $[-1,1]$ is given by

$$\int_{-1}^{1} L_m(x)L_n(x)\mathrm{d}x = \frac{2}{2n+1}\delta_{mn},$$

where δ_{mn} represents the Kronecker delta.

2.2 Shifted Legendre polynomials (SLPs)

For problem defined on the interval I, a common domain in many physical and engineering applications, the family of shifted Legendre polynomials, denoted by $\{L_n^*(x)\}_{n=0}^{\infty}$, is obtained by applying the following linear transformation to the classical LPs

$$L_n^*(x) = L_n\left(\frac{2x}{T} - 1\right).$$

The SLPs satisfy the recurrence relation:

$$(n+1)L_{n+1}^*(x) = (2n+1)\left(\frac{2x}{T} - 1\right)L_n^*(x) - nL_{n-1}^*(x), \quad n \geq 1, \tag{2.1}$$

with $L_0^*(x) = 1$ and $L_1^*(x) = \dfrac{2x}{T} - 1$. The orthogonality property of these polynomials with respect to a weight function of unity over the interval I is given by

$$\int_0^T L_m^*(x)L_n^*(x)\mathrm{d}x = \frac{T}{2n+1}\delta_{mn}. \tag{2.2}$$

For a comprehensive treatment of orthogonal polynomials and their spectral approximations, we refer the reader to the classical monograph by Canuto et al. [31].

2.3 Function approximation

We define the Hilbert space $\Phi = L^2(I)$ equipped with the standard inner product and norm

$$\langle g,h\rangle = \int_0^T g(x)h(x)\mathrm{d}x, \qquad \|g\|_2 = \left(\int_0^T |g(x)|^2 dx\right)^{1/2}. \tag{2.3}$$

We consider the set of SLPs $\{L_i^*(x)\}_{i=0}^{n}$. Let $\Phi_n \subset \Phi$ be the finite-dimensional subspace spanned by these polynomials

$$\Phi_n = \mathrm{span}\{L_0^*(x), L_1^*(x), \ldots, L_n^*(x)\}$$

For any function $g(x) \in \Phi$, its best approximation $g_n^*(x) \in \Phi_n$ in the L^2-norm sense satisfies $\|g - g_n^*\|_2 \leq \|g - f\|_2$ for all $f \in \Phi_n$. We define the vector of SLPs as

$$\mathbf{L}^*(x) = [L_0^*(x), L_1^*(x), \ldots, L_n^*(x)]^T, \tag{2.4}$$

Thus, this best approximation can be expressed in a compact vector form as

$$g(x) \simeq g_n^s(x) = \sum_{i=0}^{n} g_i^s L_i^*(x) = G_s^T \mathbf{L}^*(x), \quad x \in I, \tag{2.5}$$

where

$$G_s = [g_0^s, g_1^s, \ldots, g_n^s]^T, \tag{2.6}$$

By employing the orthogonality relation of SLPs from Eq. (2.2), the coefficients g_i^s are defined by

$$g_i^s = \frac{\langle g(x), L_i^*(\cdot)\rangle}{\|L_i^*\|_2^2} = \frac{2i+1}{T}\int_0^T g(x)L_i^*(x)\mathrm{d}x. \tag{2.7}$$

This formula follows directly from the orthogonality property (2.2) of the SLPs. Similarly, for each discontinuous kernel $k_l(x,t) \in L^2(I_l \times I_l)$, $l = 1,2,\ldots,n$, we seek its best approximation in the tensor product space SLPs basis. This approximation, denoted by $k_{l,n}(x,t)$, is structured as

$$k_l(x,t) \simeq k_l^s(x,t) = \mathbf{L}^{*T}(x)K_{sl}\mathbf{L}^*(t), \quad 1 \le l \le n, \tag{2.8}$$

where $K_{sl} = [(k_{sl})_{ij}]_{(n+1)\times(n+1)}$, is the matrix of coefficients. Employing the orthogonality property of SLPs, the matrix entries $(k_{sl})_{ij}$ are determined by

$$(k_{sl})_{ij} = \frac{\langle\langle k_l(\cdot,t), L_i^*(\cdot)\rangle, L_j^*(t)\rangle}{\|L_i^*\|_2^2\|L_j^*\|_2^2} = \frac{(2i+1)(2j+1)}{T^2}\int_0^T\int_0^T k_l(x,t)L_i^*(x)L_j^*(t)\mathrm{d}x\mathrm{d}t. \tag{2.9}$$

Note that the double integral in (2.9) is well-defined under standard regularity assumptions on the kernel pieces $k_l(x,t)$, which are assumed to be sufficiently smooth on each subdomain between the discontinuity curves $\xi_l(x)$.

2.4 Operational matrices for SLPs

Operational matrices allow integration, differentiation, and multiplication operators to be represented in simple algebraic form. In the following, we examine the operational matrices required for our formulation.

We can represent the operation of integration from 0 to x with respect to the SLPs basis vector $\mathbf{L}^*(x)$ as follows

$$\int_0^x \mathbf{L}^*(t)\mathrm{d}t = H_s\mathbf{L}^*(x), \quad x \in I, \tag{2.10}$$

where

$$H_s = \frac{T}{2}H. \tag{2.11}$$

This matrix is derived directly from the integral operational matrix H associated with the classical LPs. This follows from the recurrence relation

$$\int_{-1}^x L_i(t)dt = \frac{L_{i+1}(x) - L_{i-1}(x)}{2i+1},$$

and the change of variable $x = \frac{T}{2}(z+1)$. For a rigorous derivation see [32]. The matrix H of size $(n+1)\times(n+1)$ is given by

$$H_{i,j} = \begin{cases} 1, & i = j = 0, \\ \dfrac{1}{2i+1}, & j = i+1,\ i = 0,1,\ldots,n-1, \\ -\dfrac{1}{2i-1}, & j = i-1,\ i = 1,2,\ldots,n, \\ 0, & \text{otherwise.} \end{cases}$$

Equivalently,

$$H = \begin{bmatrix} 1 & 1 & 0 & 0 & \cdots & 0 \\ -\frac{1}{3} & 0 & \frac{1}{3} & 0 & \cdots & 0 \\ 0 & -\frac{1}{5} & 0 & \frac{1}{5} & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & -\frac{1}{2n-1} & 0 & \frac{1}{2n-1} \\ 0 & \cdots & 0 & 0 & -\frac{1}{2n+1} & 0 \end{bmatrix}.$$

The product of SLPs can be represented through the operational product matrix as [32]

$$\mathbf{L}^*(x)\mathbf{L}^{*T}(x)G_s \simeq \widehat{G}_s\mathbf{L}^*(x), \tag{2.12}$$

where

$$\widehat{G}_s = \frac{T}{2}\widehat{G}, \tag{2.13}$$

and $\widehat{G}$ is defined as $\widehat{G} = [\widehat{G}_{h,j}]_{(n+1)\times(n+1)}$, with row index h and column index j, where each entry is given by

$$\widehat{G}_{h,j} = \frac{2h+1}{2}\sum_{i=0}^{n} g_i^s \int_0^T L_i^*(x)L_j^*(x)L_h^*(x)\mathrm{d}x; \quad h,j = 0,1,\ldots,n. \tag{2.14}$$

The integrals $\int_0^T L_i^*(x)L_j^*(x)L_h^*(x)\mathrm{d}x$ are proportional to the triple product integrals of classical LPs; they vanish unless $|i-j| \le h \le i+j$ and $(i+j+h)$ is even. For explicit formulas, see [32].

The derivative of the SLPs basis satisfies

$$\frac{d}{dx}\mathbf{L}^*(x) = D\mathbf{L}^*(x),$$

where D is an $(n+1)\times(n+1)$ matrix. To derive D, we recall the derivative property of classical LPs on $[-1,1]$

$$L_k'(x) = \sum_{\substack{m=0 \\ m\equiv k-1 \pmod 2}}^{k-1} (2m+1)L_m(x), \qquad k \ge 1.$$

This means that the derivative of $L_k(x)$ is a linear combination of LPs of lower degree with the same parity as $(k-1)$. Applying the chain rule to the affine transformation $x \mapsto \frac{2x}{T} - 1$, we obtain

$$\frac{d}{dx}L_k^*\left(\frac{2x}{T}-1\right) = \frac{2}{T}\cdot L_k'\left(\frac{2x}{T}-1\right).$$

Consequently, the entries of the differentiation matrix D are given by

$$D_{ij} = \frac{2}{T}\cdot\begin{cases} 2j+1, & \text{if } i>j \text{ and } (i+j) \text{ is odd}, \\ 0, & \text{otherwise}, \end{cases} \qquad i,j = 0,1,\ldots,n.$$

For illustration, with $n = 4$ we obtain

$$D = \frac{2}{T}\begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 & 0 \\ 1 & 0 & 5 & 0 & 0 \\ 0 & 3 & 0 & 7 & 0 \end{bmatrix}.$$

For a complete derivation of the differentiation matrix for SLPs using this recurrence-based approach, see [33].

The computational cost of the operational matrix approach is primarily determined by the solution of the $(n+1)\times(n+1)$ linear system, which requires $\mathscr{O}(n^3)$ operations using standard Gaussian elimination. The construction of the matrices themselves requires $\mathscr{O}(n^2)$ operations, except for the product matrix which involves triple product integrals and requires $\mathscr{O}(n^3)$ operations. The method is stable for moderate values of n, as the condition number of the collocation matrix grows polynomially with n. The memory requirement is $\mathscr{O}(n^2)$ for storing the dense coefficient and operational matrices.

3 Methodology description

Throughout this Section, we assume that the discontinuity curves $\xi_l(x)$ are affine functions of x, i.e., $\xi_l(x) = a_l x$ with $0 = a_0 < a_1 < \cdots < a_n = 1$. This assumption ensures that $\xi_l(x)$ maps the interval I onto I and that $\mathbf{L}^*(\xi_l(x))$ remains a polynomial in x of degree at most n.

The affine assumption is essential for the operational matrix formulation presented in this work. Indeed, the use of the integration matrix H_s in (3.6) relies on the fact that $\mathbf{L}^*(\xi_l(x))$ is a polynomial of degree at most n, which allows the exact evaluation of integrals of the form $\int_0^{\xi_l(x)} \mathbf{L}^*(t)dt$. If the discontinuity curves were nonlinear, $\mathbf{L}^*(\xi_l(x))$ would no longer be a polynomial, and the operational matrix approach could not be applied directly. In such cases, one would need to evaluate the integrals $\int_{\xi_{l-1}(x)}^{\xi_l(x)} \mathbf{L}^*(t)dt$ numerically using Gaussian quadrature or adaptive quadrature rules, which would increase the computational cost. This extension, while conceptually straightforward, is beyond the scope of the present work and is left for future investigation.

Regarding the Gibbs phenomenon, we emphasize that the discontinuous kernel $k_l(x,t)$ is not approximated globally by a single polynomial expansion. Instead, the method exploits the piecewise structure of the integration limits $\xi_l(x)$. The approximation $k_l^s(x,t) = \mathbf{L}^{*T}(x)K_{sl}\mathbf{L}^*(t)$ in (2.8) is applied to each smooth piece of the kernel on each subdomain. The discontinuities are handled by the limits of integration, not by global polynomial approximation. This approach effectively avoids the Gibbs phenomenon because the polynomial approximation is only required on subdomains where the kernel is smooth.

In this Section, we utilize SLPs collocation technique to solve Eq. (1.1). For this purpose, the unknown function $g(x)$ is approximated by a truncated SLPs expansion

$$g(x) \simeq G_s^T \mathbf{L}^*(x), \tag{3.1}$$

and its first derivative is represented via the operational derivative matrix D as

$$g'(x) \simeq G_s^T D\mathbf{L}^*(x), \tag{3.2}$$

Here, vector $\mathbf{L}^*(x)$, the coefficient vector G_s, and the operational matrix of derivative D are defined in Eqs. (2.4), (2.6), and above, respectively. The known function $f(x)$ is expanded in the same basis to ensure spectral consistency of the discretization

$$f(x) \simeq F_s^T \mathbf{L}^*(x), \tag{3.3}$$

where the coefficients vector F_s is given by

$$F_s = [f_0^s, f_1^s, \ldots, f_n^s]^T. \tag{3.4}$$

Substituting the approximations (2.8), (3.1), (3.2) and (3.3) into the VIDEs (1.1), we obtain

$$G_s^T D\mathbf{L}^*(x) = F_s^T \mathbf{L}^*(x) + \sum_{l=1}^{n} \lambda_l \int_{\xi_{l-1}(x)}^{\xi_l(x)} \mathbf{L}^{*T}(x) K_{sl} \mathbf{L}^*(t) G_s^T \mathbf{L}^*(t) \mathrm{d}t, \tag{3.5}$$

Applying the operational product and integration matrices defined in subsection 2.4, and exploiting the linearity of the integral together with the independence of $\mathbf{L}^{*T}(x)$, K_{sl}, and $\widehat{G}_s$ from the integration variable t, the integral term in (3.5) can be transformed as follows

$$\begin{aligned}
\sum_{l=1}^{n} \lambda_l \int_{\xi_{l-1}(x)}^{\xi_l(x)} \mathbf{L}^{*T}(x) K_{sl} \mathbf{L}^*(t) G_s^T \mathbf{L}^*(t) \mathrm{d}t &= \sum_{l=1}^{n} \lambda_l \int_{\xi_{l-1}(x)}^{\xi_l(x)} \mathbf{L}^{*T}(x) K_{sl} \mathbf{L}^*(t) \mathbf{L}^{*T}(t) G_s \mathrm{d}t \\
&\simeq \sum_{l=1}^{n} \lambda_l \int_{\xi_{l-1}(x)}^{\xi_l(x)} \mathbf{L}^{*T}(x) K_{sl} \widehat{G}_s \mathbf{L}^*(t) \mathrm{d}t \\
&= \sum_{l=1}^{n} \lambda_l \mathbf{L}^{*T}(x) K_{sl} \widehat{G}_s \int_{\xi_{l-1}(x)}^{\xi_l(x)} \mathbf{L}^*(t) \mathrm{d}t \\
&= \sum_{l=1}^{n} \lambda_l \mathbf{L}^{*T}(x) K_{sl} \widehat{G}_s \left(\int_0^{\xi_l(x)} \mathbf{L}^*(t) \mathrm{d}t - \int_0^{\xi_{l-1}(x)} \mathbf{L}^*(t) \mathrm{d}t \right) \\
&= \sum_{l=1}^{n} \lambda_l \mathbf{L}^{*T}(x) K_{sl} \widehat{G}_s H_s \left(\mathbf{L}^*(\xi_l(x)) - \mathbf{L}^*(\xi_{l-1}(x)) \right) \\
&= \sum_{l=1}^{n} \lambda_l \frac{T^2}{4} \mathbf{L}^{*T}(x) K_{sl} \widehat{G} H \left(\mathbf{L}^*(\xi_l(x)) - \mathbf{L}^*(\xi_{l-1}(x)) \right),
\end{aligned} \tag{3.6}$$

where K_{sl}, H_s and $\widehat{G}_s$ are defined in Eqs. (2.9), (2.11) and (2.13), respectively. It is worth emphasizing that the passage from the fourth to the fifth line in (3.6), where the integration matrix H_s is applied, is valid precisely because the functions $\xi_l(x)$ are affine. For non-affine discontinuity curves, this step would require replacing H_s with a numerical integration scheme. Inserting (3.6) into (3.5) yields the fully spectral form of the governing equation

$$G_s^T D\mathbf{L}^*(x) = F_s^T \mathbf{L}^*(x) + \sum_{l=1}^{n} \lambda_l \frac{T^2}{4} \mathbf{L}^{*T}(x) K_{sl} \widehat{G} H \left(\mathbf{L}^*(\xi_l(x)) - \mathbf{L}^*(\xi_{l-1}(x)) \right). \tag{3.7}$$

Let $\{x_i\}_{i=1}^{n+1}$ be the roots of $L_{n+1}^*(x)$ (the shifted Legendre-Gauss points). The collocation scheme requires that equation (1.1) holds at these points. To incorporate the initial condition $g(0) = 0$, we replace the first collocation equation by

$$G_s^T \mathbf{L}^*(0) = 0.$$

We use the $(n+1)$ shifted Legendre-Gauss roots $x_1, x_2, \ldots, x_{n+1}$ of $L^*_{n+1}(x)$ for collocation. To enforce the initial condition $g(0) = 0$, we discard the collocation equation at the smallest node x_1 and replace it with the initial condition $G_s^T \mathbf{L}^*(0) = 0$, evaluated at $x = 0$ rather than at a Gauss node (note that Gauss nodes are strictly interior to the interval, so $x = 0$ is not one of them).

Now, we collocate (3.7) at the points $x_0, x_1, \ldots, x_n$, which are the roots of SLPs $L^*_{n+1}(x)$. Thus, we obtain the following system of linear equations with $(n+1)$ unknowns and $(n+1)$ equations

$$G_s^T D\mathbf{L}^*(x_i) = F_s^T \mathbf{L}^*(x_i) + \sum_{l=1}^{n} \lambda_l \frac{T^2}{4} \mathbf{L}^{*T}(x_i) K_{sl} \widehat{G} H \left(\mathbf{L}^*(\xi_l(x_i)) - \mathbf{L}^*(\xi_{l-1}(x_i)) \right), \quad i = 0, 1, \ldots, n. \tag{3.8}$$

An alternative equation can be derived by substituting $g(0)$ in the initial condition given in Eq. (1.1) as follows

$$g(0) \simeq G_s^T \mathbf{L}^*(0) = 0. \tag{3.9}$$

Although we assume $g(0) = 0$ in this work, the method can be applied to problems with $g(0) \neq 0$ by the simple translation $\tilde{g}(x) = g(x) - g(0)$, which reduces the problem to the homogeneous case $\tilde{g}(0) = 0$. Subsequently, the equation obtained from the initial condition (3.9) is incorporated by replacing one of the equations in the linear system (3.8), thereby maintaining the square structure of the system with $(n+1)$ equations and $(n+1)$ unknowns. This substitution preserves the square, non-singular nature of the resulting algebraic system, as the collocation equations are linearly independent and the replacement of one equation by the initial condition does not introduce linear dependence. The resulting system of algebraic equations is then solved to determine the unknown coefficients g_i^s, $i = 0, 1, \ldots, n$. Consequently, the approximate solution $g_n^s(x)$ of the problem is obtained.

4 Error bound and convergence analysis

In this Section, we perform a detailed error analysis for the numerical scheme. An upper bound for the error is obtained by employing the L^2-norm, as introduced in Section 2.3.

The following theorems establish the convergence rates of the proposed method under the stated regularity assumptions. The error bounds exhibit exponential decay with respect to the polynomial degree n, characterized by terms of the form $\frac{T^{n+1}(n+1)!}{(2n+2)!}$, which indicates rapid convergence for smooth kernels. This behavior is guaranteed only under the smoothness assumptions stated in the theorems.

Theorem 1. *Let $g(x)$ be a sufficiently smooth real-valued function defined on the interval I. Suppose that*

$$g_n^s(x) = G_s^T \mathbf{L}^*(x),$$

denotes the best L^2 approximation of $g(x)$ obtained by the SLPs expansion of degree n. Define $Q = \max_{x \in I} |g^{(n+1)}(x)|$. Then the following error estimate holds

$$\|g - g_n^s\|_2 \leq \sqrt{T}\, Q \frac{T^{n+1}(n+1)!}{(2n+2)!}.$$

Proof. Let $P_n(x)$ denote the interpolating polynomial of degree n for the function $g(x)$ at the nodes x_i, $i = 0, 1, \ldots, n$, which are the zeros of the SLPs of degree $(n+1)$ on the interval I (i.e., the shifted

Legendre-Gauss points). According to the interpolation remainder formula, we have

$$g(x)-P_n(x)=\frac{g^{(n+1)}(\alpha)}{(n+1)!}\prod_{i=0}^{n}(x-x_i),$$

where $\alpha=\alpha(x)\in I$ depends on x and on the interpolation nodes. Since the SLP nodes on the interval I satisfy the bound

$$\left|\prod_{i=0}^{n}(x-x_i)\right|\leq\frac{T^{n+1}((n+1)!)^2}{(2n+2)!},$$

it follows that

$$|g(x)-P_n(x)|\leq\frac{QT^{n+1}(n+1)!}{(2n+2)!}.$$

Since $g_n^s(x)$ represents the best approximation of $g(x)$ in the space spanned by the first $(n+1)$ SLPs with respect to the L^2-norm, we have

$$\|g-g_n^s\|_2\leq\|g-P_n\|_2.$$

Hence

$$\|g-g_n^s\|_2^2\leq\int_0^T|g(x)-P_n(x)|^2\mathrm{d}x\leq T\left[Q\frac{T^{n+1}(n+1)!}{(2n+2)!}\right]^2.$$

Taking the square root on both sides completes the proof. □

Theorem 2. *Assume that the kernel $k_l(x,t)\in C^{2n+2}(I_l\times I_l)$, $l=1,2,\ldots,n$ is a sufficiently smooth real-valued function. Let $k_l^s(x,t)=\mathbf{L}^{*T}(x)K_{sl}\mathbf{L}^*(t)$, be its approximation obtained using SLPs, where the degree of these polynomials is at most n. Define the constants*

1. $Q_1=\max_{(x,t)\in I_l\times I_l}\left|\frac{\partial^{n+1}k_l(x,t)}{\partial x^{n+1}}\right|$,
2. $Q_2=\max_{(x,t)\in I_l\times I_l}\left|\frac{\partial^{n+1}k_l(x,t)}{\partial t^{n+1}}\right|$,
3. $Q_3=\max_{(x,t)\in I_l\times I_l}\left|\frac{\partial^{2n+2}k_l(x,t)}{\partial x^{n+1}\partial t^{n+1}}\right|$.

Then, the error of the approximation in the L^2-norm satisfies

$$\|k_l-k_l^s\|_2\leq T\left[(Q_1+Q_2)\frac{T^{n+1}(n+1)!}{(2n+2)!}+Q_3\frac{T^{2n+2}((n+1)!)^2}{((2n+2)!)^2}\right],\tag{4.1}$$

Proof. Let $P_{n,n}(x,t)$ denote the bivariate interpolation polynomial of $k_l(x,t)$ constructed at the nodal points (x_i,t_j), $i,j=0,1,\ldots,n$, where $\{x_i\}_{i=0}^n$ and $\{t_j\}_{j=0}^n$ are the SLPs nodes of order $(n+1)$ on the

interval I. For this polynomial, the remainder formula yields

$$\begin{aligned} k_l(x,t) - P_{n,n}(x,t) = & \frac{1}{(n+1)!}\frac{\partial^{n+1}k_l(\alpha,t)}{\partial x^{n+1}}\prod_{i=0}^{n}(x-x_i) \\ & + \frac{1}{(n+1)!}\frac{\partial^{n+1}k_l(x,\beta)}{\partial t^{n+1}}\prod_{j=0}^{n}(t-t_j) \\ & - \frac{1}{(n+1)!(n+1)!}\frac{\partial^{2n+2}k_l(\alpha',\beta')}{\partial x^{n+1}\partial t^{n+1}}\prod_{i=0}^{n}(x-x_i)\prod_{j=0}^{n}(t-t_j), \end{aligned} \tag{4.2}$$

where $\{\alpha,\beta,\alpha',\beta'\} \in I_l$, $l = 1,2,\ldots,n$. The bivariate interpolation remainder formula (4.2) is standard in multivariate approximation theory. For a direct reference, see [34], where the remainder of the Lagrange interpolation formula for two variables is expressed in terms of partial derivatives. Taking absolute values on both sides and employing the definitions of the constants Q_1, Q_2 and Q_3, and further, utilizing the properties of the SLPs nodes, we obtain

$$|k_l(x,t) - P_{n,n}(x,t)| \leq (Q_1+Q_2)\frac{T^{n+1}(n+1)!}{(2n+2)!} + Q_3\frac{T^{2n+2}((n+1)!)^2}{((2n+2)!)^2}, \tag{4.3}$$

Since $k_l^s(x,t)$ is the best approximation of $k_l(x,t)$ in the space generated by SLPs with respect to the $L^2(I_l \times I_l)$ norm, we have

$$\|k_l - k_l^s\|_2 \leq \|k_l - P_{n,n}\|_2. \tag{4.4}$$

Hence

$$\begin{aligned} \|k_l - k_l^s\|_2^2 &= \int_0^T\int_0^T |k_l(x,t) - k_l^s(x,t)|^2 \mathrm{d}x\,\mathrm{d}t \\ &\leq \int_0^T\int_0^T |k_l(x,t) - P_{n,n}(x,t)|^2 \mathrm{d}x\,\mathrm{d}t \\ &\leq T^2\left[(Q_1+Q_2)\frac{T^{n+1}(n+1)!}{(2n+2)!} + Q_3\frac{T^{2n+2}((n+1)!)^2}{((2n+2)!)^2}\right]^2. \end{aligned} \tag{4.5}$$

Taking the square root on both sides completes the proof. □

Theorem 3. *Let $g(x)$ and $g_n^s(x)$ denote the exact solution and the numerical approximation of Eq. (1.1), respectively, where the approximate solution $g_n^s(x)$ is constructed via interpolation based on SLPs. Assume that*

$$1 - \frac{2}{3}T^2\sum_{l=1}^{n}|\lambda_l|M_l > 0. \tag{4.6}$$

Define the constants

1. $\delta = \max_{x\in I}|f(x) - f_n^s(x)|$,
2. $\gamma = \max_{x\in I}|g_n^s(x)|$,
3. $M_l = \max_{(x,t)\in I_l\times I_l}|k_l(x,t)|, \quad \forall\, l = 1,2,\ldots,n$,

4. $\eta_l = \max_{(x,t)\in I_l\times I_l} |k_l(x,t) - k_l^s(x,t)|, \quad \forall\, l = 1,2,\ldots,n.$

Here δ, γ, M_l, and η_l, are fixed positive constants. Then, the error between the exact solution $g(x)$ and its numerical approximation $g_n^s(x)$ satisfies the bound

$$\|g - g_n^s\|_2 \leq \frac{\delta T^{\frac{3}{2}} + \gamma \frac{T^{\frac{5}{2}}}{2} \sum_{l=1}^{n} |\lambda_l| \eta_l}{1 - \frac{2}{3} T^2 \sum_{l=1}^{n} |\lambda_l| M_l}, \tag{4.7}$$

Proof. Suppose that the functions $g(x)$, $f(x)$ and the kernels $k_l(x,t)$, $l = 1,2,\ldots,n$, are approximated according to (2.5), (3.3) and (2.8), respectively. To analyze the error between the exact solution $g(x)$ and its approximation $g_n^s(x)$, we consider the difference between the original equation and its approximated form

$$g'(z) - g_n^{s\prime}(z) = f(z) - f_n^s(z) + \sum_{l=1}^{n} \lambda_l \int_{\xi_{l-1}(z)}^{\xi_l(z)} k_l(z,t) g(t) \mathrm{d}t - \sum_{l=1}^{n} \lambda_l \int_{\xi_{l-1}(z)}^{\xi_l(z)} k_l^s(z,t) g_n^s(t) \mathrm{d}t. \tag{4.8}$$

By integrating both sides of (4.8) with respect to z over the interval $[0,x]$, we obtain

$$g(x) - g_n^s(x) = \int_0^x (f(z) - f_n^s(z)) \mathrm{d}z + \int_0^x \sum_{l=1}^{n} \lambda_l \int_{\xi_{l-1}(z)}^{\xi_l(z)} (k_l(z,t) g(t) - k_l^s(z,t) g_n^s(t)) \, \mathrm{d}t \, \mathrm{d}z, \tag{4.9}$$

Taking the absolute values of both sides, we have

$$|g(x) - g_n^s(x)| \leq \int_0^x |f(z) - f_n^s(z)| \mathrm{d}z + \int_0^x \sum_{l=1}^{n} |\lambda_l| \int_{\xi_{l-1}(z)}^{\xi_l(z)} |k_l(z,t) g(t) - k_l^s(z,t) g_n^s(t)| \mathrm{d}t \, \mathrm{d}z, \tag{4.10}$$

where

$$\begin{aligned} |k_l(z,t) g(t) - k_l^s(z,t) g_n^s(t)| &= |k_l(z,t) g(t) - k_l(z,t) g_n^s(t) + k_l(z,t) g_n^s(t) - k_l^s(z,t) g_n^s(t)| \\ &\leq |k_l(z,t)| |g(t) - g_n^s(t)| + |k_l(z,t) - k_l^s(z,t)| |g_n^s(t)|. \end{aligned} \tag{4.11}$$

Using the definitions of δ, γ, M_l and η_l, together with (4.10)-(4.11), we obtain

$$|g(x) - g_n^s(x)| \leq \delta T + \int_0^x \sum_{l=1}^{n} |\lambda_l| \int_{\xi_{l-1}(z)}^{\xi_l(z)} (M_l |g(t) - g_n^s(t)| + \eta_l \gamma) \, \mathrm{d}t \, \mathrm{d}z, \tag{4.12}$$

For the geometric double integral with unit integrand, using the definition of Eq. (1.1), which implies $0 \leq \xi_{l-1}(z) < \xi_l(z) \leq z \leq T$, we obtain

$$\int_0^x \int_{\xi_{l-1}(z)}^{\xi_l(z)} \mathrm{d}t \, \mathrm{d}z = \int_0^x (\xi_l(z) - \xi_{l-1}(z)) \, \mathrm{d}z \leq \int_0^x z \, \mathrm{d}z \leq \frac{T^2}{2}. \tag{4.13}$$

For the second double integral, we apply the Cauchy-Schwarz inequality to the inner integral

$$\begin{aligned} \int_{\xi_{l-1}(z)}^{\xi_l(z)} |g(t) - g_n^s(t)| \mathrm{d}t &\leq \left(\int_{\xi_{l-1}(z)}^{\xi_l(z)} 1^2 \mathrm{d}t \right)^{\frac{1}{2}} \left(\int_{\xi_{l-1}(z)}^{\xi_l(z)} |g(x) - g_n^s(x)|^2 \mathrm{d}t \right)^{\frac{1}{2}} \\ &\leq (\xi_l(z) - \xi_{l-1}(z))^{\frac{1}{2}} \|g - g_n^s\|_2. \end{aligned} \tag{4.14}$$

Using again $0 < \xi_l(z) - \xi_{l-1}(z) \le z$, we obtain

$$\begin{aligned}\int_0^x \int_{\xi_{l-1}(z)}^{\xi_l(z)} |g(t) - g_n^s(t)| \mathrm{d}t \, \mathrm{d}z &\le \|g - g_n^s\|_2 \int_0^x z^{\frac{1}{2}} \mathrm{d}z \\ &= \frac{2}{3} x^{\frac{3}{2}} \|g - g_n^s\|_2 \\ &\le \frac{2}{3} T^{\frac{3}{2}} \|g - g_n^s\|_2. \end{aligned} \tag{4.15}$$

Hence, from relations (4.12), (4.13) and (4.15), we obtain

$$|g(x) - g_n^s(x)| \le \delta T + \frac{2}{3} T^{\frac{3}{2}} \|g - g_n^s\|_2 \sum_{l=1}^{n} |\lambda_l| M_l + \gamma \frac{T^2}{2} \sum_{l=1}^{n} |\lambda_l| \eta_l, \tag{4.16}$$

We now take the L^2-norm of both sides of the inequality (4.16). Since the right-hand side is independent of x, its norm is obtained by multiplying it by $\sqrt{T}$, i.e., $\|C\|_2 = C\sqrt{T}$ for any constant C. Thus, we have

$$\|g - g_n^s\|_2 \le \sqrt{T} \left(\delta T + \frac{2}{3} T^{\frac{3}{2}} \|g - g_n^s\|_2 \sum_{l=1}^{n} |\lambda_l| M_l + \gamma \frac{T^2}{2} \sum_{l=1}^{n} |\lambda_l| \eta_l \right), \tag{4.17}$$

Therefore,

$$\|g - g_n^s\|_2 \left(1 - \frac{2}{3} T^2 \sum_{l=1}^{n} |\lambda_l| M_l \right) \le \delta T^{\frac{3}{2}} + \gamma \frac{T^{\frac{5}{2}}}{2} \sum_{l=1}^{n} |\lambda_l| \eta_l, \tag{4.18}$$

Finally, dividing both sides of relation (4.18), by the positive quantity

$$1 - \frac{2}{3} T^2 \sum_{l=1}^{n} |\lambda_l| M_l,$$

yields

$$\|g - g_n^s\|_2 \le \frac{\delta T^{\frac{3}{2}} + \gamma \frac{T^{\frac{5}{2}}}{2} \sum_{l=1}^{n} |\lambda_l| \eta_l}{1 - \frac{2}{3} T^2 \sum_{l=1}^{n} |\lambda_l| M_l},$$

This completes the proof. □

A note on the sufficient condition

The sufficient condition (4.6) involves only the problem data T, λ_l, and M_l, and is independent of the polynomial degree n. For problems where this condition is violated, convergence can still be achieved by either:

1. subdividing the interval I into smaller subintervals (a time-stepping approach), or
2. noting that $\eta_l = \mathcal{O}((n!)^{-2})$ decays exponentially as $n \to \infty$, which ensures the inequality holds for sufficiently large n.

5 Numerical experiments

To evaluate the accuracy and efficiency of the proposed method, two numerical examples are considered. All numerical computations are carried out using Maple software (version 2018) on a personal computer equipped with an Intel Core i5-4200 CPU with 4 GB of RAM. All computations were performed using Maple 2018 with double precision (IEEE 754), providing about 16 decimal digits of accuracy. Since the algorithm is fully deterministic, repeated runs give identical results; hence, single-run CPU times are reported.

It should be noted that the proposed method is developed for a specific class of Volterra integro-differential equations with discontinuous kernels, for which no direct comparable numerical method exists in the literature. The numerical examples are constructed to illustrate the effectiveness of the method in handling discontinuities. Since the exact solutions of these two examples are available, the accuracy of the proposed method can be evaluated by computing the absolute error. Let $g(t_i)$ and $g_n^s(t_i)$ denote the exact and approximate solutions at the grid points $t_i \in I$ and $T = 1$, respectively. The absolute error at the point t_i, denoted by $AE(t_i)$, is defined as

$$AE(t_i) = |g(t_i) - g_n^s(t_i)|.$$

Example 1. Consider the following linear VIDE with discontinuous kernels

$$g'(x) = f(x) + \lambda_1 \int_0^{\frac{x}{4}} (1+x+t)g(t)\mathrm{d}t + \lambda_2 \int_{\frac{x}{4}}^{\frac{x}{2}} (2+xt)g(t)\mathrm{d}t + \lambda_3 \int_{\frac{x}{2}}^{x} (1+x+t)g(t)\mathrm{d}t, \quad x \in [0,1],$$

to the initial condition $g(0) = 0$, where $f(x)$ and the parameters λ_1, λ_2, and λ_3 are given by

$$f(x) = \frac{3}{8}x^2 - \frac{271}{8192}x^4 - \frac{1099}{20480}x^5 - \frac{31}{40960}x^6, \quad \lambda_1 = \lambda_2 = \lambda_3 = 1,$$

In this example, the exact solution is given by $g(x) = \frac{x^3}{8}$. Table 1 reports the absolute errors for several values of n and x, illustrating the convergence behavior with respect to the polynomial degree. Table 2 presents the corresponding CPU times for different choices of n, further confirming the computational efficiency of the approach. Figure 1 displays the exact and approximate solutions for $n = 8$, showing an excellent agreement between them, while Figure 2 illustrates the absolute error distribution for the same value of n.

Remark 1. *For $n = 2$, the absolute error is of order 10^{-2}. This is expected because the kernel approximations $k_l^s(x,t)$ in (2.8) are of degree 2 in each variable, whereas the exact kernels require higher-degree expansions to represent the product $k_l(x,t)g(t)$ accurately. As n increases, the error decreases spectrally, confirming the convergence analysis of Section 4.*

Example 2. Consider the following linear VIDE with discontinuous kernels

$$g'(x) = f(x) + \lambda_1 \int_0^{\frac{x}{2}} (x-t)\sin(t)g(t)\mathrm{d}t + \lambda_2 \int_{\frac{x}{2}}^{x} \sin(t)g(t)\mathrm{d}t, \quad x \in [0,1],$$

subject to the initial condition $g(0) = 0$, where

$$\begin{aligned} f(x) \;=\; & \cos(x) - \tfrac{1}{16}\lambda_1\Big(-2+3x^2+2\cos(x)-2x\sin(x) \\ & -\tfrac{1}{4}\lambda_2\big(x+\sin(x)-2\cos(x)\sin(x)\big)\Big), \end{aligned}$$

Table 1: Absolute errors for various values of n and x in Example 1

x	n=2	n=4	n=8
0.0	0.00	0.00	5.30E-12
0.1	1.13E-02	2.33E-07	3.30E-11
0.2	1.93E-02	3.32E-07	3.30E-11
0.3	2.45E-02	3.55E-07	2.70E-11
0.4	2.78E-02	3.47E-07	1.20E-11
0.5	3.00E-02	3.40E-07	1.00E-18
0.6	3.17E-02	3.56E-07	1.00E-18
0.7	3.37E-02	4.03E-07	1.00E-18
0.8	3.68E-02	4.78E-07	1.00E-11
0.9	4.17E-02	5.64E-07	3.00E-11
1.0	4.92E-02	6.34E-07	1.00E-18

Table 2: CPU time(s) corresponding to different values of n in Example 1

n	1	2	3	4	5	6	7	8	9	10
CPU time(s)	0.016	0.015	0.031	0.032	0.047	0.078	0.094	0.109	0.171	0.187

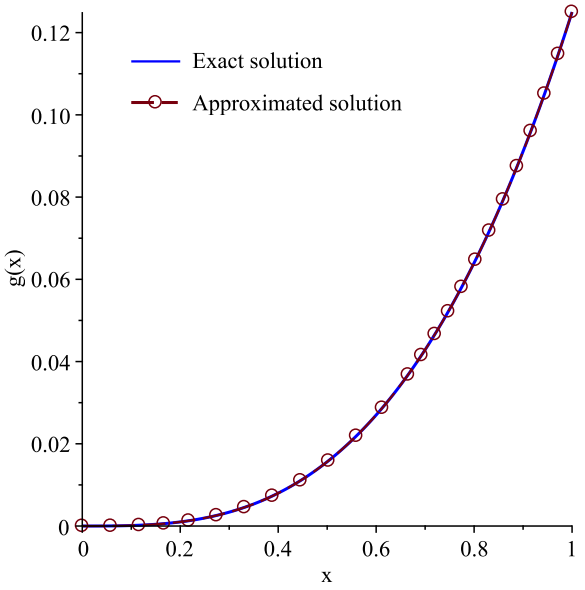

Figure 1: Graph of the exact and approximate solutions for $n = 8$ in Example 1

and

$$\lambda_1 = \lambda_2 = \frac{1}{100}.$$

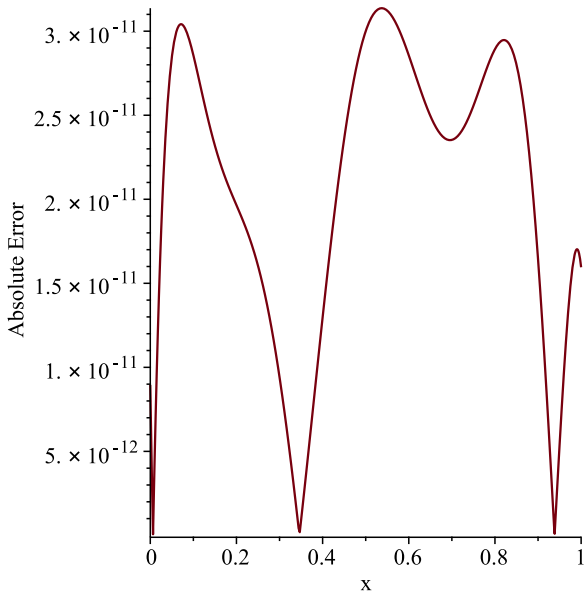

Figure 2: Graph of the absolute error of the proposed method for $n = 8$ in Example 1

Table 3: Absolute errors for different values of n in Example 2

x	n=2	n=4	n=8
0.0	0.00	0.00	1.00E-18
0.1	1.64E-02	1.96E-04	1.47E-07
0.2	2.75E-02	2.50E-04	1.17E-07
0.3	3.43E-02	2.45E-04	1.22E-07
0.4	3.76E-02	2.30E-04	1.94E-07
0.5	3.85E-02	2.24E-04	2.54E-07
0.6	3.79E-02	2.29E-04	2.32E-07
0.7	3.65E-02	2.36E-04	1.40E-07
0.8	3.53E-02	2.38E-04	7.30E-08
0.9	3.48E-02	2.32E-04	1.12E-07
1.0	3.59E-02	2.33E-04	1.62E-07

Assume that the exact solution for this example is given by $g(x) = \sin(x)$. Table 3 reports the absolute errors for various values of n and x. Table 4 presents the CPU times corresponding to different value of n. Figure 3 displays the graph of the exact and approximate solutions for $n = 8$. Figure 4 depicts the corresponding absolute error distribution. All numerical results obtained in this example confirm the accuracy, stability, and computational efficiency of the proposed method.

The numerical results presented in Tables 1-4 and Figures 1-4 demonstrate that the proposed spectral collocation method achieves high accuracy with moderate values of n. The exponential decay of the

Table 4: CPU time(s) for various values of n in Example 2

n	1	2	3	4	5	6	7	8	9	10
CPU time(s)	0.047	0.093	0.171	0.265	0.499	0.609	0.921	1.248	1.482	2.043

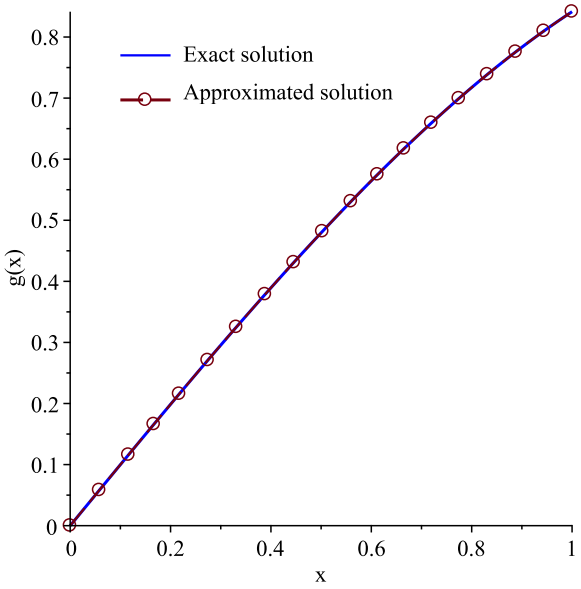

Figure 3: Graph of the exact and approximate solutions for $n = 8$ in Example 2

error as n increases confirms the rapid (exponential) convergence predicted by the theoretical analysis in Section 4, consistent with the error bounds derived therein. Furthermore, the CPU times reported in Tables 2 and 4 indicate that the method is computationally efficient, with $n = 10$ requiring less than 0.2 seconds in Example 1 and 2 seconds in Example 2. Regarding the sensitivity of the method to computational parameters, the polynomial degree n is the primary parameter controlling accuracy. The method is not sensitive to other computational parameters, such as the quadrature tolerance, due to its deterministic nature and the use of double-precision arithmetic.

6 Conclusions

In this paper, a spectral collocation method based on orthogonal polynomials has been developed for the numerical solution of linear Volterra integro-differential equations with discontinuous kernels. The method relies on the expansion of the unknown solution and the kernel in terms of shifted Legendre polynomials together with the construction of operational matrices for integration, differentiation, and polynomial products. This framework transforms the original problem into a finite system of linear algebraic equations for the expansion coefficients, which can be solved efficiently. An explicit L^2-error bound for the proposed approximation has been derived under suitable regularity assumptions. The theoretical error estimates confirm exponential convergence with respect to the polynomial degree for smooth

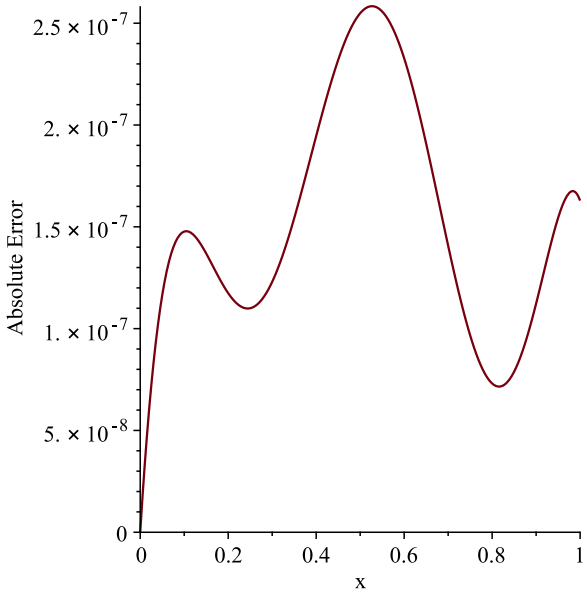

Figure 4: Graph of the absolute error of the proposed method for $n = 8$ in Example 2

kernels. Numerical experiments further verify the theoretical results and demonstrate that the method achieves high accuracy with a relatively small number of basis functions while maintaining low computational cost. A key practical advantage of the proposed method is its ability to handle discontinuous kernels directly through the piecewise integration limits, without requiring kernel regularization or global smoothness assumptions, making it particularly suitable for problems arising in applications where such discontinuities naturally occur. These results suggest that the proposed spectral scheme provides an efficient approach for the numerical treatment of Volterra integro-differential equations with discontinuous kernels, within the constraints discussed below.

The proposed method has several limitations that should be acknowledged. First, the discontinuity curves $\xi_l(x)$ are assumed to be affine functions $\xi_l(x) = a_l x$ (see Section 3). This assumption is necessary for the operational matrix formulation, as it ensures that $\mathbf{L}^*(\xi_l(x))$ remains a polynomial of degree at most n, thereby enabling the exact use of the integration matrix H_s. For non-affine discontinuity curves, the method can be extended by replacing the operational matrix with numerical quadrature, although this would increase the computational cost. Second, the method is restricted to linear problems. Third, the error analysis in Theorem 2 requires smooth kernel pieces $k_l(x,t) \in C^{2n+2}(I_l \times I_l)$ for $l = 1,2,\ldots,n$. Fourth, the method is developed for the one-dimensional case. Fifth, the initial condition is restricted to $g(0) = 0$. Extension to more general curves, nonlinear problems, multidimensional settings, and the development of an adaptive version of the method are subjects of ongoing research.

Ethical approval

This study is purely theoretical and computational in nature and does not involve any experiments with human participants or animals. Therefore, institutional ethical approval was not required.

Availability of supporting data

All results presented in this paper are derived analytically or through self-contained numerical computations described within the manuscript. No external datasets were generated or utilized in support of the findings.

Conflict of interest

The authors declare that they have no competing financial or personal interest that could potentially influence the scholarly content, methodology, or conclusions of this work.

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References

[1] J.B.J. Fourier, *Théorie analytique de la chaleur*, Firmin Didot, Paris, 1822.

[2] V. Volterra, *Leçons sur les équations intégrales et les équations intégro-différentielles: Leçons professées à la Faculté des sciences de Rome en 1910*, Gauthier-Villars, Paris, 1913.

[3] Y.S. Choi, R. Lui, *An integro-differential equation arising from an electrochemistry model*, Quart. Appl. Math. **4** (1997) 677-686.

[4] J.A. Cummings, A.D. Fitt, M.J. S. Mphaka, A. Nagamine, *A singular integro-differential equation model for dryout in LMFBR boiler tubes*, IMA J. Appl. Math. **75** (2009) 269-290.

[5] C.M. Cushing, *Integro-differential Equations and Delay Models in Populations Dynamics*, in: Lecture Notes in Biomathematics, vol. 20, Springer, New York, 1977.

[6] H. Thieme, *A model for the spatial spread of an epidemic*, J. Math. Biol. **4** (1977) 337-351.

[7] X.Q. Zhao, *Dynamical Systems in Population Biology*, CMS Books in Mathematics, vol. 16, Springer, 2003.

[8] I. Muftahov, A. Tynda, D. Sidorov, *Volterra integral equations of the first kind with discontinuous kernels*, J. Comput. Appl. Math. **313** (2017) 119-128.

[9] E.V. Markova, D.N. Sidorov, *Volterra integral equations of the first kind with piecewise continuous kernels in the theory of evolving systems modeling*, IIGU Ser. Mat. **5(2)** (2012) 31-45.

[10] I.R. Muftahov, D.N. Sidorov, *Solvability and numerical solution of systems of nonlinear Volterra integral equations of the first kind with piecewise continuous kernels*, Vestn. YuUrGU Ser. Mat. Model. Program. **9(1)** (2016) 130-136.

[11] S. Noeiaghdam, S. Micula, *A novel method for solving second kind Volterra integral equations with discontinuous kernel*, Mathematics **9** (2021) 2172.

[12] S. Noeiaghdam, A. Dreglea, J.H. He, Z. Avazzadeh, M. Suleman, M.A. Fariborzi Araghi, D. Sidorov, N. Sidorov, *Error estimation of the homotopy perturbation method to solve second kind Volterra integral equations with piecewise smooth kernels, application of the CADNA library*, Symmetry **12(10)** (2020) 1-16.

[13] S. Noeiaghdam, D. Sidorov, A.M. Wazwaz, V. Sizikov, *The numerical validation of the Adomian decomposition method for solving Volterra integral equation with discontinuous kernel using the CESTAC method*, Mathematics **9(3)** (2021) 1-15.

[14] G.C. Evans, *Integral equation of the second kind with discontinuous kernel*, Trans. Am. Math. Soc. **11** (1910) 393-413.

[15] P.M. Anselone, *Uniform approximation theory for integral equations with discontinuous kernels*, SIAM J. Numer. Anal. **4** (1967) 245-253.

[16] A.M. Denisov, A. Lorenzi, *On a special Volterra integral equation of the first kind*, Boll. U.M.I. **7** (1995) 443-457.

[17] D. N. Sidorov, *On parametric families of solutions of Volterra integral equations of the first kind with piecewise smooth kernel*, Differ. Equ. **49** (2013) 210-216.

[18] E.V. Markova, D. N. Sidorov, *On one integral Volterra model of developing dynamical systems*, Autom. Remote Control **75** (2014) 413-421.

[19] D.N. Sidorov, *Solvability of systems of Volterra integral equations of the first kind with piecewise continuous kernels*, Russ. Math. **57** (2013) 54-63.

[20] N. Hritonenko, Y. Yatsenko, *Mathematical Modeling in Economics, Ecology and the Environment*, 2nd ed., Springer, Berlin, 2013.

[21] N. Hritonenko, Y. Yatsenko, *Nonlinear integral models with delays: Recent developments and applications*, J. King Saud Univ. Sci. **32** (2020) 726-731.

[22] D.V. Ivanov, O.V. Khamisov, I.V. Karaulova, E.V. Markova, V.V. Trufanov, *Control of power grid development: Numerical solutions*, Autom. Remote Control **65** (2004) 472-482.

[23] M.N. Botoroeva, M.V. Bulatov, *Applications and methods for the numerical solution of a class of integral-algebraic equations with variable limits of integration*, Bull. Irkutsk State Univ. **20** (2017) 3-16.

[24] D. Sidorov, I. Muftahov, N. Tomin, D. Karamov, D. Panasetsky, A. Dreglea, F. Liu, *A dynamic analysis of energy storage with renewable and diesel generation using Volterra equations*, IEEE Trans. Ind. Inform. **16(1)** (2020) 55-64.

[25] S. Noeiaghdam, D. Sidorov, A.M. Wazwaz, N. Sidorov, N. Sizikov, *The numerical validation of the Adomian decomposition method for solving Volterra integral equation with discontinuous kernel using the CESTAC method*, Mathematics **9(260)** (2021).

[26] E. Hashemizadeh, M.A. Ebadi, S. Noeiaghdam, *Matrix method by Genocchi polynomials for solving nonlinear Volterra integral equations with weakly singular kernel*, Symmetry **12** (2020) 2105.

[27] S. Noeiaghdam, D. Sidorov, V. Sizikov, N. Sidorov, *Control of accuracy on Taylor-collocation method to solve the weakly regular Volterra integral equations of the first kind by using the CESTAC method*, Appl. Comput. Math. Int. J. **19** (2020) 81-105.

[28] S. Noeiaghdam, M.A. Fariborzi Araghi, *Homotopy regularization method to solve the singular Volterra integral equations of the first kind*, Jordan J. Math. Stat. **11** (2018) 1-12.

[29] S. Noeiaghdam, D. Sidorov, A. Dreglea, *A novel numerical optimality technique to find the optimal results of Volterra integral equation of the second kind with discontinuous kernel*, Appl. Numer. Math. **186** (2023) 202-212.

[30] F. Mirzaee, *Solving a class of nonlinear two-dimensional Volterra integral equations by using two-dimensional triangular orthogonal functions*, J. Math. Model. **1(1)** (2013) 28-40.

[31] C. Canuto, M.Y. Hussaini, A. Quarteroni, T.A. Zang, *Spectral Methods: Fundamentals in Single Domains*, Springer, 2006.

[32] K. Maleknejad, S. Sohrabi, *Legendre polynomial solutions of nonlinear Volterra-Fredholm integral equations*, IUST Int. J. Eng. Sci. **19** (2008) 49-52.

[33] A. Saadatmandi, M. Dehghan, *A new operational matrix for solving fractional-order differential equations*, Comput. Math. Appl. **59(3)** (2010) 1326-1336.

[34] D.D. Stancu, *The remainder of certain linear approximation formulas in two variables*, J. SIAM Numer. Anal. Ser. B **1(1)** (1964) 137-163.