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Positive periodic solutions for a class of iterative Nicholson-type equations with nonlinear iterative harvesting effects

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Abstract. This paper is devoted to the study of the existence and uniqueness of positive periodic solutions for a generalized Nicholson-type differential equation with iterative arguments and nonlinear harvesting effects. By means of the Green's function corresponding to the associated linear periodic problem, the equation is reformulated as an equivalent integral equation. Subsequently, an appropriate operator is constructed in a Banach space of periodic functions. Applying Krasnoselskii's fixed point theorem and Banach's contraction principle, along with suitable analytical estimates, we derive sufficient conditions for the existence and uniqueness of positive periodic solutions. The results obtained generalize and improve several known contributions concerning Nicholson-type equations with delays and iterative terms. Finally, an example is presented to illustrate the effectiveness of the theoretical results.

Keywords: Nicholson equation, iterative differential equation, positive periodic solution, nonlinear harvesting, Krasnoselskii's fixed point theorem
AMS Subject Classification 2010: 34K13, 34K14, 34K18, 47H10.

1 Introduction

Nicholson's blowflies equation is one of the most widely studied mathematical models in population dynamics. Originally introduced to describe the evolution of the Australian sheep-blowfly population, it incorporates two important biological mechanisms: delayed maturation and nonlinear reproduction. Since delayed effects naturally arise in numerous biological, ecological, physiological, and engineering processes, Nicholson-type equations have become a central topic in the theory of functional differential equations.

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Over the past several decades, considerable attention has been devoted to the qualitative analysis of Nicholson-type models. Fundamental questions concerning the existence, uniqueness, periodicity, almost periodicity, oscillation, persistence, and stability of solutions have been extensively investigated. A variety of analytical techniques have been employed, including Lyapunov functional, coincidence degree theory, monotone iterative methods, semigroup theory, and fixed point techniques. Among these approaches, fixed point theory has proven to be particularly effective in establishing the existence and multiplicity of positive solutions.

In 2012, Zhao et al. [26], investigated the delayed Nicholson's blowflies model with harvesting effects

$$x'(t)=\sum_{i=1}^{n}P_i(t)\,x(t-\tau_i)\,e^{-x(t-\tau_i)}-\delta(t)\,x(t)-H(t,x(t)),$$

and established sufficient conditions for the existence of positive periodic solutions by applying Krasnoselskii's fixed point theorem.

Subsequently, Liu and Ding [20] studied the nonautonomous Nicholson equation with time-varying delays

$$x'(t)=-\alpha(t)\,x(t)-\sum_{j=1}^{n}\beta_j(t)\,x(t-\tau_j(t))\,e^{-\gamma_j(t)x\left(t-\tau_j(t)\right)},$$

and obtained existence and local exponential stability results for positive almost periodic solutions through a combination of fixed point arguments and Lyapunov functional techniques. Related existence results for positive almost periodic solutions of time-delayed biological models can also be found in Prasad et al. [24].

In another contribution, Liu and his coauthors [18] considered a diffusive Nicholson model subject to Dirichlet boundary conditions and proved the existence of nontrivial nonnegative periodic solutions by means of Schauder's fixed point theorem. Later, Li et al. [19] extended these investigations to Nicholson models with patch structure and multiple time-varying delays on almost periodic time scales, deriving existence and global exponential stability criteria for positive almost periodic solutions.

More recently, Zhao and his collaborators [27] analyzed Nicholson-type equations with nonlinear decimation terms in periodic environments. By employing Krasnoselskii's fixed point theorem, they established the existence of multiple positive periodic solutions and highlighted the complex dynamical behavior generated by nonlinear harvesting mechanisms. Furthermore, Bai and Li [2] investigated generalized Nicholson equations involving iterative arguments and delay effects, proving the existence and uniqueness of positive almost periodic solutions by means of Banach's contraction principle.

Alongside the rapid development of Nicholson-type models, iterative differential equations have also attracted considerable attention in recent years. Fixed point techniques have played a fundamental role in both areas, providing powerful tools for establishing the existence, uniqueness, periodicity, and stability of solutions. A substantial body of literature is now available on Nicholson-type equations as well as on differential equations involving iterative arguments. For representative contributions and recent developments in these directions, we refer the reader to [1, 3, 11, 21, 23] and [4–6, 8–10, 12–15, 17, 22].

Several authors have recently focused on Nicholson-type equations containing iterative terms. In particular, Bouakkaz and Khemis [7] investigated the Nicholson model involving higher-order iterates and nonlinear harvesting effects

$$z'(t) = -\alpha(t) z(t) + \beta(t) z(t-\tau) e^{-\gamma(t) z(t-\tau)}$$

$$-qz(t-\tau) E\left(t, z(t), z^{[2]}(t), \ldots, z^{[n]}(t)\right),$$

where $z^{[2]}(t) = z(z(t)), \ldots, z^{[n]}(t) = z^{[n-1]}(z(t))$. By applying Krasnoselskii's fixed point theorem, the author established the existence of positive periodic solutions. Later, Khemis [16] studied the iterative Nicholson equation with variable coefficients and delay terms

$$z'(t) = -\alpha(t) z(t) + \beta(t) z(t-\tau(t)) e^{-\gamma(t) z^{[2]}(t)},$$

and obtained new existence results for positive periodic solutions.

The purpose of the present paper is to investigate the existence and uniqueness of positive periodic solutions for the following generalized Nicholson-type differential equation involving iterative arguments and nonlinear harvesting effects:

$$z'(t) = -\delta_1(t) z(t) + \delta_2(t) z(t-\tau(t)) e^{-\delta_3(t) z^{[2]}(t)} - \delta_4(t) z(t-\tau(t)) E\left(t, z(t), z^{[2]}(t)\right), \tag{1}$$

where $\delta_i \in \mathscr{C}([0,w], (0,+\infty))$, $i = 1, \ldots, 4$ are $w-$periodic functions, $\tau \in C(\mathbb{R}, [0,+\infty))$ is a $w-$periodic function satisfying $\tau(t) \geq 0$ for all $t \in \mathbb{R}$, so that $s - \tau(s) \in \mathbb{R}$ for every s, $z^{[2]}(t) = z(z(t))$ denotes the second iterate of the unknown function and the harvesting function $E \in \mathscr{C}\left([0,w] \times \mathbb{R}_+^2, (0,+\infty)\right)$ is periodic with respect to its first variable and satisfies the following Lipschitz condition:

$$|E(t,x_1,x_2) - E(t,y_1,y_2)| \leq \mu_1 |x_1 - y_1| + \mu_2 |x_2 - y_2|, \ \mu_1, \mu_2 > 0. \tag{2}$$

The biological motivation for the iterative arguments lies in the modeling of delayed feedback effects and complex life cycles. In many ecological systems, current population dynamics depend not only on the current population density but also on past population levels that influence current behavior or spatial distribution. The iterative term $z^{[2]}(t) = z(z(t))$ should not be interpreted as a conventional temporal delay. Rather, it represents a successive state-dependent feedback mechanism, in which the current population state $z(t)$ influences an intermediate feedback state that is subsequently incorporated through the composition $z(z(t))$. This formulation may be relevant to populations with complex life cycles or successive biological interactions.

The Lipschitz condition (2) on the harvesting function is biologically reasonable, as harvesting functions in practice are typically smooth (or at least Lipschitz) with respect to population density. It implies that small changes in population density lead to proportionally small changes in harvesting pressure.

Moreover, it is assumed that the production term $\delta_2(t) z(t-\tau(t))$ dominates the harvesting term $\delta_4(t) z(t-\tau(t)) E\left(t, z(t), z^{[2]}(t)\right)$, so that the net growth contribution remains positive. This assumption, which will be stated rigorously as Assumption **(H)** in Section 2, is essential for the existence of positive solutions.

Despite the substantial progress achieved in recent years, an important gap still remains in the literature. Specifically, Zhao et al. [26] considered linear harvesting without iterative terms; Liu and Ding [20] studied time-varying delays but without harvesting or iterative arguments; Bai and Li [2] incorporated iterative terms but did not consider harvesting; Bouakkaz and Khemis [7] included nonlinear harvesting and higher-order iterates but with fixed delays only; and Khemis [16] allowed time-varying delays and iterative terms but without harvesting. In contrast, our model combines all three features simultaneously: iterative arguments, nonlinear harvesting effects, and time-varying periodic coefficients. The main novelty of this work lies precisely in this combination, which has not been investigated in the existing literature.

More precisely, our approach differs from previous studies in several fundamental ways that require new analytical estimates and cannot be directly adapted from existing work. First, the nonlinear harvesting term, which depends on the iterative argument, requires new estimates for the associated nonlinear operator, as discussed in Section 3. Second, the presence of the functional composition $z^{[2]}(t) = z(z(t))$ requires a dedicated Lipschitz estimate, which is essential for the contraction argument. Third, proving that the composite operator defined in Section 3 maps the admissible set $\mathbb{M}$, defined in Section 2, into itself requires additional estimates that account for the interaction between the iterative argument and the nonlinear harvesting term. Fourth, the sufficient conditions for uniqueness are derived from these new estimates and are therefore specific to the proposed model.

The interaction among these components leads to a richer mathematical structure and requires the development of an appropriate analytical framework. By constructing a suitable Green-function-based operator and employing Krasnoselskii's fixed point theorem in a carefully chosen Banach space, we derive sufficient conditions ensuring the existence of positive periodic solutions. Moreover, uniqueness is established by applying the Banach fixed point theorem.

Although each of the above mechanisms has been investigated separately, their combined influence on the periodic dynamics of Nicholson-type equations has not yet received adequate attention. This gap motivates the present study. Consequently, the present work contributes to the growing theory of iterative functional differential equations and provides new insights into the qualitative behavior of nonlinear population models with memory and feedback effects.

The remainder of the paper is organized as follows. Section 2 presents the preliminary concepts, definitions, and auxiliary results required throughout the paper. Section 3 contains the main existence and uniqueness results together with their proofs. Section 4 provides an illustrative example demonstrating the applicability of the theoretical findings. Finally, Section 5 presents a concluding discussion and outlines possible directions for future research.

2 Mathematical background

Let
$$\mathbb{X} = \{z \in \mathscr{C}(\mathbb{R},\mathbb{R}),\ z(t+w) = z(t),\ \forall t \in \mathbb{R}\},$$
be the Banach space of continuous $w-$periodic functions from $\mathbb{R}$ to $\mathbb{R}$, equipped with the supremum norm
$$\|z\| = \sup_{t\in[0,w]} |z(t)|.$$
Next, define the subset
$$\mathbb{M} = \{z \in \mathbb{X} :\ 0 \le z(t) \le \ell_1,\ |z(t_2) - z(t_1)| \le \ell_2 |t_2 - t_1|, \forall t, t_1, t_2 \in \mathbb{R}\},$$
where $\ell_1, \ell_2 > 0$ are given constants. The condition $0 \le z(t) \le \ell_1$ implies that $\mathbb{M}$ is uniformly bounded, while the condition $|z(t_2) - z(t_1)| \le \ell_2 |t_2 - t_1|$ ensures that $\mathbb{M}$ is equicontinuous. Therefore, by the

Arzelà–Ascoli theorem, $\mathbb{M}$ is relatively compact in $\mathbb{X}$. Since $\mathbb{M}$ is closed, it follows that $\mathbb{M}$ is a nonempty, compact, and convex subset of $\mathbb{X}$.
In the sequel, we use the following notation:

$$\begin{aligned}
&\underline{\delta_i} = \min_{t\in[0,w]} |\delta_i(t)|,\ \overline{\delta_i} = \max_{t\in[0,w]} |\delta_i(t)|,\ i=\overline{1,4},\ \overline{E} = \max_{t\in[0,w]} E(t,0,0),\\
&\rho = \int_0^w \delta_1(\vartheta)\,d\vartheta,\ \underline{g} = \frac{e^{-\rho}}{e^{\rho}-1},\ \overline{g} = \frac{e^{\rho}}{e^{\rho}-1},\\
&\mu_3 = \ell_1(\mu_1+\mu_2)+\overline{E},\ \mu_4 = w\overline{g}\,\overline{\delta_2}\left(1+\overline{\delta_3}\ell_1(1+\ell_2)\right),\\
&\mu_5 = w\overline{g}\,\overline{\delta_4}\left(\mu_3+\ell_1(\mu_1+\mu_2(1+\ell_2))\right)\\
&\mu_6 = w\overline{g}\,\overline{\delta_2},\ \mu_7 = \overline{g}\ell_1\left(\overline{\delta_2}+\mu_3\overline{\delta_4}\right)\left(w\overline{\delta_1}+2\right),\\
&\mu_8 = \mu_4+\mu_5.
\end{aligned}$$

Throughout this paper, we assume the following hypothesis
(H)

$$\min_{t\in\mathbb{R},\ z\in\mathbb{M}}\left\{\delta_2(t)\,z(t-\tau(t))\,e^{-\delta_3(t)z^{[2]}(t)} - \delta_4(t)\,z(t-\tau(t))\,E\left(t,z(t),z^{[2]}(t)\right)\right\} > 0.$$

Remark 1. *In [16], we find the following inequalities*

$$\left\|z_1^{[2]} - z_2^{[2]}\right\| \le (1+\ell_2)\,\|z_1-z_2\|, \tag{3}$$

and

$$\left|z_1(s-\tau(s))\,e^{-\delta_3(s)z_1^{[2]}(s)} - z_2(s-\tau(s))\,e^{-\delta_3(s)z_2^{[2]}(s)}\right| \le \left(1+\overline{\delta_3}\ell_1(1+\ell_2)\right)\|z_1-z_2\|, \tag{4}$$

for all $z_1,z_2\in\mathbb{M}$. Indeed, for $z_1,z_2\in\mathbb{M}$, we have

$$\begin{aligned}
&\left|z_1(s-\tau(s))\,e^{-\delta_3(s)z_1^{[2]}(s)} - z_2(s-\tau(s))\,e^{-\delta_3(s)z_2^{[2]}(s)}\right|\\
&\le z_1(s-\tau(s))\left|e^{-\delta_3(s)z_1^{[2]}(s)} - e^{-\delta_3(s)z_2^{[2]}(s)}\right|\\
&+e^{-\delta_3(s)z_2^{[2]}(s)}\left|z_1(s-\tau(s)) - z_2(s-\tau(s))\right|.
\end{aligned}$$

By the mean value theorem applied to $f(z)=e^{-z}$, there exists $\zeta(s)$ between $\delta_3(s)\,z_1^{[2]}(s)$ and $\delta_3(s)\,z_2^{[2]}(s)$ such that

$$\left|e^{-\delta_3(s)z_1^{[2]}(s)} - e^{-\delta_3(s)z_2^{[2]}(s)}\right| \le e^{-\zeta(s)}\delta_3(s)\left|z_1^{[2]}(s) - z_2^{[2]}(s)\right|.$$

Since $e^{-\zeta(s)}\le 1$ and $\overline{\delta_3} = \max_{t\in[0,w]}|\delta_3(t)|$, it follows that

$$\left|e^{-\delta_3(s)z_1^{[2]}(s)} - e^{-\delta_3(s)z_2^{[2]}(s)}\right| \le \overline{\delta_3}\left|z_1^{[2]}(s) - z_2^{[2]}(s)\right|.$$

Moreover, since $0\le z_1(s-\tau(s))\le \ell_1$ and $e^{-\delta_3(s)z_2^{[2]}(s)}\le 1$, using (3), we obtain

$$\begin{aligned}
\left|z_1(s-\tau(s))\,e^{-\delta_3(s)z_1^{[2]}(s)} - z_2(s-\tau(s))\,e^{-\delta_3(s)z_2^{[2]}(s)}\right| &\le \overline{\delta_3}\ell_1(1+\ell_2)\|z_1-z_2\| + \|z_1-z_2\|\\
&= \left(1+\overline{\delta_3}\ell_1(1+\ell_2)\right).
\end{aligned}$$

Thus, (4) follows.

Remark 2. *It follows from (2) and (3) that*

$$E\left(t, z(t), z^{[2]}(t)\right) \leq \mu_3. \tag{5}$$

Lemma 1. *([25]) It holds that*

$$\mathbb{M} = \{z \in \mathbb{X},\ 0 \leq z(t) \leq \ell_1,\ |z(t_2) - z(t_1)| \leq \ell_2 |t_2 - t_1|, \forall t, t_1, t_2 \in \mathbb{R} \in [0, w]\}.$$

3 Main findings

The equivalence stated in the following lemma follows directly by applying the classical Green's function associated with the linear operator $\mathscr{L}z = z'(t) + \delta_1(t) z(t)$ under periodic boundary conditions.

Lemma 2. *A function $z \in \mathbb{M} \cap \mathscr{C}^1(\mathbb{R}, \mathbb{R})$ is a solution of equation (1) if and only if it satisfies the following integral equation:*

$$\begin{aligned} z(t) = & \int_t^{t+w} \delta_2(s) z(s - \tau(s)) e^{-\delta_3(s) z^{[2]}(s)} G(t,s)\, ds \\ & - \int_t^{t+w} \delta_4(s) z(s - \tau(s)) E\left(s, z(s), z^{[2]}(s)\right) G(t,s)\, ds, \end{aligned} \tag{6}$$

where

$$G(t,s) = \frac{\exp\left(\int_t^s \delta_1(\vartheta)\, d\vartheta\right)}{e^\rho - 1}, \text{ for } s \in [t, t+w]. \tag{7}$$

Proof. Let

$$F(t) = \delta_2(t) z(t - \tau(t)) e^{-\delta_3(t) z^{[2]}(t)} - \delta_4(t) z(t - \tau(t)) E(t, z(t), z^{[2]}(t)).$$

Then equation (1) can be rewritten as

$$z'(t) + \delta_1(t) z(t) = F(t).$$

Suppose first that $z \in \mathbb{M} \cap \mathscr{C}^1(\mathbb{R}, \mathbb{R})$ is a solution of the above equation. Multiplying both sides by

$$\exp\left(\int_t^s \delta_1(\vartheta))\, d\vartheta\right),$$

and integrating from t to $t+w$, we obtain

$$z(t+w) \exp\left(\int_t^{t+w} \delta_1(\vartheta)\, d\vartheta\right) - z(t) = \int_t^{t+w} F(s) \exp\left(\int_t^s \delta_1(\vartheta))\, d\vartheta\right) ds.$$

Since z and δ_1 are $w-$periodic and

$$\int_0^w \delta_1(\vartheta)\, d\vartheta = \rho,$$

we have

$$z(t+w) = z(t), \quad \int_t^{t+w} \delta_1(\vartheta)\, d\vartheta = \rho.$$

Consequently,

$$z(t)\left(e^{\rho}-1\right)=\int_t^{t+w}F(s)\exp\left(\int_t^s \delta_1(\vartheta))d\vartheta\right)ds.$$

Thus,

$$z(t)=\int_t^{t+w}F(s)G(t,s)\,ds,$$

where

$$G(t,s)=\frac{\exp\left(\int_t^s \delta_1(\vartheta)\,d\vartheta\right)}{e^{\rho}-1},\ s\in[t,t+w].$$

Substituting the definition of F, we obtain

$$\begin{aligned}z(t)=&\int_t^{t+w}\delta_2(s)\,z(s-\tau(s))\,e^{-\delta_3(s)z^{[2]}(s)}G(t,s)\,ds\\&-\int_t^{t+w}\delta_4(s)\,z(s-\tau(s))\,E\left(s,z(s),z^{[2]}(s)\right)G(t,s)\,ds,\end{aligned}$$

which is exactly (6).

Conversely, suppose that $z\in\mathbb{M}$ satisfies (6). Differentiating (6) with respect to t, using Leibniz's rule and

$$\frac{\partial G(t,s)}{\partial t}=-\delta_1(t)G(t,s)),$$

together with the w-periodicity of F, yields

$$z'(t)+\delta_1(t)\,z(t)=F(t).$$

Substituting the definition of $F(t)$, we obtain

$$z'(t)=-\delta_1(t)\,z(t)+\delta_2(t)\,z(t-\tau(t))\,e^{-\delta_3(t)z^{[2]}(t)}-\delta_4(t)\,z(t-\tau(t))\,E\left(t,z(t),z^{[2]}(t)\right),$$

which is equation (1). Hence, (1) and (6) are equivalent. □

Remark 3. *The function G satisfies the periodicity property*

$$G(w+t,w+s)=G(t,s),\forall s\in[t,t+w],\ t\in\mathbb{R}. \tag{8}$$

Moreover, G is bounded between two positive constants:

$$0<\underline{g}\le G(t,s)\le\overline{g}. \tag{9}$$

Furthermore, the following estimate holds:

$$\int_{t_1}^{t_1+w}|G(t_2,s)-G(t_1,s)|\,ds\le w\overline{\delta_1}\,\overline{g}\,|t_2-t_1|, \tag{10}$$

for all $t_2,t_1\in\mathbb{R}$.

In view of Lemma 2, we introduce two integral operators $\mathscr{S}_1, \mathscr{S}_2 : \mathbb{M} \to \mathbb{X}$ defined by

$$(\mathscr{S}_1 z)(t) = \int_t^{t+w} G(t,s)\,\delta_2(s)\,z(s-\tau(s))\,e^{-\delta_3(s)z^{[2]}(s)}ds, \tag{11}$$

and

$$(\mathscr{S}_2 z)(t) = -\int_t^{t+w} \delta_4(s)\,z(s-\tau(s))\,E\left(s, z(s), z^{[2]}(s)\right) G(t,s)\,ds. \tag{12}$$

Consequently, the fixed points of the operator $\mathscr{S} = \mathscr{S}_1 + \mathscr{S}_2$ correspond exactly to the solutions of equation (1) and vice versa.

Now, using Krasnoselskii's fixed point theorem, we establish the existence of positive periodic solutions.

Lemma 3. *$\mathscr{S}_1$ is a continuous and compact operator on $\mathbb{M}$.*

Proof. Using (8) and the periodicity of the functions δ_2 and δ_3, we obtain

$$(\mathscr{S}_1 z)(t+w) = (\mathscr{S}_1 z)(t),$$

for all $z \in \mathbb{M}$, and so $\mathscr{S}_1(\mathbb{M}) \subset \mathbb{X}$.
Now, let $z_1, z_2 \in \mathbb{M}$, we have

$$\begin{aligned}|(\mathscr{S}_1 z_1)(t) - (\mathscr{S}_1 z_2)(t)| \le \int_t^{t+w} \delta_2(s) \Big| z_1(s-\tau(s))\,e^{-\delta_3(s)z_1^{[2]}(s)} \\ -z_2(s-\tau(s))\,e^{-\delta_3(s)z_2^{[2]}(s)}\Big|\,G(t,s)\,ds.\end{aligned}$$

It follows from (4) and (9) that

$$\begin{aligned}\|\mathscr{S}_1 z_1 - \mathscr{S}_1 z_2\| &\le w\overline{g}\,\overline{\delta_2}\left(1+\overline{\delta_3}\ell_1(1+\ell_2)\right)\|z_1 - z_2\| \\ &= \mu_4 \|z_1 - z_2\|. \end{aligned} \tag{13}$$

Thus, $\mathscr{S}_1$ is a continuous operator as a μ_4-Lipschitz mapping.
By the Arzelà–Ascoli theorem, as justified in the definition of $\mathbb{M}$, $\mathbb{M}$ is a compact subset of $\mathbb{X}$. Since $\mathscr{S}_1$ is continuous, $\mathscr{S}_1(\mathbb{M})$ is compact in $\mathbb{X}$. Consequently, $\mathscr{S}_1$ is a compact operator on $\mathbb{M}$. □

Lemma 4. *If $\mu_5 < 1$, then $\mathscr{S}_2$ is a contraction mapping on $\mathbb{M}$.*

Proof. It is easy to show that $\mathscr{S}_2(\mathbb{M}) \subset \mathbb{X}$. Let $z_1, z_2 \in \mathbb{M}$. Then

$$\begin{aligned}|(\mathscr{S}_2 z_1)(t) - (\mathscr{S}_2 z_2)(t)| \le \int_t^{t+w} \delta_4(s) \Big| z_1(s-\tau(s))\,E\left(s, z_1(s), z_1^{[2]}(s)\right) \\ -z_2(s-\tau(s))\,E\left(s, z_2(s), z_2^{[2]}(s)\right)\Big|\,G(t,s)\,ds.\end{aligned}$$

We have

$$
\begin{aligned}
&\left|z_1(s-\tau(s))E\left(s,z_1(s),z_1^{[2]}(s)\right)-z_2(s-\tau(s))E\left(s,z_2(s),z_2^{[2]}(s)\right)\right|\\
&\le\left|z_1(s-\tau(s))E\left(s,z_1(s),z_1^{[2]}(s)\right)-z_2(s-\tau(s))E\left(s,z_1(s),z_1^{[2]}(s)\right)\right|\\
&+\left|z_2(s-\tau(s))E\left(s,z_1(s),z_1^{[2]}(s)\right)-z_2(s-\tau(s))E\left(s,z_2(s),z_2^{[2]}(s)\right)\right|\\
&\le E\left(s,z_1(s),z_1^{[2]}(s)\right)|z_1(s-\tau(s))-z_2(s-\tau(s))|\\
&+z_2(s-\tau(s))\left|E\left(s,z_1(s),z_1^{[2]}(s)\right)-E\left(s,z_2(s),z_2^{[2]}(s)\right)\right|.
\end{aligned}
$$

From (2), (3) and (5) we obtain

$$
\begin{aligned}
&\left|z_1(s-\tau(s))E\left(s,z_1(s),z_1^{[2]}(s)\right)-z_2(s-\tau(s))E\left(s,z_2(s),z_2^{[2]}(s)\right)\right|\\
&\le(\mu_3+\ell_1(\mu_1+\mu_2(1+\ell_2)))\|z_1-z_2\|,
\end{aligned}
$$

and hence

$$
\begin{aligned}
\|\mathscr{S}_2z_1-\mathscr{S}_2z_2\|&\le w\overline{g}\overline{\delta_4}(\mu_3+\ell_1(\mu_1+\mu_2(1+\ell_2)))\|z_1-z_2\|\\
&=\mu_5\|z_1-z_2\|.
\end{aligned}\tag{14}
$$

Since $\mu_5<1$, it follows that $\mathscr{S}_2$ is a contraction mapping on $\mathbb{M}$. □

Lemma 5. *Suppose that $\mu_6\le 1$ and $\mu_7\le\ell_2$. Then $\mathscr{S}_1z_1+\mathscr{S}_2z_2\in\mathbb{M}$ for all $z_1,z_2\in\mathbb{M}$.*

Proof. Let $z_1,z_2\in\mathbb{M}$.
By assumption (**H**), the production term dominates the harvesting term. Moreover, $G(t,s)>0$ by (9). Since the integral of a strictly positive function over an interval of positive length is strictly positive, we conclude

$$(\mathscr{S}_1z_1)(t)+(\mathscr{S}_2z_2)(t)>0.$$

We also have the upper bound:

$$
\begin{aligned}
(\mathscr{S}_1z_1)(t)+(\mathscr{S}_2z_2)(t)&\le\int_t^{t+w}G(t,s)\delta_2(s)z_1(s-\tau(s))e^{-\delta_3(s)z_1^{[2]}(s)}ds\\
&-\int_t^{t+w}\delta_4(s)z_2(s-\tau(s))E\left(s,z_2(s),z_2^{[2]}(s)\right)G(t,s)ds\\
&\le\int_t^{t+w}G(t,s)\delta_2(s)z_1(s-\tau(s))e^{-\delta_3(s)z_1^{[2]}(s)}ds\\
&\le\int_t^{t+w}G(t,s)\delta_2(s)z_1(s-\tau(s))ds.\\
&\le w\overline{g}\overline{\delta_2}\ell_1=\mu_6\ell_1.
\end{aligned}
$$

Since $\mu_6\le 1$, it follows that

$$0<(\mathscr{S}_1z_1)(t)+(\mathscr{S}_2z_2)(t)\le\ell_1.\tag{15}$$

Now, for $t_1, t_2 \in [0, w]$, we have

$$
\begin{aligned}
&\left|\left(\mathscr{S}_1 z_1\right)(t_2) + \left(\mathscr{S}_2 z_2\right)(t_2) - \left(\left(\mathscr{S}_1 z_1\right)(t_1) + \left(\mathscr{S}_2 z_2\right)(t_1)\right)\right| \\
&\leq \int_{t_2}^{t_1} G(t_2, s)\, \delta_2(s)\, z_1(s - \tau(s))\, e^{-\delta_3(s) z_1^{[2]}(s)} ds \\
&+ \int_{t_1+w}^{t_2+w} G(t_2, s)\, \delta_2(s)\, z_1(s - \tau(s))\, e^{-\delta_3(s) z_1^{[2]}(s)} ds \\
&+ \int_{t_1}^{t_1+w} \left|G(t_2, s) - G(t_1, s)\right| \delta_2(s)\, z_1(s - \tau(s))\, e^{-\delta_3(s) z_1^{[2]}(s)} ds \\
&+ \int_{t_2}^{t_1} \delta_4(s)\, z_2(s - \tau(s))\, E\left(s, z_2(s), z_2^{[2]}(s)\right) G(t_2, s)\, ds \\
&+ \int_{t_1+w}^{t_2+w} \delta_4(s)\, z_2(s - \tau(s))\, E\left(s, z_2(s), z_2^{[2]}(s)\right) G(t_2, s)\, ds \\
&+ \int_{t_1}^{t_1+w} \left|G(t_2, s) - G(t_1, s)\right| \delta_4(s)\, z_2(s - \tau(s))\, E\left(s, z_2(s), z_2^{[2]}(s)\right) ds.
\end{aligned}
$$

Using Lemma 1, (5), (9) and (10), we obtain

$$
\begin{aligned}
&\left|\left(\mathscr{S}_1 z_1\right)(t_2) + \left(\mathscr{S}_2 z_2\right)(t_2) - \left(\left(\mathscr{S}_1 z_1\right)(t_1) + \left(\mathscr{S}_2 z_2\right)(t_1)\right)\right| \\
&\leq \overline{g}\,\overline{\delta_2}\,\ell_1 \left(w\overline{\delta_1} + 2\right) |t_2 - t_1| + \overline{g}\mu_3 \overline{\delta_4}\,\ell_1 \left(w\overline{\delta_1} + 2\right) |t_2 - t_1| \\
&= \overline{g}\,\ell_1 \left(\overline{\delta_2} + \mu_3 \overline{\delta_4}\right) \left(w\overline{\delta_1} + 2\right) |t_2 - t_1| \\
&= \mu_7 |t_2 - t_1|
\end{aligned}
$$

Since $\mu_7 \leq \ell_2$, and in light of Lemma 1, we obtain

$$
\left|\left(\mathscr{S}_1 z_1\right)(t_2) + \left(\mathscr{S}_2 z_2\right)(t_2) - \left(\left(\mathscr{S}_1 z_1\right)(t_1) + \left(\mathscr{S}_2 z_2\right)(t_1)\right)\right| \leq \ell_2 |t_2 - t_1|, \tag{16}
$$

for all $t_1, t_2 \in \mathbb{R}$.
From (15) and (16), we conclude that $\mathscr{S}_1 z_1 + \mathscr{S}_2 z_2 \in \mathbb{M}$ for all $z_1, z_2 \in \mathbb{M}$. $\square$

Theorem 1. *Assume that $\mu_5 < 1$, $\mu_6 \leq 1$ and $\mu_7 \leq \ell_2$. Then equation (1) has at least one positive periodic solution in $\mathbb{M}$.*

Proof. In view of Lemmas 3, 4 and 5, we can apply Krasnoselskii's fixed point theorem to ensure that the operator $\mathscr{S} = \mathscr{S}_1 + \mathscr{S}_2$ has at least one fixed point in $\mathbb{M}$. By Lemma 2, the obtained fixed points are solutions of equation (1). $\square$

Theorem 2. *Suppose that all conditions of Theorem 1 hold. Furthermore, assume that $\mu_8 < 1$. Then equation (1) admits exactly one positive periodic solution belonging to $z \in \mathbb{M}$.*

Proof. Using the Lipschitz estimates from Lemmas 3 and 4 together with $\mu_8 = \mu_4 + \mu_5 < 1$, we conclude that

$$
\begin{aligned}
\left\|\mathscr{S} z_1 - \mathscr{S} z_2\right\| &\leq \mu_4 \left\|z_1 - z_2\right\| + \mu_5 \left\|z_1 - z_2\right\| \\
&= \mu_8 \left\|z_1 - z_2\right\|.
\end{aligned}
$$

Hence, $\mathscr{S} = \mathscr{S}_1 + \mathscr{S}_2$ is a contraction mapping on $\mathbb{M}$. Furthermore, for $z \in \mathbb{M}$, we have

$$(\mathscr{S}z)(t) = \int_t^{t+w} \left\{ \delta_2(s) z(s-\tau(s)) e^{-\delta_3(s) z^{[2]}(s)} - \delta_4(s) z(s-\tau(s)) E\left(s, z(s), z^{[2]}(s)\right) \right\} G(t,s)\, ds$$

By hypothesis **(H)** and relation (9) , we have

$$\min_{t\in\mathbb{R},\, z\in\mathbb{M}} \left\{ \delta_2(t) z(t-\tau(t)) e^{-\delta_3(t) z^{[2]}(t)} - \delta_4(t) z(t-\tau(t)) E\left(t, z(t), z^{[2]}(t)\right) \right\} > 0,$$

and $\underline{g} > 0$. Hence,

$$\begin{aligned}(\mathscr{S}z)(t) &\geq \min_{t\in\mathbb{R},\, z\in\mathbb{M}} \left\{ \delta_2(t) z(t-\tau(t)) e^{-\delta_3(t) z^{[2]}(t)} - \delta_4(t) z(t-\tau(t)) E\left(t, z(t), z^{[2]}(t)\right) \right\} \int_t^{t+w} G(t,s)\, ds \\ &\geq \min_{t\in\mathbb{R},\, z\in\mathbb{M}} \left\{ \delta_2(t) z(t-\tau(t)) e^{-\delta_3(t) z^{[2]}(t)} - \delta_4(t) z(t-\tau(t)) E\left(t, z(t), z^{[2]}(t)\right) \right\} w\underline{g}\end{aligned}$$

Therefore,

$$(\mathscr{S}z)(t) > \min_{t\in\mathbb{R},\, z\in\mathbb{M}} \left\{ \delta_2(t) z(t-\tau(t)) e^{-\delta_3(t) z^{[2]}(t)} - \delta_4(t) z(t-\tau(t)) E\left(t, z(t), z^{[2]}(t)\right) \right\} w\underline{g} > 0.$$

Thus, $\mathscr{S}$ is strictly positive on $\mathbb{M}$, and the trivial function $z = 0$ cannot be a fixed point of $\mathscr{S}$. Since $\mathscr{S}$ is a contraction mapping on $\mathbb{M}$, the Banach contraction mapping theorem guarantees the existence of a unique fixed point $z \in \mathbb{M}$. Consequently, the considered problem admits a unique positive $w-$periodic solution. □

Remark 4. *The assumptions imposed in Theorems 1 and 2 are sufficient but not necessary. We leave the investigation of weaker conditions ensuring the existence and uniqueness of positive periodic solutions to future work.*

4 Example

This section presents an illustrative example whose primary purpose is to verify that the hypotheses of the main theoretical results can be satisfied.

Example 1. Let $\ell_1 = 0.08$, $\ell_2 = 0.0052$, $w = 38$, and define the following periodic functions:

$$\begin{aligned}\delta_1(t) &= 0.01 + 0.01 \sin^2\left(\frac{\pi}{38} t\right), \\ \delta_2(t) &= 0.01 + 0.0014 \cos^2\left(\frac{\pi}{38} t\right), \\ \delta_3(t) &= 0.004 + 0.008 \cos^2\left(\frac{\pi}{38} t\right), \\ \delta_4(t) &= 0.003 + 0.003 \cos^2\left(\frac{\pi}{38} t\right), \\ E(t,x,y) &= 0.003 + 0.037x + 0.037y.\end{aligned}$$

The parameter values in this example are chosen to be biologically plausible and to satisfy the assumptions of Theorems 1 and 2. The period $w = 38$ corresponds approximately to 5-6 weeks, which is consistent with the developmental cycle of many insect populations, including the Australian sheep-blowfly (Lucilia cuprina), for which the classical Nicholson model was originally developed. Indeed, experimental studies on blowfly populations indicate that generation times typically range from 35 to 40 days, making $w = 38$ a biologically realistic choice. The periodic functions $\delta_1(t)$, $\delta_2(t)$, $\delta_3(t)$, and $\delta_4(t)$ represent seasonal variations in mortality, reproduction, competition, and harvesting rates, respectively. The harvesting function $E(t,x,y) = 0.003 + 0.37x + 0.37y$ models a density-dependent harvesting pressure, where the coefficients 0.37 reflect the intensity of harvesting in response to both current population density and the iterated population density. The small values of $\ell_1 = 0.08$ and $\ell_2 = 0.0052$ correspond to a population exhibiting small-amplitude periodic fluctuations, which is typical for stable ecological systems under moderate environmental forcing.
For these choices, we obtain the following bounds:

$$\begin{aligned}
&\underline{\delta_1} = 0.01,\ \overline{\delta_1} = 0.02,\ \underline{\delta_2} = 0.01,\ \overline{\delta_2} = 0.0114,\\
&\underline{\delta_3} = 0.004,\ \overline{\delta_3} = 0.012,\ \underline{\delta_4} = 0.003,\ \overline{\delta_4} = 0.006,\ \overline{E} = 0.003,\\
&\underline{g} \approx 0.7361,\ \overline{g} \approx 2.30163,\ \mu_1 = \mu_2 = 0.037,\ \mu_3 \approx 0.00892,\\
&\mu_4 \approx 0.99802,\ \mu_5 \approx 0.0077957 < 1,\\
&\mu_6 \approx 0.99706 < 1,\ \mu_7 \approx 5.8207 \times 10^{-3} \leq \ell_2.
\end{aligned}$$

Moreover, to ensure that the production term dominates the harvesting term, we assume $z(t) \geq 0.0014$, $\forall t \in [0,38]$. Then we compute

$$\min_{t\in[0,38]} z_1(s-\tau(s))\, e^{-\delta_3(s) z_1^{[2]}(s)} \approx 1.3987 \times 10^{-3}$$

and

$$\max_{t\in[0,38]} \delta_4(s)\, z(s-\tau(s))\, E\left(s, z(s), z^{[2]}(s)\right) \approx 4.2816 \times 10^{-6},$$

which proves that

$$\min_{t\in\mathbb{R},\, z\in\mathbb{M}} \left\{ (\delta_2(t))\,(z(t-\tau(t)))\, e^{-(\delta_3(t)) z^{[2]}(t)} - \delta_4(t)\, z(t-\tau(t))\, E\left(t, z(t), z^{[2]}(t)\right) \right\} \approx 1.3944 \times 10^{-3} > 0,$$

for all $z \in \mathbb{M}$, where

$$\mathbb{M} = \{ z \in \mathbb{X},\ 0.0014 \leq z(t) \leq 0.08,\ |z(t_2) - z(t_1)| \leq 0.0052\,|t_2 - t_1|\,, \forall t, t_1, t_2 \in \mathbb{R} \}.$$

Since all conditions of Theorem 1 are satisfied, equation (1) admits at least one positive periodic solution in $\mathbb{M}$. We have also

$$\mu_8 = \mu_4 + \mu_5 \approx 0.99802 + 0.0077957 \approx 1.00582 > 1.$$

Thus, the condition $\mu_8 < 1$ of Theorem 2 is not satisfied. Consequently, the obtained solution is not necessarily unique.
We now modify the functions δ_2 and δ_4 as follows:

$$\begin{aligned}
\delta_2(t) &= 0.0001 + 0.0001 \cos^2 \frac{\pi}{38} t,\\
\delta_4(t) &= 0.00003 + 0.00003 \cos^2 \frac{\pi}{38} t,
\end{aligned}$$

while keeping all other variables unchanged. This yields the following new values:

$$\underline{\delta_2} = 0.0001,\ \overline{\delta_2} = 0.0002, \underline{\delta_4} = 0.00003,\ \overline{\delta_4} = 0.00006,$$
$$\mu_4 \approx 0.017509,\ \mu_5 \approx 7.79569 \times 10^{-5} < 1,\ \mu_6 \approx 0.01749239 \le 1,$$
$$\mu_7 \approx 1.01912 \times 10^{-4} \le \ell_2 = 0.0052.$$

Moreovere

$$\min_{t\in[0,38]} z_1(s-\tau(s))\, e^{-\delta_3(s) z_1^{[2]}(s)} \approx 1.398657 \times 10^{-3},$$
$$\max_{t\in[0,38]} \delta_4(s)\, z(s-\tau(s))\, E\left(s, z(s), z^{[2]}(s)\right) \approx 4.2816 \times 10^{-8},$$

which proves that

$$\min_{t\in\mathbb{R},\, z\in\mathbb{M}} \left\{ (\delta_2(t))\,(z(t-\tau(t)))\, e^{-(\delta_3(t)) z^{[2]}(t)} - \delta_4(t)\, z(t-\tau(t))\, E\left(t, z(t), z^{[2]}(t)\right) \right\} \approx 1.39861 \times 10^{-3} > 0,$$

Thus,

$$\mu_8 = \mu_4 + \mu_5 = 0.017509 + 7.79569 \times 10^{-5} = 0.01758 < 1.$$

Therefore, the condition $\mu_8 < 1$ of Theorem 2 is satisfied. Consequently, the obtained solution is unique.

5 Conclusion

In this paper, we have established the existence and uniqueness of positive $w-$periodic solutions for a generalized Nicholson-type differential equation with iterative arguments and nonlinear harvesting effects. Using Krasnoselskii's fixed point theorem and Banach's contraction principle, we derived sufficient conditions for the existence and uniqueness of positive periodic solutions. The results generalize and improve several existing contributions by combining iterative arguments, nonlinear harvesting, and time-varying periodic coefficients in a unified framework. An illustrative example is provided to verify that the assumptions of the main theorems can be satisfied for a concrete model, thereby demonstrating the applicability of the theoretical framework.

The numerical simulation of iterative differential equations presents significant challenges due to the functional composition $z^{[2]}(t) = z(z(t))$, which introduces strong nonlinearities that complicate the implementation of standard numerical schemes. The development of efficient numerical methods for Nicholson-type models with nonlinear iterative arguments constitutes an important direction for future research.

Several promising directions for future research include stability and bifurcation analysis of the obtained solutions, extension to almost periodic and pseudo-almost periodic settings, incorporation of higher-order iterative arguments, combination with spatial diffusion, inclusion of stochastic perturbations, and relaxation of the Lipschitz condition on the harvesting function. Indeed, although the Lipschitz assumption (2) is mathematically convenient and biologically reasonable, as harvesting functions in practice are typically smooth with respect to population density, it could potentially be replaced by weaker regularity conditions, such as Caratheodory-type assumptions or monotonicity conditions, while still preserving the existence and uniqueness results through alternative fixed point arguments. Exploring such generalizations would extend the applicability of our approach to a wider class of harvesting functions.

Conflict of Interest

The authors declare that they have no conflict of interest.

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