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On balancing and Lucas-balancing numbers expressible as a product of two k-Lucas numbers

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Abstract. A positive integer n is called a balancing number if $1+2+\cdots+(n-1) = (n+1)+(n+2)+\cdots+(n+r)$ holds for some positive integer r. Then r is called a balancer corresponding to the balancing number n. It is well known that if n is a balancing number, then $8n^2+1$ is a perfect square, and its positive square root is called a Lucas-balancing number. For any integer $k \geq 2$, let $\{L_n^{(k)}\}_{n\geq -(k-2)}$ denote k-generalized Lucas sequence which starts with $0, \ldots, 2, 1$(k terms) where each next term is the sum of the k preceding terms. In this paper, we look at all the balancing and Lucas-balancing numbers that can be represented as the product of two k-Lucas numbers.

Keywords: k-Lucas numbers, Balancing numbers, Lucas-balancing numbers, Linear forms in logarithms, Reduction method.
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1 Introduction

A positive integer n is called a balancing number if

$$1+2+\cdots+(n-1) = (n+1)+(n+2)+\cdots+(n+r)$$

holds for some positive integer r. Then r is called a balancer corresponding to the balancing number n (see [4]). The balancing sequence $\{B_n\}_{n\geq 0}$ is defined by $B_0 = 0, B_1 = 1$, and

$$B_{n+1} = 6B_n - B_{n-1}, \quad \text{for all} \quad n \geq 1.$$

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The first few terms of the balancing sequence are

$$0, 1, 6, 35, 204, 1189, 6930, 40391, 235416, 1372105, 7997214, 46611179, \ldots$$

Corresponding to each balancing number n, the positive value of $\sqrt{8n^2+1}$ is known as a Lucas-balancing number which is denoted by C_n (see [13, 14]). Let $\{C_n\}_{n\geq 0}$ be the Lucas-balancing sequence following the same recursive pattern as the balancing sequence, but with the initial conditions $C_0 = 1, C_1 = 3$. Its first few terms are

$$1, 3, 17, 99, 577, 3363, 19601, 114243, 665857, 3880899, 22619537, 131836323, \ldots$$

The balancing and Lucas-balancing sequences are the sequences A001109 and A001541 respectively, in the online encyclopedia of integer sequences (OEIS). The characteristic equation for both of these sequences is $x^2 - 6x + 1 = 0$, which has roots $\gamma = 3 + 2\sqrt{2}$ and $\delta = 3 - 2\sqrt{2}$. With these values of γ and δ, the Binet formulas for the sequences of balancing and Lucas-balancing, numbers are respectively given by

$$B_n = \frac{\gamma^n - \delta^n}{4\sqrt{2}} \quad \text{and} \quad C_n = \frac{\gamma^n + \delta^n}{2} \quad \text{for all} \quad n \geq 1. \tag{1}$$

Using induction, it can be seen that the inequalities

$$\gamma^{n-1} \leq B_n \leq \gamma^n \qquad \text{and} \qquad \gamma^n \leq 2C_n \leq \gamma^{n+1} \quad \text{hold for all} \quad n \geq 1. \tag{2}$$

The Lucas sequence $\{L_n\}_{n\geq 0}$ is the binary recurrence sequence defined by

$$L_{n+2} = L_{n+1} + L_n \ \text{ for } n \geq 0$$

with initials $L_0 = 2$ and $L_1 = 1$.

Let $k \geq 2$ be an integer. We consider a generalization of the Lucas sequence $\{L_n^{(k)}\}_{n\geq -(k-2)}$ defined as

$$L_n^{(k)} = L_{n-1}^{(k)} + L_{n-2}^{(k)} + \cdots + L_{n-k}^{(k)} = \sum_{i=1}^{k} L_{n-i}^{(k)} \ \text{ for all } n \geq 2,$$

with the initial conditions $L_{-(k-2)}^{(k)} = L_{-(k-3)}^{(k)} = \cdots = L_{-1}^{(k)} = 0, L_0^{(k)} = 2$ and $L_1^{(k)} = 1$. This sequence $L_n^{(k)}$ is called the k-generalized Lucas sequence or the k-Lucas sequence. This generalization is a family of sequences, with each new choice of k producing a unique sequence. For example, if $k = 2$, we get $L_n^{(2)} = L_n$, the classical Lucas sequence.

Diophantine equations have a long history involving balancing numbers, Lucas-balancing numbers, and the k-Lucas sequence. In recent years, the study of k-Lucas sequences has attracted considerable attention from many researchers. One can go through [3, 5, 16, 19] for more details. Similarly, balancing and Lucas-balancing numbers have been studied from many perspectives; for example, see [15, 17]. In 2019, Ddamulira [8] searched for X-coordinates of Pell equations, which are products of two Lucas numbers. Then, Erduvan and Keskin in [10] determined all repdigits that are products of two Lucas numbers. In the same year, Erduvan et al. [11] tackled the problem of finding repdigits base b as products of two Lucas numbers. Recently, Adédji et al. [1] found Padovan and Perrin numbers, which are products of two generalized Lucas numbers. Motivated by the above works, the present work aims to determine all balancing and Lucas-balancing numbers that are products of two k-Lucas numbers. To be more precise, we show the following results.

Theorem 1. *All the solutions of the Diophantine equation*

$$B_l = L_m^{(k)} L_n^{(k)} \tag{3}$$

in positive integers l, m, n and k with $k \geq 2$ and $1 \leq m \leq n$ are

$$B_1 = L_1^{(k)} L_1^{(k)} \quad \text{for all} \quad k \geq 2, \quad B_2 = L_1^{(k)} L_3^{(k)} \quad \text{for all} \quad k \geq 3 \text{ and } B_3 = L_1^{(3)} L_6^{(3)}.$$

Theorem 2. *All the solutions of the Diophantine equation*

$$C_l = L_m^{(k)} L_n^{(k)} \tag{4}$$

in positive integers l, m, n and k with $k \geq 2$ and $1 \leq m \leq n$ are $C_1 = L_1^{(k)} L_2^{(k)}$ for all $k \geq 2$.

To show our main results, we proceed as follows: First, we rewrite equation (3) and (4) in two suitable ways in order to obtain two different forms in logarithms of algebraic numbers, which are both nonzero and small. Next, we use twice a lower bound on such nonzero linear forms in logarithms of algebraic numbers due to Matveev [12] to bound n polynomially in terms of k. When k is small, we use a reduction algorithm due to Dujella and Pethö [9] to reduce the upper bounds to a size that can be easily handled. When k is large, we use the fact that the dominant root of the k-Lucas sequence is exponentially close to 2. So, we use this estimation in our further calculation with linear forms in logarithms to obtain absolute upper bounds for n and k, which can be reduced by using Dujella and Pethö's result [9]. In this way, we complete the proof of our main results. Our proof relies on a few preliminary results, which are extensively discussed in the subsequent section.

2 A few preliminaries

This section presents the definitions, notation, properties, and results that will be used in the rest of this study.

2.1 Linear forms in logarithms

Let γ be an algebraic number of degree d with a minimal primitive polynomial

$$f(Y) := b_0 Y^d + b_1 Y^{d-1} + \cdots + b_d = b_0 \prod_{j=1}^{d} (Y - \gamma^{(j)}) \in \mathbb{Z}[Y],$$

where the b_j's are relatively prime integers, $b_0 > 0$, and the $\gamma^{(j)}$'s are conjugates of γ. Then the *logarithmic height* of γ is given by

$$h(\gamma) = \frac{1}{d} \left(\log b_0 + \sum_{j=1}^{d} \log \left(\max\{|\gamma^{(j)}|, 1\} \right) \right).$$

With the above notation, Matveev (see [12] or [7, Theorem 9.4]) proved the following result.

Theorem 3. *Let $\eta_1, \ldots, \eta_s$ be positive real algebraic numbers in a real algebraic number field $\mathbb{L}$ of degree $d_{\mathbb{L}}$. Let $a_1, \ldots, a_s$ be non-zero integers such that*

$$\Lambda := \eta_1^{a_1} \cdots \eta_s^{a_s} - 1 \neq 0.$$

Then

$$-\log|\Lambda| \leq 1.4 \cdot 30^{s+3} \cdot s^{4.5} \cdot d_{\mathbb{L}}^2 (1 + \log d_{\mathbb{L}})(1 + \log D) \cdot B_1 \cdots B_s,$$

where

$$D \geq \max\{|a_1|, \ldots, |a_s|\},$$

and

$$B_j \geq \max\{d_{\mathbb{L}} h(\eta_j), |\log \eta_j|, 0.16\}, \text{ for all } j = 1, \ldots, s.$$

2.2 The reduction method

Our next tool is a version of the reduction method of Baker and Davenport (see [2]). Here, we use a slight variant of the version given by Dujella and Pethö (see [9]). For a real number x, we write $||x||$ for the distance from x to the nearest integer.

Lemma 1. *Let M be a positive integer, p/q be a convergent of the continued fraction of the irrational τ such that $q > 6M$, and A, B, μ be some real numbers with $A > 0$ and $B > 1$. Furthermore, let*

$$\epsilon := ||\mu q|| - M \cdot ||\tau q||.$$

If $\epsilon > 0$, then there is no solution to the inequality

$$0 < |u\tau - v + \mu| < AB^{-w}$$

in positive integers u, v and w with

$$u \leq M \quad \text{and} \quad w \geq \frac{\log(Aq/\epsilon)}{\log B}.$$

2.3 Properties of k-Lucas sequence

The characteristic polynomial of the k- Lucas sequence is

$$\Psi_k(x) = x^k - x^{k-1} - \cdots - x - 1.$$

The above polynomial is irreducible over $\mathbb{Q}[x]$ and it has one positive real root that is $\varphi := \varphi(k)$ which is located between $2(1 - 2^{-k})$ and 2, lies outside the unit circle (see [22]). The other roots are firmly contained within the unit circle. To simplify the notation, we will omit the dependence on k of φ whenever no confusion may arise. The Binet formula for $L_n^{(k)}$ is

$$L_n^{(k)} = \sum_{i=1}^{k} (2\varphi_i - 1) f_k(\varphi_i) \varphi_i^{n-1} = \sum_{i=1}^{k} \frac{(2\varphi_i - 1)(\varphi_i - 1)}{2 + (k+1)(\varphi_i - 2)} \varphi_i^{n-1},$$

where $\varphi := \varphi_1, \varphi_2, \ldots, \varphi_k$ are the roots of the characteristic polynomial $\Psi_k(x)$. It was also shown in [6, Lemma 2] that the approximation

$$\left| L_n^{(k)} - (2\varphi - 1) f_k(\varphi)\varphi^{n-1} \right| < \frac{3}{2},$$

holds for all $n \geq 1$ and $k \geq 2$. So, for $n \geq 1$ and $k \geq 2$, we have

$$L_n^{(k)} = (2\varphi - 1) f_k(\varphi)\varphi^{n-1} + e_k(n), \quad \text{where} \quad |e_k(n)| \leq \frac{3}{2}. \tag{5}$$

Furthermore, it was shown by Bravo and Luca in [6, Lemma 2] that the inequality

$$\varphi^{n-1} \leq L_n^{(k)} \leq 2\varphi^n \text{ holds for all } n \geq 1 \text{ and } k \geq 2. \tag{6}$$

Lemma 2 ([18], Lemma 2.4))**.** *For every positive integer $n \geq 2$, we have*

$$L_n^{(k)} \leq 3 \cdot 2^{n-2} \textit{ holds for all } 2 \leq n \leq k+1.$$

Lemma 3 ([6], Lemma 2)**.** *Let $k \geq 2$ be an integer. Then we have*

$$\frac{1}{2} < f_k(\varphi) < \frac{3}{4} \quad \textit{and} \quad \left| f_k(\varphi^{(i)}) \right| < 1, \quad 2 \leq i \leq k.$$

So, the number $f_k(\varphi)$ is not an algebraic integer.

Furthermore, Bravo et al. in [6] showed that the logarithmic height of $f_k(\varphi)$ satisfies

$$h(f_k(\varphi)) < \log(k+1) + \log 4 \quad \text{for all} \quad k \geq 2. \tag{7}$$

Lemma 4 ([18], Lemma 2.6)**.** *For all $k \geq 2$ and $n \geq k+1$, we have*

$$L_n^{(k)} = 3 \cdot 2^{n-2}(1+\xi), \quad \textit{where} \quad |\xi| < \frac{1}{2^{k/2}}. \tag{8}$$

2.4 Other useful lemmas

We conclude this section by recalling two lemmas that we will need in this work.

Lemma 5 ([21], Lemma 2.2)**.** *Let $a, x \in \mathbb{R}$. If $0 < a < 1$ and $|x| < a$, then*

$$|\log(1+x)| < \frac{-\log(1-a)}{a} \cdot |x|$$

and

$$|x| < \frac{a}{1 - e^{-a}} \cdot |e^x - 1|.$$

Lemma 6 ([20], Lemma 7)**.** *If $m \geq 1$, $S \geq (4m^2)^m$ and $\frac{x}{(\log x)^m} < S$, then $x < 2^m S(\log S)^m$.*

3 Proof of Theorem 1

We begin our analysis of (3) for $1 \leq n \leq k+1$. In this case, it is known that $L_n^{(k)} = 3 \cdot 2^{n-2}$, thus the equation (3) transforms into

$$B_l = 3^2 \cdot 2^{m+n-4},$$

which has no solution in the range $1 \leq n \leq k+1$. From now on, we consider that $n \geq k+2$ and $k \geq 2$.

3.1 An upper bound for l versus n

Combining inequalities (2) and (6) with the equation (3), we have

$$\gamma^{l-1} \leq B_l = L_m^{(k)} L_n^{(k)} \leq 4\varphi^{m+n} < 4\varphi^{2n}.$$

This leads us to the inequality

$$l - 1 \leq \left(\frac{\log 4}{\log \gamma}\right) + 2n\left(\frac{\log \varphi}{\log \gamma}\right).$$

By using this and taking into account that $2(1-2^{-k}) < \varphi(k) < 2$ for all $k \geq 2$, it results

$$l < 0.8n + 1.79. \tag{9}$$

3.2 Bounding n in terms of k

In this subsection, we will bound n in terms of k. Namely, we will show the following lemma.

Lemma 7. *If (l, m, n, k) is an integer solution of (3) with $k \geq 2$ and $n \geq k+2$, then the inequalities*

$$n < 5.8 \times 10^{32} k^8 \log^5 k \tag{10}$$

hold.

Proof. Using Binet's formulas (1) and (5), we rewrite the equation (3) as

$$\frac{\gamma^l - \delta^l}{4\sqrt{2}} = \left((2\varphi - 1)f_k(\varphi)\varphi^{m-1} + e_k(m)\right)\left((2\varphi - 1)f_k(\varphi)\varphi^{n-1} + e_k(n)\right), \tag{11}$$

which implies

$$\frac{\gamma^l}{4\sqrt{2}} - (2\varphi-1)^2 f_k^2(\varphi)\varphi^{m+n-2} = (2\varphi-1)f_k(\varphi)\varphi^{n-1}e_k(m) + (2\varphi-1)f_k(\varphi)\varphi^{m-1}e_k(n)$$
$$+ e_k(n)e_k(m) + \frac{\delta^l}{4\sqrt{2}}.$$

Thus, we obtain

$$\left|\frac{\gamma^l}{4\sqrt{2}} - (2\varphi-1)^2 f_k^2(\varphi)\varphi^{m+n-2}\right| < \frac{3(2\varphi-1)f_k(\varphi)}{2}\varphi^{n-1} + \frac{3(2\varphi-1)f_k(\varphi)}{2}\varphi^{m-1} + \frac{9}{4} + \frac{|\delta|^l}{4\sqrt{2}}. \tag{12}$$

Dividing the above inequality by $(2\varphi-1)^2 f_k^2(\varphi)\varphi^{m+n-2}$ and using the fact $f_k(\varphi) > 1/2$, we get

$$\left|\gamma^l\varphi^{-(m+n-2)}\frac{(2\varphi-1)^{-2}f_k^{-2}(\varphi)}{4\sqrt{2}} - 1\right| < \frac{1.5}{\varphi^{m-1}} + \frac{1.5}{\varphi^{n-1}} + \frac{2.5}{\varphi^{m+n-2}} < \frac{5.5}{\varphi^{m-1}}. \tag{13}$$

Let

$$\Lambda_1 := \gamma^l\varphi^{-(m+n-2)}\frac{(2\varphi-1)^{-2}f_k^{-2}(\varphi)}{4\sqrt{2}} - 1. \tag{14}$$

From (13), we have

$$|\Lambda_1| < 5.5\cdot\varphi^{-(m-1)}. \tag{15}$$

Suppose that $\Lambda_1 = 0$, then we get

$$f_k^2(\varphi) = \frac{\gamma^l\varphi^{-(m+n-2)}}{4\sqrt{2}(2\varphi-1)^2},$$

and so $f_k^2(\varphi)$ is an algebraic integer, which is impossible. Hence $\Lambda_1 \neq 0$. Therefore, we apply Theorem 3 to get a lower bound for Λ_1 defined by (14) with the parameters:

$$\eta_1 := \gamma, \quad \eta_2 := \varphi, \quad \eta_3 := \frac{(2\varphi-1)^{-2}f_k^{-2}(\varphi)}{4\sqrt{2}},$$

and

$$a_1 := l, \quad a_2 := -(m+n-2), \quad a_3 := 1.$$

Note that the algebraic numbers η_1, η_2, η_3 belongs to the field $\mathbb{L} := \mathbb{Q}(\varphi, \sqrt{2})$, so we can assume $d_{\mathbb{L}} = [\mathbb{L} : \mathbb{Q}] \leq 2k$. Since $h(\eta_1) = (\log\gamma)/2$ and $h(\eta_2) = (\log\varphi)/k < (\log 2)/k$, it follows that

$$\max\{2kh(\eta_1), |\log\eta_1|, 0.16\} = k\log\gamma := B_1$$

and

$$\max\{2kh(\eta_2), |\log\eta_2|, 0.16\} = 2\log 2 := B_2.$$

Using the estimate (7) and the properties of logarithmic height, it follows that for all $k \geq 2$

$$\begin{aligned} h(\eta_3) &\leq 2h(2\varphi-1) + 2h(f_k(\varphi)) + h(4\sqrt{2}) \\ &< 2\log 3 + 2\left(\log(k+1) + \log 4\right) + \frac{\log 32}{2} \\ &< 12.9\log k. \end{aligned}$$

Thus, we obtain

$$\max\{2kh(\eta_3), |\log \eta_3|, 0.16\} = 25.8k \log k := B_3.$$

Finally, the fact that $m \leq n$ and the inequality (9) imply that we can take $D := 2n$. Therefore, according to Theorem 3, it comes that

$$\log |\Lambda_1| > -1.432 \times 10^{11} (2k)^2 (1 + \log 2k)(1 + \log 2n)(k \log \gamma)(2 \log 2)(25.8k \log k). \tag{16}$$

Comparing the lower bound (16) with the upper bound (15) of $|\Lambda_1|$ yields

$$(m-1) \log \varphi - \log 5.5 < 3.62 \times 10^{13} k^4 \log k (1 + \log 2k)(1 + \log 2n).$$

Using the facts $1 + \log 2k < 3.5 \log k$ for all $k \geq 2$ and $1 + \log 2n < 2.3 \log n$ for all $n \geq 4$, we conclude that

$$m \log \varphi < 2.95 \times 10^{14} k^4 \log^2 k \log n. \tag{17}$$

To apply Theorem 3 for a second time, we return to (3) and formulate it as

$$\frac{\gamma^l - \delta^l}{4\sqrt{2}} = \left((2\varphi - 1) f_k(\varphi) \varphi^{n-1} + e_k(n)\right) L_m^{(k)}.$$

From the above, it follows

$$\left| \frac{\gamma^l}{4\sqrt{2} L_m^{(k)}} - (2\varphi - 1) f_k(\varphi) \varphi^{n-1} \right| = \left| \frac{\delta^l}{4\sqrt{2} L_m^{(k)}} + e_k(n) \right| \leq 1.7. \tag{18}$$

If we divide through by $(2\varphi - 1) f_k(\varphi) \varphi^{n-1}$, we obtain

$$|\Lambda_2| \leq \frac{1.7}{(2\varphi - 1) f_k(\varphi) \varphi^{n-1}} < \frac{2.6}{\varphi^n}, \tag{19}$$

where

$$\Lambda_2 := \gamma^l \varphi^{-(n-1)} \frac{(2\varphi - 1)^{-1} f_k^{-1}(\varphi)}{4\sqrt{2} L_m^{(k)}} - 1. \tag{20}$$

In a similar manner used to show that $\Lambda_1 \neq 0$, one can verify that $\Lambda_2 \neq 0$. Now, let us apply Theorem 3 with

$$(\eta_1, a_1) := (\gamma, l), \; (\eta_2, a_2) := (\varphi, -(n-1)), \text{ and } (\eta_3, a_3) := \left(\frac{(2\varphi - 1)^{-1} f_k^{-1}(\varphi)}{4\sqrt{2} L_m^{(k)}}, 1 \right).$$

The number field containing η_1, η_2, η_3 is $\mathbb{L} := \mathbb{Q}(\varphi, \sqrt{2})$, which has degree $d_{\mathbb{L}} = [\mathbb{L} : \mathbb{Q}] = 2k$. As calculated before, we take

$$B_1 = k \log \gamma, \qquad B_2 = 2 \log 2.$$

However, we need to determine $h(\eta_3)$ and subsequently B_3. Using logarithmic properties and (17), we determine that for all $k \geq 2$,

$$h(\eta_3) \leq h\left(\frac{(2\varphi - 1)^{-1} f_k^{-1}(\varphi)}{4\sqrt{2} L_m^{(k)}} \right)$$

$$\begin{aligned}
&\leq h(2\varphi-1)+h(f_k(\varphi))+h(4\sqrt{2})+h(L_m^{(k)})\\
&< \log 3+\log(k+1)+\log 4+\frac{\log 32}{2}+\log 2+m\log\varphi\\
&< 2.96\times 10^{14}k^4\log^2 k\log n.
\end{aligned}$$

Thus, we conclude that

$$\max\{2kh(\eta_3), |\log\eta_3|, 0.16\} = 5.92\times 10^{14}k^5\log^2 k\log n := B_3.$$

Finally, by the inequalities (9) and $l < 0.8n+1.79 < 1.3n$ which hold for all $n\geq 4$, we can take $D := 1.3n$. Therefore, applying Theorem 3 and comparing the resulting inequality with (19), we obtain

$$n < 1.45\times 10^{28}k^8\log^3 k\log^2 n,$$

where we have used the facts $1+\log 2k < 3.5\log k$ and $1+\log(1.3n) < 2\log n$ for all $k\geq 2$ and $n\geq 4$. Therefore, we obtain

$$\frac{n}{\log^2 n} < 1.45\times 10^{28}k^8\log^3 k. \tag{21}$$

Thus, putting $S := 1.45\times 10^{28}k^8\log^3 k$ in (21) and using Lemma 6 with the fact $64.84+8\log k+3\log(\log k) < 100\log k$ for all $k\geq 2$, gives

$$\begin{aligned}
n &< 2^2\left(1.45\times 10^{28}k^8\log^3 k\right)\left(\log\left(1.45\times 10^{28}k^8\log^3 k\right)\right)^2\\
&< 2^2\left(1.45\times 10^{28}k^8\log^3 k\right)\left(64.84+8\log k+3\log(\log k)\right)^2\\
&< 5.8\times 10^{32}k^8\log^5 k.
\end{aligned}$$

This establishes (10) and completes the proof of Lemma 7. □

3.3 The case when $2\leq k\leq 600$

In this subsection, we will show the following result.

Lemma 8. *If (l,m,n,k) is an integer solution of (3) with $2\leq k\leq 600$ and $n\geq k+2$, then our variables are bounded as follows:*

$$n\leq 448 \quad \textit{and} \quad l\leq 360.$$

Proof. To apply Lemma 1, we define

$$\Gamma_1 := \log(\Lambda_1+1) = l\log\gamma-(m+n-2)\log\varphi+\log\left(\frac{(2\varphi-1)^{-2}f_k^{-2}(\varphi)}{4\sqrt{2}}\right). \tag{22}$$

Suppose that $m\geq 6$, then by (15), we have $|\Lambda_1| < 0.73$. Choosing $a := 0.73$, we obtain the inequality

$$|\Gamma_1| < \frac{-\log(1-0.73)}{0.73}\cdot\frac{5.5}{\varphi^{m-1}} < \frac{9.9}{\varphi^{m-1}}$$

by Lemma 5. Thus, it follows that

$$\left| l\log\gamma-(m+n-2)\log\varphi+\log\left(\frac{(2\varphi-1)^{-2}f_k^{-2}(\varphi)}{4\sqrt{2}}\right)\right|<\frac{9.9}{\varphi^{m-1}}.$$

Dividing the above inequality by $\log\varphi$, we get

$$\left| l\left(\frac{\log\gamma}{\log\varphi}\right)-(m+n)+2+\frac{\log\left(\frac{(2\varphi-1)^{-2}f_k^{-2}(\varphi)}{4\sqrt{2}}\right)}{\log\varphi}\right|<\frac{24.5}{\varphi^{m-1}}. \tag{23}$$

In order to apply Lemma 1, we set

$$\tau=\frac{\log\gamma}{\log\varphi},\quad \mu=2+\frac{\log\left(\frac{(2\varphi-1)^{-2}f_k^{-2}(\varphi)}{4\sqrt{2}}\right)}{\log\varphi},\quad A:=24.5\quad\text{and}\quad B:=\varphi.$$

We have $\tau\notin\mathbb{Q}$. Indeed, if we assume there exist coprime integers a and b such that $\tau=a/b$, then we get that $\varphi^a=\gamma^b$. Let $\sigma\in Gal(\mathbb{K}/\mathbb{Q})$, the Galois group of the extension $\mathbb{K}/\mathbb{Q}$, such that $\sigma(\varphi)=\varphi_i$, for some $i\in\{2,\dots,k\}$. Applying this to the above relation and taking absolute values we get $1<2^a=|\varphi_i|<1$, which is a contradiction. Moreover, we note that $M_k:=\lfloor 5.8\times10^{32}k^8\log^5 k\rfloor$, which is an upper bound of l from Lemma 7. For each $k\in[2,600]$, we find a good approximation of τ and a convergent p_l/q_l of the continued fraction of τ such that $q=q(k)>6M_k$ is a denominator of a convergent of the continued fraction of τ with $\epsilon=\epsilon(k):=\|\mu q\|-M_k\|\tau q\|>0$. A computation with *Mathematica* revealed that if $k\in[2,600]$, then the maximum value of $\log(Aq/\epsilon)/\log B$ is 431.767, which is according to Lemma 1, is an upper bound on $m-1$.

Now, we fix $1\le m\le 432$ and we consider

$$\Gamma_2:=\log(\Lambda_2+1)=l\log\gamma-(n-1)\log\varphi+\log\left(\frac{(2\varphi-1)^{-1}f_k^{-1}(\varphi)}{4\sqrt{2}L_m^{(k)}}\right). \tag{24}$$

It is easy to see that $|\Lambda_2|<0.78$ for $n\ge3$. As a result, using inequality (19) and Lemma 5, we obtain

$$|\Gamma_2|<\frac{-\log(1-0.78)}{0.78}\cdot\frac{2.6}{\varphi^n}<\frac{5.1}{\varphi^n}. \tag{25}$$

Replacing (25) by (24) and dividing through by $\log\varphi$, we obtain

$$\left| l\left(\frac{\log\gamma}{\log\varphi}\right)-(n-1)+\frac{\log\left(\frac{(2\varphi-1)^{-1}f_k^{-1}(\varphi)}{4\sqrt{2}L_m^{(k)}}\right)}{\log\varphi}\right|<\frac{12.6}{\varphi^n}. \tag{26}$$

To apply Lemma 1 to (26), this time for $1\le m\le 432$, we set

$$\tau=\frac{\log\gamma}{\log\varphi},\quad \mu=1+\frac{\log\left(\frac{(2\varphi-1)^{-1}f_k^{-1}(\varphi)}{4\sqrt{2}L_m^{(k)}}\right)}{\log\varphi},\quad A:=12.6\quad\text{and}\quad B:=\varphi.$$

Again, for $(k,l) \in [2,600] \times [1,432]$, we find a good approximation of τ and a convergent p_l/q_l of the continued fraction of τ such that $\epsilon = \epsilon(k) := \|\mu q\| - M_k\|\tau q\| > 0$, where $M_k := \lfloor 5.8 \times 10^{32} k^8 \log^5 k \rfloor$ which is an upper bound of l from Lemma 7. After doing this, we use Lemma 1 on inequality (26). Again, a program in *Mathematica* revealed that the maximum value of $\log(Aq/\epsilon)/\log B$ over all $(k,m) \in [2,600] \times [1,432]$ is 448.584, which according to Lemma 1, is an upper bound of n.

Hence, we deduce that the possible solutions (l,m,n,k) of (3) for which $k \in [2,600]$ satisfy $k+2 \le m \le n \le 448$. Therefore, we use inequality (9) to obtain $l \le 360$. □

3.4 The case when $k > 600$

In this subsection, our primary goal is to show the following lemmas that aim to prove that there does not exist any solution for the case when $k > 600$ and $n \ge k+2$.

Lemma 9. *If (l,m,n,k) is a solution of the Diophantine equation (3) with $k > 600$ and $n \ge k+2$, then k and n are bounded as*

$$k < 3.12 \times 10^{17} \quad \textit{and} \quad n < 5.6 \times 10^{180}.$$

Proof. For $k > 600$, the following inequalities hold

$$n < 5.8 \times 10^{32} k^8 \log^5 k < 2^{k/2}.$$

Hence, from Lemma 4, we have

$$L_m^{(k)} L_n^{(k)} = 9 \cdot 2^{m+n-4}(1+\xi)^2. \tag{27}$$

Inserting (27) and (1) in (3), we obtain

$$\frac{\gamma^l}{4\sqrt{2}} - 9 \cdot 2^{m+n-4} = 9 \cdot 2^{m+n-4}\left(2\xi + \xi^2\right) + \frac{\delta^l}{4\sqrt{2}}.$$

The above equality and the fact $\left(2\xi + \xi^2\right) < 3/2^{k/2}$ together with $n \ge m \ge k+2$ yields that

$$\left|\gamma^l 3^{-2} 2^{-\left(m+n-\frac{3}{2}\right)} - 1\right| < \frac{3}{2^{k/2}} + \frac{1}{2^{2k}} < \frac{4}{2^{k/2}}. \tag{28}$$

In order to use Theorem 3, we take

$$(\eta_1, a_1) := (\gamma, l), \qquad (\eta_2, a_2) := (3, -2), \qquad \text{and} \qquad (\eta_3, a_3) := \big(2, -(m+n-3/2)\big).$$

The number field containing η_1, η_2, η_3 is $\mathbb{L} := \mathbb{Q}(\sqrt{2})$, which has degree $d_{\mathbb{L}} = [\mathbb{L} : \mathbb{Q}] = 2$. Here

$$\Lambda_3 := \gamma^l 3^{-2} 2^{-\left(m+n-\frac{3}{2}\right)} - 1, \tag{29}$$

is nonzero. If $\Lambda_3 = 0$, then

$$3^2 \cdot 2^{m+n-\frac{3}{2}} = \gamma^l,$$

which is impossible because the left-hand side is rational, whereas it can be seen that the right-hand side is irrational. Therefore, $\Lambda_3 \neq 0$. Moreover, since

$$h(\eta_1) = \frac{\log\gamma}{2}, \qquad h(\eta_2) = \log 3, \qquad h(\eta_3) = \log 2.$$

Therefore, we may take

$$B_1 := \log\gamma, \quad B_2 := 2\log 3 \quad \text{and} \quad B_3 := 2\log 2.$$

Finally, the fact that $m \leq n$ and the inequality (9) imply that we can take $D := 2n$. Thus, taking into account inequality (28) and applying Theorem 3, we obtain

$$k < 3.76 \times 10^{13} \log n, \tag{30}$$

where we have used the fact that $1 + \log 2n < 2.3 \log n$ for $n \geq 4$. On the other hand, from Lemma 7 we get

$$\log n < \log(5.8 \times 10^{32} k^8 \log^5 k) < 75.44 + 8\log k + 5\log(\log k) < 115 \log k$$

for $k \geq 2$. So, from (30) we obtain

$$k < 4.33 \times 10^{15} \log k.$$

Further using Lemma 6 and Lemma 7 the above inequality leads to

$$k < 3.12 \times 10^{17} \quad \text{and} \quad n < 5.6 \times 10^{180}.$$

This completes the proof of Lemma 9. □

The bounds obtained for n and k from the above lemma are very large. Now, our next goal is to reduce this upper bound to a reasonable range for which we will prove the following lemma.

Lemma 10. *The equation* (3) *has no solutions for* $k > 600$ *and* $n \geq k + 2$.

Proof. Here, we attempt to reduce the upper bound on n and k. To do so, we first consider

$$\Gamma_3 := -\left(m + n - \frac{3}{2}\right)\log 2 + l\log\gamma - 2\log 3.$$

Since $k > 600$, then from (28) we have $|\Lambda_3| < 0.01$. Choosing $a := 0.01$, we obtain the inequality

$$|\Gamma_3| = |\log(\Lambda_3 + 1)| < \frac{-\log(1 - 0.01)}{0.01} \cdot \frac{4}{2^{k/2}} < \frac{4.1}{2^{k/2}}. \tag{31}$$

by Lemma 5. Thus, it follows that

$$\left|\left(m + n - \frac{3}{2}\right)\log 2 - l\log\gamma + \log 9\right| < \frac{4.1}{2^{k/2}}.$$

Dividing the above inequality by $\log\gamma$, we get

$$\left|\left(m+n-\frac{3}{2}\right)\frac{\log 2}{\log\gamma}-l+\frac{\log 9}{\log\gamma}\right|<\frac{2.33}{2^{k/2}}. \tag{32}$$

Applying Lemma 1 with parameters

$$\tau:=\frac{\log 2}{\log\gamma},\quad \mu:=\frac{\log 9}{\log\gamma},\quad A:=2.33\quad\text{and}\quad B:=2.$$

The fact that $m+n-\frac{3}{2}<2n$ and $n<5.6\times10^{180}$ imply that we can take $M:=1.12\times10^{181}$ by Lemma 1. We found that q_{339}, the denominator of the $339th$ convergent of τ exceeds $6M$. Furthermore, a quick computation with *Mathematica* gives us the value

$$\frac{\log(Aq_{339}/\epsilon)}{\log B}$$

is less than 613. So, if the inequality (32) has a solution, then

$$\frac{k}{2}<\frac{\log(Aq_{339}/\epsilon)}{\log B}<613,$$

which implies that $k\le 1226$.
With the above upper bound for k and by Lemma 7, we have

$$n<5.4\times10^{61}.$$

We continue as previously, but with $M:=1.1\times10^{62}$, we get that $k<432$, which contradicts our assumption $k>600$. This completes the proof of Lemma 10. □

3.5 The final step

According to Lemma 8 and 10, if (l,m,n,k) is a solution of the Diophantine equation (3) then

$$2\le k\le 600,\quad k+2\le n\le 448,\quad 1\le m\le n\quad\text{and}\quad 1\le l\le 360.$$

Using *Mathematica*, we verified that all the solutions of the Diophantine equation (3) are those listed in the statement of Theorem 1. This concludes the proof of Theorem 1.

4 Proof of Theorem 2

We begin our analysis of (4) for $1\le n\le k+1$. In this case, it is known that $L_n^{(k)}=3\cdot2^{n-2}$, thus the equation (4) transforms into

$$C_l=3^2\cdot2^{m+n-4},$$

which has no solution in the range $1\le n\le k+1$. From now on, we consider that $n\ge k+2$ and $k\ge2$.

4.1 An upper bound for l versus n

The inequalities (2) and (6) together with the equation (4), imply

$$\frac{\gamma^l}{2} \leq C_l = L_m^{(k)} L_n^{(k)} \leq 4\varphi^{m+n} < 4\varphi^{2n},$$

so we deduce that

$$l \leq \left(\frac{\log 8}{\log \gamma}\right) + 2n\left(\frac{\log \varphi}{\log \gamma}\right).$$

Using the fact that $2(1-2^{-k}) < \varphi(k) < 2$ for all $k \geq 2$, we obtain

$$l < 0.8n + 1.18. \tag{33}$$

4.2 Bounding n in terms of k

In this step, we will give an inequality for n in terms of k. More precisely, we will show the following lemma.

Lemma 11. *If (l, m, n, k) is an integer solution of (4) with $k \geq 2$ and $n \geq k+2$, then the inequalities*

$$n < 4.72 \times 10^{32} k^8 \log^5 k \tag{34}$$

hold.

Proof. We use identities (1) and (5) to express (4) into the form

$$\frac{\gamma^l + \delta^l}{2} = \big((2\varphi - 1) f_k(\varphi)\varphi^{m-1} + e_k(m)\big)\big((2\varphi - 1) f_k(\varphi)\varphi^{n-1} + e_k(n)\big), \tag{35}$$

which we rewrite as

$$\frac{\gamma^l}{2} - (2\varphi-1)^2 f_k^2(\varphi)\varphi^{m+n-2} = (2\varphi-1) f_k(\varphi)\varphi^{n-1} e_k(m) + (2\varphi-1) f_k(\varphi)\varphi^{m-1} e_k(n)$$
$$+ e_k(n)e_k(m) + \frac{\delta^l}{2}.$$

Therefore, we get

$$\left|\frac{\gamma^l}{2} - (2\varphi-1)^2 f_k^2(\varphi)\varphi^{m+n-2}\right| < \frac{3(2\varphi-1) f_k(\varphi)}{2}\varphi^{n-1} + \frac{3(2\varphi-1) f_k(\varphi)}{2}\varphi^{m-1} + \frac{9}{4} + \frac{|\delta|^l}{2}. \tag{36}$$

Dividing both sides by $(2\varphi-1)^2 f_k^2(\varphi)\varphi^{m+n-2}$ and using the fact $f_k(\varphi) > 1/2$, we obtain

$$|\Lambda_4| < \frac{1.5}{\varphi^{m-1}} + \frac{1.5}{\varphi^{n-1}} + \frac{2.75}{\varphi^{m+n-2}} < \frac{5.75}{\varphi^{m-1}}, \tag{37}$$

where

$$\Lambda_4 := \gamma^l \varphi^{-(m+n-2)} \frac{(2\varphi-1)^{-2} f_k^{-2}(\varphi)}{2} - 1. \tag{38}$$

To apply Theorem 3, we need to show that $\Lambda_4 \neq 0$. Indeed, $\Lambda_4 = 0$, implies

$$f_k^2(\varphi) = \frac{\gamma^l \varphi^{-(m+n-2)}}{2(2\varphi-1)^2},$$

and so $f_k^2(\varphi)$ is an algebraic integer, which is false. To apply Theorem 3, we set

$$\eta_1 := \gamma, \quad \eta_2 := \varphi, \quad \eta_3 := \frac{(2\varphi-1)^{-2} f_k^{-2}(\varphi)}{2},$$

and

$$a_1 := l, \quad a_2 := -(m+n-2), \quad a_3 := 1.$$

The number field containing η_1, η_2, η_3 is $\mathbb{L} := \mathbb{Q}(\varphi, \sqrt{2})$, which has degree $d_{\mathbb{L}} = [\mathbb{L} : \mathbb{Q}] = 2k$. As before, we have

$$h(\eta_1) = \frac{\log \gamma}{2} \quad \text{and} \quad h(\eta_2) = \frac{\log \varphi}{k} < \frac{\log 2}{k}.$$

Moreover

$$\max\{2kh(\eta_1), |\log \eta_1|, 0.16\} = k \log \gamma := B_1$$

and

$$\max\{2kh(\eta_2), |\log \eta_2|, 0.16\} = 2 \log 2 := B_2.$$

Using the properties of logarithmic height and the estimate (7), we obtain

$$\begin{aligned} h(\eta_3) &\leq 2h((2\varphi-1)) + 2h(f_k(\varphi)) + h(2) \\ &< 2\log 3 + 2\left(\log(k+1) + \log 4\right) + \log 2 \\ &< 11.4 \log k \end{aligned}$$

for $k \geq 2$. So, we can take

$$\max\{2kh(\eta_3), |\log \eta_3|, 0.16\} = 22.8k \log k := B_3.$$

Finally, the fact that $m \leq n$ and the inequality (33) implies that we can take $D := 2n$. Therefore, due to Theorem 3 one obtains

$$\log |\Lambda_4| > -1.432 \times 10^{11} (2k)^2 (1 + \log 2k)(1 + \log 2n)(k \log \gamma)(2 \log 2)(22.8k \log k). \tag{39}$$

Comparing the lower bound (39) with the upper bound (37) of $|\Lambda_4|$ yields

$$m \log \varphi < 2.6 \times 10^{14} k^4 \log^2 k \log n, \tag{40}$$

where we have used the facts $1 + \log 2k < 3.5 \log k$ for all $k \geq 2$ and $1 + \log 2n < 2.3 \log n$ for all $n \geq 4$.

To apply Theorem 3 the second time, we go back to (4) and we rewrite it as

$$\frac{\gamma^l + \delta^l}{2} = \left((2\varphi-1) f_k(\varphi) \varphi^{n-1} + e_k(n)\right) L_m^{(k)},$$

which yields

$$\left|\frac{\gamma^l}{2L_m^{(k)}} - (2\varphi-1)f_k(\varphi)\varphi^{n-1}\right| = \left|\frac{\delta^l}{2L_m^{(k)}} + e_k(n)\right| \leq 2. \tag{41}$$

If we divide through by $(2\varphi-1)f_k(\varphi)\varphi^{n-1}$, we obtain

$$|\Lambda_5| \leq \frac{2}{(2\varphi-1)f_k(\varphi)\varphi^{n-1}} < \frac{3}{\varphi^n}, \tag{42}$$

where

$$\Lambda_5 := \gamma^l \varphi^{-(n-1)} \frac{(2\varphi-1)^{-1} f_k^{-1}(\varphi)}{2L_m^{(k)}} - 1. \tag{43}$$

One can see that $\Lambda_5 \neq 0$ by a similar method used to show that $\Lambda_4 \neq 0$. We apply Theorem 3 to Λ_5 defined by (43). To do this, we take

$$(\eta_1, a_1) := (\gamma, l)\,,\ (\eta_2, a_2) := (\varphi, -(n-1))\,,\ \text{and } (\eta_3, a_3) := \left(\frac{(2\varphi-1)^{-1} f_k^{-1}(\varphi)}{2L_m^{(k)}}, 1\right).$$

Clearly $\mathbb{L} := \mathbb{Q}(\varphi, \sqrt{2})$ contains η_1, η_2, η_3 and $d_{\mathbb{L}} \leq 2k$. As calculated before, we can take $B_1 = k\log\gamma$ and $B_2 = 2\log 2$. We need to compute B_3. Using the estimates (6), (40) and properties of logarithmic height, we have for all $k \geq 2$,

$$\begin{aligned}
h(\eta_3) &\leq h\left(\frac{(2\varphi-1)^{-1} f_k^{-1}(\varphi)}{2L_m^{(k)}}\right) \\
&\leq h((2\varphi-1)) + h(f_k(\varphi)) + h(2) + h(L_m^{(k)}) \\
&< \log 3 + \log(k+1) + \log 4 + 2\log 2 + m\log\varphi \\
&< 2.7 \times 10^{14} k^4 \log^2 k \log n.
\end{aligned}$$

Thus, we deduce that

$$\max\{2kh(\eta_3), |\log\eta_3|, 0.16\} = 5.4 \times 10^{14} k^5 \log^2 k \log n := B_3.$$

The inequality (33) and the fact $l < 0.8n + 1.18 < 1.1n$, which hold for all $n \geq 4$, allow us to select $D := 1.1n$. Therefore, applying Theorem 3 and comparing the resulting inequality with (42), we obtain

$$\frac{n}{\log^2 n} < 1.18 \times 10^{28} k^8 \log^3 k, \tag{44}$$

where we have used that $1 + \log 2k < 3.5\log k$ for all $k \geq 2$ and $1 + \log(1.1n) < 1.8\log n$ for all $n \geq 4$. Thus, putting $S := 1.18 \times 10^{28} k^8 \log^3 k$ in (44) and using Lemma 6 together with the fact $64.63 + 8\log k + 3\log(\log k) < 100\log k$ which holds for all $k \geq 2$, implies

$$\begin{aligned}
n &< 2^2\left(1.18 \times 10^{28} k^8 \log^3 k\right)\left(\log\left(1.18 \times 10^{28} k^8 \log^3 k\right)\right)^2 \\
&< 2^2\left(1.18 \times 10^{28} k^8 \log^3 k\right)\left(64.63 + 8\log k + 3\log(\log k)\right)^2 \\
&< 4.72 \times 10^{32} k^8 \log^5 k.
\end{aligned}$$

This establishes (34) and completes the proof of Lemma 11. $\square$

4.3 The case when $2 \leq k \leq 630$

In this subsection, we will show the following result.

Lemma 12. *If (l, m, n, k) is an integer solution of (4) with $2 \leq k \leq 630$ and $n \geq k+2$, then our variables are bounded as follows:*

$$n \leq 428 \quad \textit{and} \quad l \leq 343.$$

Proof. Let us define

$$\Gamma_4 := \log(\Lambda_4 + 1) = l \log \gamma - (m+n-2) \log \varphi + \log \left(\frac{(2\varphi - 1)^{-2} f_k^{-2}(\varphi)}{2} \right). \tag{45}$$

Assuming $m \geq 6$, from (37), we have $|\Lambda_4| < 0.76$. Choosing $a := 0.76$, we obtain the inequality

$$|\Gamma_4| < \frac{-\log(1-0.76)}{0.76} \cdot \frac{5.75}{\varphi^{m-1}} < \frac{10.8}{\varphi^{m-1}}$$

by Lemma 5. Thus, it follows that

$$\left| l \log \gamma - (m+n-2) \log \varphi + \log \left(\frac{(2\varphi - 1)^{-2} f_k^{-2}(\varphi)}{2} \right) \right| < \frac{10.8}{\varphi^{m-1}}.$$

Dividing the above inequality by $\log \varphi$, we get

$$\left| l \left(\frac{\log \gamma}{\log \varphi} \right) - (m+n) + 2 + \frac{\log \left(\frac{(2\varphi-1)^{-2} f_k^{-2}(\varphi)}{2} \right)}{\log \varphi} \right| < \frac{26.65}{\varphi^{m-1}}. \tag{46}$$

We apply Lemma 1 with the parameters

$$\tau = \left(\frac{\log \gamma}{\log \varphi} \right), \quad \mu = 2 + \frac{\log \left(\frac{(2\varphi-1)^{-2} f_k^{-2}(\varphi)}{2} \right)}{\log \varphi}, \quad A := 26.65 \quad \text{and} \quad B := \varphi.$$

As seen before $\tau \notin \mathbb{Q}$. We take $M_k := \lfloor 4.72 \times 10^{32} k^8 \log^5 k \rfloor$, which is an upper bound of l from Lemma 11. For each $k \in [2, 630]$, we find a good approximation of τ and a convergent p_l/q_l of the continued fraction of τ such that $q_l = q(k) > 6M_k$ is a denominator of a convergent of the continued fraction of τ with $\epsilon = \epsilon(k) := \|\mu q\| - M_k \|\tau q\| > 0$. A computer search with *Mathematica* revealed that if $k \in [2, 630]$, then the maximum value of $\log(Aq/\epsilon)/\log B$ is 412.739, which is according to Lemma 1, is an upper bound on $m-1$.

Now, we fix $1 \leq m \leq 413$ and we consider

$$\Gamma_5 := \log(\Lambda_5 + 1) = l \log \gamma - (n-1) \log \varphi + \log \left(\frac{(2\varphi - 1)^{-1} f_k^{-1}(\varphi)}{2L_m^{(k)}} \right). \tag{47}$$

Suppose that $n \geq 3$. Then from (42), we have $|\Lambda_5| < 0.89$. Choosing $a := 0.89$ and using Lemma 5, we obtain

$$|\Gamma_5| < \frac{-\log(1-0.89)}{0.89} \cdot \frac{3}{\varphi^n} < \frac{7.5}{\varphi^n}. \tag{48}$$

Replacing (48) by (47) and dividing across by $\log\varphi$, we obtain

$$\left| l\left(\frac{\log\gamma}{\log\varphi}\right) - (n-1) + \frac{\log\left(\frac{(2\varphi-1)^{-1}f_k^{-1}(\varphi)}{2L_m^{(k)}}\right)}{\log\varphi} \right| < \frac{18.5}{\varphi^n}. \tag{49}$$

To apply Lemma 1 to (49), this time for $1 \leq m \leq 413$, we set

$$\tau = \frac{\log\gamma}{\log\varphi}, \quad \mu = 1 + \frac{\log\left(\frac{(2\varphi-1)^{-1}f_k^{-1}(\varphi)}{2L_m^{(k)}}\right)}{\log\varphi}, \quad A := 18.5 \quad \text{and} \quad B := \varphi.$$

Again, for $(k,m) \in [2,630] \times [1,413]$, we find a good approximation of τ and a convergent p_l/q_l of the continued fraction of τ such that $\epsilon = \epsilon(k) := \|\mu q\| - M_k\|\tau q\| > 0$, where $M_k := \lfloor 4.72 \times 10^{32} k^8 \log^5 k \rfloor$ which is an upper bound of l from Lemma 11. After doing this, we use Lemma 1 on inequality (49). Again, a program in *Mathematica* revealed that the maximum value of $\log(Aq/\epsilon)/\log B$ over all $(k,m) \in [2,630] \times [1,413]$ is 428.947, which according to Lemma 1, is an upper bound of n.

Hence, we deduce that the possible solutions (l,m,n,k) of (4) for which $k \in [2,630]$ satisfy $k+2 \leq m \leq n \leq 428$. Therefore, we use inequality (33) to obtain $l \leq 343$. □

4.4 The case when $k > 630$

This subsection aims to establish that there are no solutions for the case where $k > 630$ and $n \geq k+2$.

Lemma 13. *If (l,m,n,k) is a solution of the Diophantine equation (4) with $k > 630$ and $n \geq k+2$, then k and n are bounded as*

$$k < 3.1 \times 10^{17} \quad \text{and} \quad n < 4.27 \times 10^{180}.$$

Proof. For $k \geq 630$, it is easy to see that

$$n < 4.72 \times 10^{32} k^8 \log^5 k < 2^{k/2}.$$

Inserting (1) and (8) in (4), we obtain

$$\frac{\gamma^l}{2} - 9 \cdot 2^{m+n-4} = 9 \cdot 2^{m+n-4}\left(2\xi + \xi^2\right) - \frac{\delta^l}{2}.$$

As $n \geq m \geq k+2$, this and the fact $\left(2\xi + \xi^2\right) < 3/2^{k/2}$ yield

$$|\Lambda_6| = \left|\gamma^l 3^{-2} 2^{-(m+n-3)} - 1\right| < \frac{3}{2^{k/2}} + \frac{1}{2^{2k}} < \frac{4}{2^{k/2}}. \tag{50}$$

But Λ_6 is not zero. Indeed, if Λ_6 were zero, we would then get that $\gamma^l = 3^2 \cdot 2^{m+n-3} \in \mathbb{Z}$, which is impossible. Therefore, we can apply Theorem 3 with

$$(\eta_1, a_1) := (\gamma, l), \qquad (\eta_2, a_2) := (3, -2), \quad \text{and} \quad (\eta_3, a_3) := (2, -(m+n-3)).$$

Since η_1, η_2, η_3 are elements of the field $\mathbb{L} := \mathbb{Q}(\sqrt{2})$, then $d_{\mathbb{L}} = [\mathbb{L} : \mathbb{Q}] = 2$. The fact that $m \leq n$ and the inequality (33) implies that we can choose $D := 2n$ because $m + n - 3 < 2n$. On the other hand, since

$$h(\eta_1) = \frac{\log \gamma}{2}, \qquad h(\eta_2) = \log 3, \qquad h(\eta_3) = \log 2.$$

Thus, we can take

$$B_1 := \log \gamma, \quad B_2 := 2 \log 3 \quad \text{and} \quad B_3 := 2 \log 2.$$

Thus, considering inequality (50) and using Theorem 3, we get

$$k < 3.76 \times 10^{13} \log n, \tag{51}$$

where we have used the fact that $1 + \log 2n < 2.3 \log n$ for $n \geq 4$. In contrast, Lemma 11 leads to

$$\log n < \log(4.72 \times 10^{32} k^8 \log^5 k) < 75.23 + 8 \log k + 5 \log(\log k) < 114 \log k$$

for $k \geq 2$. Thus, from (51), we get

$$k < 4.3 \times 10^{15} \log k.$$

Further using Lemma 6 and Lemma 11 the above inequality leads to

$$k < 3.1 \times 10^{17} \quad \text{and} \quad n < 4.27 \times 10^{180}.$$

This finishes the proof of Lemma 13. □

The above lemma provides very large bounds for n and k. Our next goal is to minimize this upper bound to a suitable range, for which we shall prove the following lemma.

Lemma 14. *The equation (4) has no solutions for $k > 630$ and $n \geq k + 2$.*

Proof. Now, we lower the bound on n and k. We define

$$\Gamma_6 := -(m + n - 3) \log 2 + l \log \gamma - 2 \log 3.$$

Since $k > 630$, then from (50) we have $|\Lambda_6| < 0.01$. Therefore by Lemma 5, we have

$$|\Gamma_6| = |\log(\Lambda_6 + 1)| < \frac{-\log(1 - 0.01)}{0.01} \cdot \frac{4}{2^{k/2}} < \frac{4.1}{2^{k/2}}. \tag{52}$$

Thus, it follows that

$$|(m + n - 3) \log 2 - l \log \gamma + 2 \log 3| < \frac{4.1}{2^{k/2}}.$$

Dividing the above inequality by $\log \gamma$, we get

$$\left|(m+n-3)\frac{\log 2}{\log \gamma} - l + \frac{\log 9}{\log \gamma}\right| < \frac{2.33}{2^{k/2}}. \tag{53}$$

To apply Lemma 1, we set parameters

$$\tau := \frac{\log 2}{\log \gamma}, \quad \mu := \frac{\log 9}{\log \gamma}, \quad A := 2.33 \quad \text{and} \quad B := 2.$$

The fact $m+n-3 < 2n$ and $n := 4.27\times10^{180}$ imply that we can take $M := 8.54\times10^{180}$ by Lemma 1. Using *Mathematica*, we find that q_{340} satisfies the hypothesis of Lemma 1. Furthermore, according to Lemma 1, if the inequality (53) has a solution, then

$$\frac{k}{2} < \frac{\log(Aq_{340}/\epsilon)}{\log B} < 612,$$

which implies that $k \leq 1222$.
With this new upper bound on k and by Lemma 11, we have

$$n < 4.26 \times 10^{61}.$$

We repeat the same process with $M := 8.52 \times 10^{61}$. In this application, we obtain that $k < 432$, which contradicts our assumption $k > 630$. This completes the proof of Lemma 14. □

4.5 The final step

According to Lemma 12 and 14, if (l, m, n, k) is a solution of the Diophantine equation (4) then

$$2 \leq k \leq 630, \quad k+2 \leq n \leq 428, \quad 1 \leq m \leq n \quad \text{and} \quad 1 \leq l \leq 343.$$

Using *Mathematica*, we checked that all the solutions of the Diophantine equation (4) are those listed in the statement of Theorem 2. This completes the proof of Theorem 2.

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