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Dominions and varieties of commutative semigroups

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Abstract. In the seminal work of Isbell [16], he demonstrated that the property of commutativity is preserved under dominions. In particular, this implies that commutativity is also preserved under epimorphisms. In this paper, we show that the subvarieties $\mathcal{V}_1 = [pqr = pqr^n]$(for any integer $n > 2$) and $\mathcal{V}_2 = [p^{n_1}q^{n_2} = 0]$(for any integers $n_1,\ n_2 > 0$) of the variety of commutative semigroups are closed under dominions.

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1 Introduction

The broader investigation into which varieties remain closed under epis has been a subject of interest in semigroup theory, ring theory, and related fields [7]. For instance, Gardner demonstrated in [8] that certain identities weaker than commutativity fail to be preserved under epis in rings, even though the variety of commutative rings is closed under epis [6]. In semigroups, Higgins [12] established that identities where each side contains a repeated variable are not preserved under epis. Despite these advances, a comprehensive classification of all semigroup identities preserved under epis remains an unresolved challenge.

The preservation of commutativity under dominions was shown by Isbell [16] which in particular implies that commutativity is also preserved under epimorphisms. Khan [17–19] extended this result in several significant directions. He established that all subvarieties of the variety of commutative semigroups are closed under epimorphisms. He also proved that all permutation identities are preserved under epimorphisms and all subvarieties of variety of permutative semigroups are closed under epimorphisms. In this direction, Ahanger, Shah and Khan [2, 4, 5] extended these results to the varieties of commutative and permutative posemigroups. Also, Ahanger, Nabi and Shah [3] extended the classical work of Isbell, Higgins and Khan, leading to the discovery of several new saturated and closed varieties of semigroups. Furthermore,

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Khan [17] advanced the theory by showing that any non-trivial permutation identity, apart from commutativity, cannot be preserved under dominions. This leads to the conclusion that non-trivial permutative varieties, excluding commutative varieties, cannot be closed under dominions. Though the variety of commutative semigroups is closed under dominions but whether all the subvarieties of the variety of commutative semigroups are closed under dominions is still an open problem. Therefore, it is natural to inquire which subvarieties of the variety of commutative semigroups are closed under dominions.

In this paper, we partially answer this question by finding certain subvarieties of commutative semigroups closed under dominions. The problem is still open, so the authors are contributing partial results.

For further details on varieties of semigroups and posemigroups, the reader is referred to [1, 9–11, 13, 14].

2 Preliminaries

Consider two semigroups S and U, with the property that U is a subsemigroup of S. Consider an element d within the semigroup S. Following the framework proposed by Isbell [16], d is defined to be dominated by U if, for any pair of semigroup morphisms $\alpha,\ \beta : S \to T$, $u\alpha = u\beta$, for all $u \in U$ implies $d\alpha = d\beta$. All such elements of S that are dominated by U is called the *dominion* of U in S, denoted by $Dom(U, S)$. The condition $d_1d_2 \in Dom(U, S)$ holds for all $d_1, d_2 \in Dom(U, S)$. Therefore, $Dom(U, S)$ is a subsemigroup of S containing U. A subsemigroup U of a semigroup S is said to be closed in S if $Dom(U, S) = U$ and U is said to be dense or epimorphically embedded in S if $Dom(U, S) = S$. A semigroup U is classified as absolutely closed if U remains closed within every containing semigroup. If $Dom(U, S) \neq S$, then there exist at least two distinct morphisms on S that coincide on U. When this condition holds true for all semigroups that properly contain U, we refer to U as being saturated. A class $\mathcal{V}$ of semigroups is said to be a variety if it is closed under subsemigroups, morphic images and direct products. A class $\mathcal{V}$ of semigroups, satisfying a collection $[u_i = v_i, i \geq 1]$ of identities, defines a variety determined by these identities. The list of identities is called a presentation of $\mathcal{V}$, and we write $\mathcal{V} = [u_i = v_i, i \geq 1]$. We consider $u = 0$ (where u is a nonempty word) as a semigroup identity and it means the combination of the two identities $uy = u$ and $yu = u$, where y is a variable not occurring in the word u, Khan [17, Remark 4]. A variety $\mathcal{V}$ is absolutely closed or saturated if all of its members are so and it is closed if $Dom(U, S) = U$ for all $S, U \in \mathcal{V}$ with U as a subsemigroup of S.

A morphism $\gamma : S \to T$ in the category of all semigroups is called an *epimorphism* (*epi* for short) if, for all semigroups W and all morphisms $\alpha,\ \beta : T \to W$,

$$\gamma\alpha = \gamma\beta \implies \alpha = \beta.$$

It can be easily seen that every surjective morphism is an epimorphism in the category of semigroups and in the category of groups one has the converse result that every epimorphism is surjective. However, in the category of semigroups, there exist non-surjective epimorphisms. It is not hard to see that $\gamma : S \to T$ is an epi if and only if the inclusion map $i : S\gamma \to T$ is epi and the inclusion map $i : U \to S$ is epi if and only if $Dom(U, S) = S$.

An identity $u = v$ is said to be preserved under epis if the following holds: whenever a subsemigroup U of a semigroup S satisfies $u = v$ and such that $Dom(U, S) = S$, then S itself must satisfy $u = v$ and the identity $u = v$ is said to be preserved under dominions, if U satisfies $u = v$ implies $Dom(U, S)$ also satisfies $u = v$, where S is any semigroup containing U as a subsemigroup. Let $\mathcal{V}$ be the variety of semigroups such that whenever $U \in \mathcal{V}$ implies $Dom(U, S) \in \mathcal{V}$ for any semigroup S containing U as a subsemigroup, then we say $\mathcal{V}$ is closed under dominions.

The Isbell's Zigzag Theorem serves as a pivotal tool in the theory of semigroup dominions, offering a precise characterization of the elements of a semigroup S that reside within its dominion when considered in any containing semigroup. This theorem is critical in the analysis of dominion elements. Because of the foundational role the Isbell's Zigzag Theorem plays in the study of semigroup theory, particularly in the context of epimorphisms and dominions, it is included as a central result in texts like Howie's [15]. The proof of the theorem, drawn from Howie's work, underpins the structural understanding of semigroup dominions, providing essential insight into how certain semigroup elements interact within larger classes of semigroups. Equivalently the necessary and sufficient condition for an element of a semigroup to be in dominion is provided by the famous Isbell's Zigzag Theorem [16] and is as follows:

Theorem 1 ([16], Theorem 2.3)**.** *Let U be a subsemigroup of a semigroup S and $z \in S$. Then $z \in Dom(U, S)$ if and only if $z \in U$ or there exists a series of factorization of z as follows:*

$$\begin{aligned} z &= \beta_0\gamma_1, & \beta_0 &= \alpha_1\beta_1 \\ \beta_{2i-1}\gamma_i &= \beta_{2i}\gamma_{i+1}, & \alpha_i\beta_{2i} &= \alpha_{i+1}\beta_{2i+1} \ (i = 1,\ 2, \ldots,\ m-1) \\ \beta_{2m-1}\gamma_m &= \beta_{2m}, & \alpha_m\beta_{2m} &= z, \end{aligned} \tag{1}$$

where $\alpha_1,\ \alpha_2, \ldots,\ \alpha_m,\ \gamma_1,\ \gamma_2, \ldots,\ \gamma_m \in S$ and $\beta_0,\ \beta_1,\ \beta_2, \ldots,\ \beta_{2m} \in U$.
The above series of factorization of z is called zigzag of length m in S over U with value z and *spine* $\beta_0,\ \beta_1,\ \beta_2, \ldots,\ \beta_{2m}$. In whatever follows by zigzag equations we shall mean equations of the type (1).

Following results are also useful for our investigations.

Theorem 2 ([16], Corollary 2.5)**.** *If U is a commutative subsemigroup of a semigroup S, then $Dom(U, S)$ is also commutative.*

Theorem 3 ([20], Lemma 3.2)**.** *Let U be a commutative subsemigroup of a semigroup S. Let $z \in Dom(U, S) \setminus U$ and let (1) be the zigzag of length m over U with value z. Then for any positive integer k,*

$$z^k = \alpha_m^k \beta_{2m}^k.$$

3 Main Results

Lemma 1. *Let U be a commutative subsemigroup of a semigroup S. Let $z \in Dom(U, S) \setminus U$ and let (1) be the zigzag of length m over U with value z. Then,*

$$\prod_{k=0}^{m} \beta_{2k}^{n-1} = (\prod_{k=0}^{m} \beta_{2k}^{n-2})(\prod_{k=1}^{m} \beta_{2k-1})z,$$

where $n > 1$ is any integer.

Proof. For any integer $n > 1$ we have,

$$\begin{aligned}
\prod_{k=0}^{m} \beta_{2k}^{n-1} &= \alpha_1\beta_1\beta_0^{n-2}\prod_{k=1}^{m}\beta_{2k}^{n-1} \quad \text{(by zigzag equations (1))}\\
&= \alpha_1\beta_2(\prod_{k=0}^{1}\beta_{2k}^{n-2})(\prod_{k=2}^{m}\beta_{2k}^{n-1})\beta_1 \quad \text{(because } U \text{ is commutative)}\\
&= \alpha_2\beta_3(\prod_{k=0}^{1}\beta_{2k}^{n-2})(\prod_{k=2}^{m}\beta_{2k}^{n-1})\beta_1 \quad \text{(by zigzag equations (1))}\\
&= \alpha_2\beta_4(\prod_{k=0}^{2}\beta_{2k}^{n-2})(\prod_{k=3}^{m}\beta_{2k}^{n-1})(\prod_{k=1}^{2}\beta_{2k-1}) \quad \text{(because } U \text{ is commutative)}\\
&\;\vdots\\
&= \alpha_m\beta_{2m}(\prod_{k=0}^{m}\beta_{2k}^{n-2})(\prod_{k=1}^{m}\beta_{2k-1})\\
&= z(\prod_{k=0}^{m}\beta_{2k}^{n-2})(\prod_{k=1}^{m}\beta_{2k-1}) \quad \text{(by zigzag equations (1))}\\
&= (\prod_{k=0}^{m}\beta_{2k}^{n-2})(\prod_{k=1}^{m}\beta_{2k-1})z \quad \text{(by Theorem 2)},
\end{aligned}$$

as required. □

Lemma 2. *Let U be a commutative subsemigroup of a semigroup S. Let $z \in Dom(U,S) \setminus U$ and let (1) be the zigzag of length m over U with value z. Then,*

$$\prod_{k=0}^{m}\beta_{2k}^{n-1} = (\prod_{k=1}^{m}\beta_{2k-1}^{n-1})z^{n-1},$$

where $n > 1$ is any integer.

Proof. The proof follows by applying Lemma 1 $(n-1)$- times. □

Lemma 3. *Let U be a commutative subsemigroup of a semigroup S and U satisfies the identity*

$$pqr = pqr^n (\textit{for any integer } n > 2). \tag{2}$$

Let $z \in Dom(U,S) \setminus U$ and let (1) be the zigzag of length m over U with value z. Then

$$(\prod_{k=0}^{m}\beta_{2k}^{n-1})z = z^n.$$

Proof. We first prove that

$$(\prod_{k=0}^{m} \beta_{2k}^{n-1})z = \alpha_l \beta_{2l-1}^{n} (\prod_{k=l+1}^{m} \beta_{2k-1}^{n-1}) z^{n-1} \gamma_l \text{ for all } l = 1,\ 2, \dots,\ m-1. \tag{3}$$

To prove (3), we use induction on l. For $l = 1$ we have,

$$\begin{aligned}(\prod_{k=0}^{m} \beta_{2k}^{n-1})z &= (\prod_{k=1}^{m} \beta_{2k-1}^{n-1})z^{n} \quad \text{(by Lemma 2)}\\ &= (\prod_{k=1}^{m} \beta_{2k-1}^{n-1})z^{n-1}\beta_0\gamma_1 \quad \text{(by zigzag equations (1))}\\ &= \beta_0(\prod_{k=1}^{m} \beta_{2k-1}^{n-1})z^{n-1}\gamma_1 \quad \text{(by Theorem 2)}\\ &= \alpha_1\beta_1^{n}(\prod_{k=2}^{m} \beta_{2k-1}^{n-1})z^{n-1}\gamma_1 \quad \text{(by zigzag equations (1)).}\end{aligned}$$

Thus, (3) holds for $l = 1$. Assume inductively that (3) holds for $l = r$ ($\leq m - 2$). We show that it holds for $l = r + 1$. Now,

$$\begin{aligned}(\prod_{k=0}^{m} \beta_{2k}^{n-1})z &= \alpha_r\beta_{2r-1}^{n}(\prod_{k=r+1}^{m} \beta_{2k-1}^{n-1})z^{n-1}\gamma_r \quad \text{(by inductive hypothesis)}\\ &= \alpha_r(\prod_{k=r+1}^{m} \beta_{2k-1}^{n-1})\beta_{2r-1}^{n}z^{n-1}\gamma_r \quad \text{(because } U \text{ is commutative)}\\ &= \alpha_r(\prod_{k=r+1}^{m} \beta_{2k-1}^{n-1})\beta_{2r-1}z^{n-1}\gamma_r \quad \text{(as } U \text{ satisfies the identity (2))}\\ &= \alpha_r(\prod_{k=r+1}^{m} \beta_{2k-1}^{n-1})z^{n-1}\beta_{2r-1}\gamma_r \quad \text{(by Theorem 2)}\\ &= \alpha_r(\prod_{k=r+1}^{m} \beta_{2k-1}^{n-1})z^{n-1}\beta_{2r}\gamma_{r+1} \quad \text{(by zigzag equations (1))}\\ &= \alpha_r\beta_{2r}(\prod_{k=r+1}^{m} \beta_{2k-1}^{n-1})z^{n-1}\gamma_{r+1} \quad \text{(by Theorem 2)}\\ &= \alpha_{r+1}\beta_{2r+1}^{n}(\prod_{k=r+2}^{m} \beta_{2k-1}^{n-1})z^{n-1}\gamma_{r+1} \quad \text{(by zigzag equations (1)),}\end{aligned}$$

as required. Now, for any integer $n > 2$ we have,

$$(\prod_{k=0}^{m} \beta_{2k}^{n-1})z = \alpha_{m-1}\beta_{2m-3}^{n}(\beta_{2m-1}^{n-1})z^{n-1}\gamma_{m-1} \quad \text{(by equation (3))}$$

$$
\begin{aligned}
&= \alpha_{m-1}\beta_{2m-3}^{n}\beta_{2m-1}^{n-1}\beta_0\gamma_1 z^{n-2}\gamma_{m-1} \quad \text{(by zigzag equations (1))}\\
&= \alpha_{m-1}\beta_{2m-1}^{n-1}\beta_0\beta_{2m-3}^{n}\gamma_1 z^{n-2}\gamma_{m-1} \quad \text{(because } U \text{ is commutative)}\\
&= \alpha_{m-1}\beta_{2m-1}^{n-1}\beta_0\beta_{2m-3}\gamma_1 z^{n-2}\gamma_{m-1} \quad \text{(as } U \text{ satisfies the identity (2))}\\
&= \alpha_{m-1}\beta_{2m-3}\beta_{2m-1}^{n-1}\beta_0\gamma_1 z^{n-2}\gamma_{m-1} \quad \text{(because } U \text{ is commutative)}\\
&= \alpha_{m-1}\beta_{2m-3}\beta_{2m-1}^{n-1} z^{n-1}\gamma_{m-1} \quad \text{(by zigzag equations (1))}\\
&= \alpha_{m-1}\beta_{2m-1}^{n-1} z^{n-1}\beta_{2m-3}\gamma_{m-1} \quad \text{(by Theorem 2)}\\
&= \alpha_{m-1}\beta_{2m-1}^{n-1} z^{n-1}\beta_{2m-2}\gamma_{m} \quad \text{(by zigzag equations (1))}\\
&= \alpha_{m-1}\beta_{2m-2}\beta_{2m-1}^{n-1} z^{n-1}\gamma_{m} \quad \text{(by Theorem 2)}\\
&= \alpha_{m}\beta_{2m-1}^{n} z^{n-1}\gamma_{m} \quad \text{(by zigzag equations (1))}\\
&= \alpha_{m} z\beta_{2m-1}^{n} z^{n-2}\gamma_{m} \quad \text{(by Theorem 2)}\\
&= \alpha_{m}\alpha_m\beta_{2m}\beta_{2m-1}^{n}\beta_0\gamma_1 z^{n-3}\gamma_{m} \quad \text{(by zigzag equations (1))}\\
&= \alpha_{m}\alpha_m\beta_{2m}\beta_0\beta_{2m-1}^{n}\gamma_1 z^{n-3}\gamma_{m} \quad \text{(because } U \text{ is commutative)}\\
&= \alpha_{m}\alpha_m\beta_{2m}\beta_0\beta_{2m-1}\gamma_1 z^{n-3}\gamma_{m} \quad \text{(as } U \text{ satisfies the identity (2))}\\
&= \alpha_{m}\alpha_m\beta_{2m}\beta_{2m-1}\beta_0\gamma_1 z^{n-3}\gamma_{m} \quad \text{(because } U \text{ is commutative)}\\
&= \alpha_{m} z\beta_{2m-1} z^{n-2}\gamma_{m} \quad \text{(by zigzag equations (1))}\\
&= \alpha_{m} z^{n-1}\beta_{2m-1}\gamma_{m} \quad \text{(by Theorem 2)}\\
&= \alpha_{m} z^{n-1}\beta_{2m} \quad \text{(by zigzag equations (1))}\\
&= \alpha_{m}\beta_{2m} z^{n-1} \quad \text{(by Theorem 2)}\\
&= z^{n} \quad \text{(by zigzag equations (1)).}
\end{aligned}
$$

This proves the lemma completely. □

Corollary 1. *Let U be a commutative subsemigroup of a semigroup S and U satisfies the identity*

$$pqr = pqr^n (\textit{for any integer } n > 2).$$

Let $z \in Dom(U,S) \setminus U$ and let (1) be the zigzag of length m over U with value z. Then,

$$z(\prod_{k=0}^{m}\beta_{2k}^{n-1}) = z^n.$$

Proof. The proof follows by Theorem 2 and Lemma 3. □

Theorem 4. *A subvariety $\mathcal{V}_1 = [pqr = pqr^n](for\ any\ integer\ n > 2)$ of the variety of commutative semigroups is closed under dominions.*

Proof. Let $\mathcal{V}_1$ be the variety defined as in the statement of the Theorem. Let $U \in \mathcal{V}_1$ and S be any semigroup containing U as a subsemigroup. We show that $Dom(U,S) \in \mathcal{V}_1$ by showing that for any $p, q, r \in Dom(U,S)$,

$$pqr = pqr^n.$$

To prove the Theorem completely, we consider the following cases.

Case 1. If $p, q, r \in U$, then, $pqr = pqr^n$ holds trivially.

Case 2. If $p, q \in U$ and $r \in Dom(U, S) \setminus U$. Let (1) be a zigzag of minimum length m over U with value r. We first prove that

$$pqr = \alpha_{m-l}\Big(\prod_{k=m-l}^{m} \beta_{2k}^{n-1}\Big)pq\beta_{2(m-l)-1}\gamma_{m-l} \text{ for all } l = 0,\ 1, \ldots,\ m-1. \tag{4}$$

To prove (4), we use induction on l. For $l = 0$ we have,

$$\begin{aligned}
pqr &= rpq \quad \text{(by Theorem 2)}\\
&= \alpha_m\beta_{2m}pq \quad \text{(by zigzag equations (1))}\\
&= \alpha_m pq\beta_{2m} \quad \text{(because } U \text{ is commutative)}\\
&= \alpha_m pq\beta_{2m}^n \quad \text{(since } U \in \mathcal{V}_1)\\
&= \alpha_m\beta_{2m}^{n-1}pq\beta_{2m} \quad \text{(because } U \text{ is commutative)}\\
&= \alpha_m\beta_{2m}^{n-1}pq\beta_{2m-1}\gamma_m \quad \text{(by zigzag equations (1))}.
\end{aligned}$$

Thus, (4) holds for $l = 0$. Assume inductively that (4) holds for $l = r$ $(\leq m-2)$. We show that it holds for $l = r+1$. Now,

$$\begin{aligned}
pqr &= \alpha_{m-r}\Big(\prod_{k=m-r}^{m} \beta_{2k}^{n-1}\Big)pq\beta_{2(m-r)-1}\gamma_{m-r} \quad \text{(by inductive hypothesis)}\\
&= \alpha_{m-r}\beta_{2(m-r)-1}\Big(\prod_{k=m-r}^{m} \beta_{2k}^{n-1}\Big)pq\gamma_{m-r} \quad \text{(because } U \text{ is commutative)}\\
&= \alpha_{m-(r+1)}\beta_{2(m-r)-2}\Big(\prod_{k=m-r}^{m} \beta_{2k}^{n-1}\Big)pq\gamma_{m-r} \quad \text{(by zigzag equations (1))}\\
&= \alpha_{m-(r+1)}\Big(\prod_{k=m-r}^{m} \beta_{2k}^{n-1}\Big)pq\beta_{2(m-r)-2}\gamma_{m-r} \quad \text{(because } U \text{ is commutative)}\\
&= \alpha_{m-(r+1)}\Big(\prod_{k=m-r}^{m} \beta_{2k}^{n-1}\Big)pq\beta_{2(m-r)-2}^{n}\gamma_{m-r} \quad \text{(since } U \in \mathcal{V}_1)\\
&= \alpha_{m-(r+1)}\Big(\prod_{k=m-(r+1)}^{m} \beta_{2k}^{n-1}\Big)pq\beta_{2(m-r)-2}\gamma_{m-r} \quad \text{(because } U \text{ is commutative)}\\
&= \alpha_{m-(r+1)}\Big(\prod_{k=m-(r+1)}^{m} \beta_{2k}^{n-1}\Big)pq\beta_{2(m-(r+1))-1}\gamma_{m-(r+1)} \quad \text{(by zigzag equations (1))},
\end{aligned}$$

as required. Now,

$$\begin{aligned}
pqr &= \alpha_1(\prod_{k=1}^{m}\beta_{2k}^{n-1})pq\beta_1\gamma_1 \quad \text{(by equation (4))}\\
&= \alpha_1\beta_1(\prod_{k=1}^{m}\beta_{2k}^{n-1})pq\gamma_1 \quad \text{(because } U \text{ is commutative)}\\
&= \beta_0(\prod_{k=1}^{m}\beta_{2k}^{n-1})pq\gamma_1 \quad \text{(by zigzag equations (1))}\\
&= (\prod_{k=1}^{m}\beta_{2k}^{n-1})pq\beta_0\gamma_1 \quad \text{(because } U \text{ is commutative)}\\
&= (\prod_{k=1}^{m}\beta_{2k}^{n-1})pq\beta_0^n\gamma_1 \quad \text{(since } U \in \mathcal{V}_1)\\
&= pq(\prod_{k=0}^{m}\beta_{2k}^{n-1})\beta_0\gamma_1 \quad \text{(because } U \text{ is commutative)}\\
&= pq(\prod_{k=0}^{m}\beta_{2k}^{n-1})r \quad \text{(by zigzag equations (1))}\\
&= pqr^n \quad \text{(by Lemma 3)},
\end{aligned}$$

as required.

Case 3. If $q, r \in U$ and $p \in Dom(U, S) \setminus U$. Let (1) be a zigzag of minimum length m over U with value p. Then,

$$\begin{aligned}
pqr &= \alpha_m\beta_{2m}qr \quad \text{(by zigzag equations (1))}\\
&= \alpha_m\beta_{2m}qr^n \quad \text{(since } U \in \mathcal{V}_1)\\
&= pqr^n \quad \text{(by zigzag equations (1))},
\end{aligned}$$

as required.

Case 4. If $p, r \in U$ and $q \in Dom(U, S) \setminus U$. Then,

$$\begin{aligned}
pqr &= qpr \quad \text{(by Theorem 2)}\\
&= qpr^n \quad \text{(by Case 3)}\\
&= pqr^n \quad \text{(by Theorem 2)},
\end{aligned}$$

as required.

Case 5. If $p \in U$ and $q, r \in Dom(U, S) \setminus U$. Let (1) be a zigzag of minimum length m over U with value q. Then,

$$pqr = qpr \quad \text{(by Theorem 2)}$$

$$\begin{aligned}
&= \alpha_m\beta_{2m}pr \quad \text{(by zigzag equations (1))}\\
&= \alpha_m\beta_{2m}pr^n \quad \text{(by Case 2)}\\
&= qpr^n \quad \text{(by zigzag equations (1))}\\
&= pqr^n \quad \text{(by Theorem 2)},
\end{aligned}$$

as required.

Case 6. If $q \in U$ and $p, r \in Dom(U,S) \setminus U$. Then,

$$\begin{aligned}
pqr &= qpr \quad \text{(by Theorem 2)}\\
&= qpr^n \quad \text{(by Case 5)}\\
&= pqr^n \quad \text{(by Theorem 2)},
\end{aligned}$$

as required.

Case 7. $r \in U$ and $p, q \in Dom(U,S) \setminus U$. Let (1) be a zigzag of minimum length m over U with value p. Then,

$$\begin{aligned}
pqr &= \alpha_m\beta_{2m}qr \quad \text{(by zigzag equations (1))}\\
&= \alpha_m\beta_{2m}qr^n \quad \text{(by Case 4)}\\
&= pqr^n \quad \text{(by zigzag equations (1))},
\end{aligned}$$

as required.

Case 8. If $p, q, r \in Dom(U,S) \setminus U$. Let (1) be a zigzag of minimum length m over U with value p. Then,

$$\begin{aligned}
pqr &= \alpha_m\beta_{2m}qr \quad \text{(by zigzag equations (1))}\\
&= \alpha_m\beta_{2m}qr^n \quad \text{(by Case 5)}\\
&= pqr^n \quad \text{(by zigzag equations (1))},
\end{aligned}$$

as required. This brings the proof of the Theorem 4 to a conclusion. □

Theorem 5. *A subvariety $\mathcal{V}_2 = [p^{n_1}q^{n_2} = 0]$(for any integers $n_1,\ n_2 > 0$) of the variety of commutative semigroups is closed under dominions.*

Proof. Let $\mathcal{V}_2$ be the variety defined as in the statement of the Theorem. Let $U \in \mathcal{V}_2$ and S be any semigroup containing U as a subsemigroup. We show that $Dom(U,S) \in \mathcal{V}_2$ by showing that for any $p, q, r \in Dom(U,S)$,

$$p^{n_1}q^{n_2}r = p^{n_1}q^{n_2} = rp^{n_1}q^{n_2}. \tag{5}$$

Since, by Theorem 2, we need to prove only one side of (5). For this, we consider the following cases.

Case 1. If $p, q, r \in U$, then, $p^{n_1}q^{n_2}r = p^{n_1}q^{n_2}$ holds trivially.

Case 2. If $p, q \in U$ and $r \in Dom(U, S) \setminus U$. Let (1) be a zigzag of minimum length m over U with value r. Then,

$$\begin{aligned}
p^{n_1}q^{n_2}r &= p^{n_1}q^{n_2}\beta_0\gamma_1 \quad \text{(by zigzag equations (1))}\\
&= p^{n_1}q^{n_2}\gamma_1 \quad (\text{since } U \in \mathcal{V}_2)\\
&= p^{n_1}q^{n_2}\beta_1\gamma_1 \quad (\text{since } U \in \mathcal{V}_2)\\
&= p^{n_1}q^{n_2}\beta_2\gamma_2 \quad \text{(by zigzag equations (1))}\\
&\vdots\\
&= p^{n_1}q^{n_2}\beta_{2m-2}\gamma_m\\
&= p^{n_1}q^{n_2}\gamma_m \quad (\text{since } U \in \mathcal{V}_2)\\
&= p^{n_1}q^{n_2}\beta_{2m-1}\gamma_m \quad (\text{since } U \in \mathcal{V}_2)\\
&= p^{n_1}q^{n_2}\beta_{2m} \quad \text{(by zigzag equations (1))}\\
&= p^{n_1}q^{n_2} \quad (\text{since } U \in \mathcal{V}_2),
\end{aligned}$$

as required.

Case 3. If $q, r \in U$ and $p \in Dom(U, S) \setminus U$. Let (1) be a zigzag of minimum length m over U with value p. Then,

$$\begin{aligned}
p^{n_1}q^{n_2}r &= \alpha_m^{n_1}\beta_{2m}^{n_1}q^{n_2}r \quad \text{(by Theorem 3)}\\
&= \alpha_m^{n_1}\beta_{2m}^{n_1}q^{n_2} \quad (\text{since } U \in \mathcal{V}_2)\\
&= p^{n_1}q^{n_2} \quad \text{(by Theorem 3)},
\end{aligned}$$

as required.

Case 4. If $p, r \in U$ and $q \in Dom(U, S) \setminus U$. Then,

$$\begin{aligned}
p^{n_1}q^{n_2}r &= q^{n_2}p^{n_1}r \quad \text{(by Theorem 2)}\\
&= q^{n_2}p^{n_1} \quad \text{(by Case 3)}\\
&= p^{n_1}q^{n_2} \quad \text{(by Theorem 2)},
\end{aligned}$$

as required.

Case 5. If $r \in U$ and $p, q \in Dom(U, S) \setminus U$. Let (1) be a zigzag of minimum length m over U with value p. Then,

$$\begin{aligned}
p^{n_1}q^{n_2}r &= \alpha_m^{n_1}\beta_{2m}^{n_1}q^{n_2}r \quad \text{(by Theorem 3)}\\
&= \alpha_m^{n_1}\beta_{2m}^{n_1}q^{n_2} \quad \text{(by Case 4)}\\
&= p^{n_1}q^{n_2} \quad \text{(by Theorem 3)},
\end{aligned}$$

as required.

Case 6. If $q \in U$ and $p, r \in Dom(U, S) \setminus U$ or $p \in U$ and $q, r \in Dom(U, S) \setminus U$ or $p, q, r \in Dom(U, S) \setminus U$. As $r \in Dom(U, S) \setminus U$ we may let (1) be a zigzag of minimum length m over U with value r. Now,

$$\begin{aligned}
p^{n_1}q^{n_2}r &= p^{n_1}q^{n_2}\beta_0\gamma_1 \quad \text{(by zigzag equations (1))}\\
&= p^{n_1}q^{n_2}\gamma_1 \quad \text{(by Cases 3, 4 and 5)}\\
&= p^{n_1}q^{n_2}\beta_1\gamma_1 \quad \text{(by Cases 3, 4 and 5)}\\
&= p^{n_1}q^{n_2}\beta_2\gamma_2 \quad \text{(by zigzag equations (1))}\\
&\vdots\\
&= p^{n_1}q^{n_2}\beta_{2m-2}\gamma_m\\
&= p^{n_1}q^{n_2}\gamma_m \quad \text{(by Cases 3, 4 and 5)}\\
&= p^{n_1}q^{n_2}\beta_{2m-1}\gamma_m \quad \text{(by Cases 3, 4 and 5)}\\
&= p^{n_1}q^{n_2}\beta_{2m} \quad \text{(by zigzag equations (1))}\\
&= p^{n_1}q^{n_2} \quad \text{(by Cases 3, 4 and 5)},
\end{aligned}$$

as required. This completes the proof of the Theorem 5. □

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