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# On index divisors of septic number fields defined by $x^7 + ax^2 + bx + c$

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**Abstract.** Consider a septic number field $K$ generated by a root of an irreducible quadrinomial $F(x) = x^7 + ax^2 + bx + c \in \mathbb{Z}[x]$. Let $i(K)$ denote the index of $K$, which is the greatest common divisor of indices of all primitive elements of the ring of integers of $K$. In this paper, we show that 2 and 3 are the only primes that can divide $i(K)$. Furthermore, for $p = 2$ and 3, we give sufficient conditions on $a, b$ and $c$ for which $i(K)$ is divisible by $p$ and we determine $\nu_p(i(K))$, the highest power of $p$ dividing $i(K)$. In particular, if $i(K) \neq 1$, then $K$ cannot be monogenic. Our results provide a partial answer to Problem 22 of Narkiewicz [Elementary and Analytic Theory of Algebraic Numbers, Springer-Verlag, Berlin, Heidelberg (2004)] for these families of number fields.

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## 1 Introduction

Let $K$ be a number field of degree $n$ and $\mathcal{O}_K$ its ring of integers. For any primitive element $\gamma \in O_K$, we denote the index of the subgroup $\mathbb{Z}[\gamma]$ in $\mathcal{O}_K$ by $\mathrm{ind}(\gamma)$. The field $K$ is called monogenic if $\mathrm{ind}(\gamma) = 1$ for some primitive element $\gamma$ of $\mathcal{O}_K$, i.e., $\mathcal{O}_K = \mathbb{Z}[\gamma]$. This means that $\mathbb{B}_\gamma := \{1, \gamma, \dots, \gamma^{n-1}\}$ is an integral basis of $K$. It is called a power integral basis of $K$. If there does not exist any $\gamma \in \mathcal{O}_K$ for which $\mathrm{ind}(\gamma) = 1$, then $K$ is non-monogenic.

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For $n \geq 4$, the monogenity of number fields of degree $n$ is not completely characterized: for example the monogenity of the pure quartic number field $\mathbb{Q}(\sqrt[4]{m})$ is not completely described, even if the parameter $m$ is square-free (see the recent paper [4] by Arnóczki and Nyul).

The problem of studying the monogenity or not monogenity of number fields have been intensively studied in the literature (cf. [1, 2, 9, 10, 12, 14, 15, 17, 21, 22, 26, 34, 35]).

In [20, 21, 23, 24], Győry proved in an effective way that there are only finitely many $\mathbb{Z}$-equivalence classes of $\gamma$ which can generate a power integral basis of $\mathcal{O}_K$, and a full system of representatives can be determined effectively. This gave the first general effectif algorithm for deciding whether $K$ is monogenic or not.

Evertse and Győry's book [14] provides detailed surveys on the discriminant form and index form theory and its applications, including related Diophantine equations and monogenity of number fields.

A great number of results have been obtained by different approaches. For quartic number fields, see [2–4, 31, 34, 35]. For quintic fields, you may consult [16]. For sextic fields, we refer to [1, 10]. For multiquadratic number fields see [33]. In [25, 26] Jones et al., studied the monogenity of different families of number fields defined by trinomials. See also [5] by Ben Yakkou. For more details and efficient algorithms, the reader can consult the books [14] by Evertse and Győry, [15] by Gaál and [29] by Narkiewicz.

An efficient method to study the monogenity of number fields, particularly when the degree $n$ is larger, typically greater than 6, is to analyze their indices. Recall that the index of a number field $K$ is

$$i(K) := \gcd \{\operatorname{ind}(\gamma) \mid \gamma \in \mathcal{O}_K \text{ and } K = \mathbb{Q}(\gamma)\}. \tag{1}$$

Thus, if $i(K) > 1$, then $K$ cannot be monogenic. The prime divisor $p$ of $i(K)$ are called prime common index divisor of $K$. In 1871, Dedekind proved that the cubic field $K$ generated by a root of $x^3 - x^2 - 2x - 8$ cannot be monogenic, since 2 is a common index divisor of $K$ (see [12, §5, Page 30]). For number fields of degree $n \leq 7$, Engstrom [13] showed that $\nu_p(i(K))$ can be determined from the decomposition type of $p$ into prime ideals. In [30], Nart extended Engstrom's results by giving explicit formulas for computing $\nu_p(i(K))$ for some type of $p$-decompositions. Problem 22 of Narkiewicz [29] asks for explicit formulas for $\nu_p(i(K))$ in classes of number fields.

For various results on indices in different families of number fields defined by trinomials, we refer to [5, 6, 8, 11, 27]. In 2024, Ben Yakkou [7] studied indices in quartic number fields defined by the quadrinomial $x^4 + ax^3 + bx + c \in \mathbb{Z}[x]$.

The goal of the present paper is to study the prime powers of indices for number fields $K$ generated by roots of irreducible quadrinomials $x^7 + ax^2 + bx + c$. We show that if a prime $p$ divides $i(K)$, then $p = 2$ or 3. We give sufficient conditions on $a, b$ and $c$, so that $p \in \{2, 3\}$ is a common index divisor of $K$, and we evaluate $\nu_p(i(K))$. Our method primarily relies on a theorem of Ore [32] on the decomposition of primes in number fields, while employing Newton polygon techniques [19], and the results established by Engstrom [13].

## 2 Newton polygons and Theorem of Ore

As mentioned in the introduction, the main tool to prove our results is a theorem of Ore on the decomposition of primes in number fields, so we need some preliminary results before stating this theorem.

Let $p$ be a prime and $K$ a number field. In 1928, Ore [32] developed an efficient algorithm to factorize $p\mathcal{O}_K$. His method utilizes some Newton polygon techniques. Ore's method were deeply developed by Guàrdia et al. in a series of their papers [18] and [19], who introduced higher Newton polygons.

Now, following [19, 28], we shortly recall some fundamental notions on these techniques.

Let $\mathbb{F}_p := \mathbb{Z}/p\mathbb{Z}$ and $\nu_p$ denote the discrete valuation of $\mathbb{Q}_p(x)$. Let $F(x) \equiv \prod_{i=1}^{t} \phi_i(x)^{l_i}$ be the factorization of $F(x)$ into a product of powers of distinct monic irreducible polynomials in $\mathbb{F}_p[x]$. For every $i = 1, \ldots, t$, let $F(x) = b_{i0}(x) + b_{i1}(x)\phi(x) + \cdots + b_{iL_i}(x)\phi(x)^{L_i}$ be the $\phi_i$-adic development of $F(x)$. Also, for every $0 \leq j \leq L_i$, let $u_{ij} = \nu_p(b_{ij}(x))$. The $\phi_i$-Newton polygon of $F(x)$ (with respect to $\nu_p$), denoted by $\mathcal{N}_{\phi_i}(F)$, is the lower boundary convex envelope of the points $\{(j, u_{ij}), 0 \leq j \leq L_i, b_{ij}(x) \neq 0\}$ in the Euclidean plane. The sides of $\mathcal{N}_\phi(F)$ which have negative slopes form the $\phi_i$-principal Newton polygon of $F(x)$. This polygon will be denoted by $\mathcal{N}^-_{\phi_i}(F)$. The polygon $\mathcal{N}^-_{\phi_i}(F)$ is the union of $r_i$ different adjacent sides $S_{i1}, \ldots, S_{ir_i}$ with increasing slopes $\lambda_{i1} < \cdots < \lambda_{ir_i}$. We shall write $\mathcal{N}^-_{\phi_i}(F) = S_{i1} + S_{12} + \cdots + S_{ir_i}$.

Let $S_{ij}$ be a side of $\mathcal{N}^-_{\phi_i}(F)$ with length $l(S_{ij})$ and height $h(S_{ij})$. Then the slope of $S_{ij}$ equals $-\frac{h(S_{ij})}{l(S_{ij})} = -\frac{h_{ij}}{e_{ij}}$, where $e_{ij}$ and $h_{ij}$ are two positive coprime integers. The positive integer $e_{ij}$ is called the ramification index of $S_{ij}$ and denoted by $e(S_{ij})$. The degree of $S_{ij}$ is $d(S_{ij}) = d_{ij} = \frac{l(S_{ij})}{e_{ij}}$.

Define the finite residual field $\mathbb{F}_{\phi_i} := \mathbb{Z}[x]/(p, \phi_i(x))$. Then, we attach to $S_{ij}$ the following residual polynomial:

$$\mathcal{R}_{ij}(F)(y) := c_{s_{ij}} + c_{s_{ij}+e_{ij}} y + \cdots + c_{s_{ij}+(d_{ij}-1)e_{ij}} y^{d_{ij}-1} + c_{s_{ij}+d_{ij}e_{ij}} y^{d_{ij}} \in \mathbb{F}_{\phi_i}[y],$$

where $c_{s_{ij}+ke_{ij}} = \dfrac{a_{s_{ij}+ke_{ij}}(x)}{p^{u_{i,s_{ij}+ke_{ij}}}} \pmod{(p, \phi_i(x))}$ for $k = 1, \ldots, d_{ij}$, and $s_{ij}$ is the abscissa of the initial point of $S_{ij}$. Let $\mathcal{R}_{ij}(F)(y) = \prod_{s=1}^{s_{ij}} \psi_{ijs}^{n_{ijs}}(y)$ be the factorization of $\mathcal{R}_{ij}(F)(y)$ into a product of powers of distinct monic irreducible polynomials in $\mathbb{F}_{\phi_i}[y]$. With the above notations, we give the following two concepts of regularity:

1. The polynomial $F(x)$ is called $\phi_i$-regular with respect to $\nu_p$ if $\mathcal{R}_{ij}(F)(y)$ is separable in $\mathbb{F}_{\phi_i}[y]$ for every $j = 1, \ldots, r_i$, that is $n_{ijs} = 1$ for all $(i, j, s)$.

2. The polynomial $F(x)$ is said $p$-regular if it is $\phi_i$-regular for all $i = 1, \ldots, t$.

The following theorem, due to Ore [32], will play a significant role in the proof of our results (see [19, Theorems 1.13, 1.15 and 1.19]) and [28]:

**Theorem 1** (Theorem of Ore)**.** *With the above notations, if $F(x)$ is $p$-regular, then*

$$p\mathcal{O}_K = \prod_{i=1}^{t} \prod_{j=1}^{r_i} \prod_{s=1}^{s_{ij}} \mathfrak{p}_{ijs}^{e_{ij}},$$

*where $e_{ij} = e(S_{ij})$ and $f_{ijs} = f(\mathfrak{p}_{ijs}/p) = \deg(\phi_i) \times \deg(\psi_{ijs})$ is the residue degree of $\mathfrak{p}_{ijs}$ over $p$.*

**Corollary 1.** *With the above notations, we have:*

1. *If for some* $i = 1, \ldots, t$, $l_i = 1$, *then* $\phi_i$ *provides a unique prime ideal of* $\mathcal{O}_K$ *above* $p$ *of residue degree* $\deg(\phi_i)$ *with ramification index* 1.
2. *If for some* $i = 1, \ldots, t$, $\mathcal{N}_{\phi_i}^{-}(F) = S_{i1} + \cdots + S_{ir_i}$ *has* $r_i$ *distinct sides of degree* 1 *each, then* $\phi_i$ *provides* $r_i$ *distinct prime ideals of* $\mathcal{O}_K$ *above* $p$ *of residue degree* $\deg(\phi_i)$ *each, with respective ramification indices* $e(S_{ij}), j = 1, \ldots, r_i$.

The proof of the following numerical example in the quartic case is based on the application of Ore's Theorem.

**Example 1.** Consider $F(x) = x^4 + 76x^2 + 152x + 171 \in \mathbb{Z}[x]$. Since $F(x)$ is a 19-Eisenstein polynomial, $F(x)$ is irreducible over $\mathbb{Q}$. Let $\alpha$ be a root of $F(x)$ and set $K = \mathbb{Q}(\alpha)$. By [13], $i(K) = 2^u \cdot 3^v$, with $u \leq 2$ and $v \leq 1$. So, let us determine $u$ and $v$.

For $p = 2$, we have $F(x) \equiv (x-1)^4 \pmod 2$. Set $\phi_1 = x - 1$. The $\phi_1$-adic development of $F(x)$ is

$$F(x) = \phi_1(x)^4 + 4\phi_1(x)^3 + 82\phi_1(x)^2 + 308\phi_1(x) + 400.$$

As $\nu_2(400) = 4, \nu_2(308) = 2, \nu_2(82) = 1$ and $\nu_2(4) = 2$, $\mathcal{N}_{\phi_1}^{-}(F) = \mathcal{S}_{11} + \mathcal{S}_{12} + \mathcal{S}_{13}$ has three distinct sides joining $(0,4), (1,2), (2,1)$ and $(4,0)$ (see Figure 1). Their respective ramification indices are $e(S_{11}) = e(S_{12}) = 1$ and $e(S_{13}) = 2$. The degree of each side is 1, then their attached residual polynomials are separable in $\mathbb{F}_{\phi_1}[y] \simeq \mathbb{F}_2[y]$. So, $F(x)$ is $\phi_1$-regular, and so it is 2-regular. Applying Theorem 1, we see that $2\mathcal{O}_K = \mathfrak{p}_{111}\mathfrak{p}_{121}\mathfrak{p}_{131}^2$, where $\mathfrak{p}_{111}, \mathfrak{p}_{121}$ and $\mathfrak{p}_{131}$ are three distinct prime ideals of $\mathcal{O}_K$ having residue degree 1 each. Hence, by Engstrom's table (see [13, Page 234]), we deduce that $u = 1$.

For $p = 3$, we have $F(x) \equiv x(x-1)(x^2 - x - 1) \pmod 3$. Using Corollary 1(1), we get $3\mathcal{O}_K = \mathfrak{p}_{111}\mathfrak{p}_{121}\mathfrak{p}_{131}$, where $\mathfrak{p}_{111}$ and $\mathfrak{p}_{121}$ are two distinct prime ideals of $\mathcal{O}_K$ having residue degree 1 each, and $\mathfrak{p}_{131}$ is a prime ideal of $\mathcal{O}_K$ of residue degree 2. Therefore, by the above mentioned Engstrom's table, $v = 0$.

We conclude that $i(K) = 2$. Hence, $K$ is not monogenic.

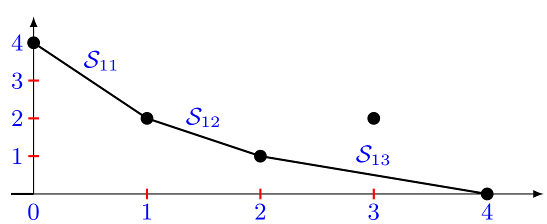


**Figure 1:** $\mathcal{N}_{\phi_1}^{-}(F)$ with respect to $\nu_2$.

## 3 Main results

In what follows, for any rational integer $r$ and a prime $p$, $\nu_p(r)$ will denote the highest power of $p$ dividing $r$. $K$ is a septic number field generated by a root of an irreducible quadrinomial

$F(x) = x^7 + ax^2 + bx + c \in \mathbb{Z}[x]$. For every prime $p$, scaling the coefficients if necessary, we lose no generality in assuming that

$$\nu_p(a) < 5 \ \text{ or } \ \nu_p(b) < 6 \ \text{ or } \ \nu_p(c) < 7. \tag{2}$$

By [29, Proposition 2.13], we have

$$\Delta(F) = \text{ind}(\alpha)^2 \cdot d_K, \tag{3}$$

where $\Delta(F)$ is the discriminant of $F(x)$ and $d_K$ is the discriminant of $K$. Therefore, if $p \mid i(K)$, then $p^2 \mid \Delta(F)$. Note that

$$\begin{aligned} \Delta(F) &= -2^2 \times 5^5 a^7 c + 5^5 a^6 b^2 + 2 \times 5^3 \times 7^4 a^3 b c^3 - 2^2 \times 3^3 \times 5^2 \times 7^3 a^2 b^3 c^2 \\ &+ 2^5 \times 3^5 \times 7^2 a b^5 c - 2^6 \times 3^6 b^7 - 7^7 b^6. \end{aligned} \tag{4}$$

For simplicity, if $p\mathcal{O}_K = \prod_{j=1}^{g} \mathfrak{p}_j^{e_j}$, where $\mathfrak{p}_j$ are distinct prime ideals of $\mathcal{O}_K$ with residue degrees $f(\mathfrak{p}_j/p) = f_j$, then we write $p\mathcal{O}_K = [f_1^{e_1}, \ldots, f_g^{e_g}]$. Note also that

$$\sum_{j=1}^{g} e_j f_j = \deg(K) = 7. \tag{5}$$

The following result in the septic case is derived immediately from the above equality and Theorems 4.33 and 4.34 and Proposition 4.35 of [29].

**Lemma 1.** *Let $K$ be a number field of degree 7. Then $5 \mid i(K)$ if and only if $L_5(1) \geq 6$, where $L_5(1)$ is the number of distinct prime ideals of $\mathcal{O}_K$ having residue degree 1 and lying above 5.*

The following table, which is copied from [13], provides the highest power of the primes 2 and 3 dividing the index of any number field $K$ of degree 7.

**Table 1:** $\nu_p(i(K))$ for $p \in \{2, 3\}$ and $\deg(K) = 7$

| $p\mathcal{O}_K = [f_1^{e_1}, \ldots, f_g^{e_g}]$ | $\nu_2(i(K))$ | $\nu_3(i(K))$ |
|---|---|---|
| $[2^1, 2^1, 1^1, 1^1, 1^1]$ | 3 | 0 |
| $[2^1, 2^1, 1^2, 1^1]$ | 2 | 0 |
| $[2^1, 2^1, 2^1, 1^1]$ | 2 | 0 |
| $[2^1, 2^2, 1^1]$ | 1 | 0 |
| $[2^2, 1^1, 1^1, 1^1]$ | 1 | 0 |
| $[4^1, 1^1, 1^1, 1^1]$ | 1 | 0 |
| $[1^2, 1^1, 1^1, 1^1, 1^1, 1^1]$ | 7 | 3 |
| $[1^2, 1^2, 1^1, 1^1, 1^1]$ | 4 | 2 |
| $[1^3, 1^1, 1^1, 1^1, 1^1]$ | 4 | 2 |
| $[2^1, 1^2, 1^1, 1^1, 1^1]$ | 2 | 1 |
| $[1^3, 1^2, 1^1, 1^1]$ | 2 | 1 |

We are now ready to prove one of our main results in this paper.

**Theorem 2.** *If $p \geq 5$ is a prime number, then $p$ is not a common index divisor of $K$; $p \nmid i(K)$.*

*Proof.* Let $p$ be a prime number. Since the degree of $K$ is 7, by [29, Proposition 4.36], if $p$ divides $i(K)$, then $p < 7$. Therefore, it is sufficient to show that $5 \nmid i(K)$. By (1), (3) and (4), $5 \mid i(K)$ if and only if $abc + b^3 \equiv 2c^2 \pmod{5}$. This means that $(a,b,c) \in \{(0,0,0),(1,0,0),(2,0,0),(3,0,0),(4,0,0),(1,1,1),(1,1,2),(4,1,3),(4,1,4),(2,2,1),$ $(0,2,2),(0,2,3),(3,2,4),(0,3,1),(1,3,2),(4,3,3),(0,3,4),(2,4,1),(3,4,2),(2,4,3),(3,4,4)\} \pmod{5}$.

We distinguish three cases.

**A1:** If $a \equiv b \equiv c \equiv 0 \pmod{5}$, then $F(x) \equiv x^7 \pmod{5}$. Let $\phi_1 = x$. According to (2), one needs to discuss five sub-cases which cover all the possible cases.

1. If $7\nu_5(a) < 5\nu_5(c)$ and $2\nu_5(b) \geq \nu_5(a)+\nu_5(c)$, then $\mathcal{N}^-_{\phi_1}(F) = \mathcal{S}_{11}+\mathcal{S}_{12}$ joining $(0,\nu_5(c)),(2,\nu_5(a))$ and $(7,0)$. The degree of the side $\mathcal{S}_{12}$ equals 2 and its ramification index is 5. By Theorem 1 and Equality (5), one has: $5\mathcal{O}_K = [1^5,1^2],[1^5,1^1,1^1]$ or $[1^5,2^1]$.

2. If $7\nu_5(a) < 5\nu_5(c)$ and $2\nu_5(b) < \nu_5(a)+\nu_5(c)$, then $\mathcal{N}^-_{\phi_1}(F) = \mathcal{S}_{11}+\mathcal{S}_{12}+\mathcal{S}_{13}$ has three distinct sides of degree 1 each joining the points $(0,\nu_5(c)),(1,\nu_5(b),(2,\nu_5(a))$ and $(7,0)$. Their respective ramification indices are: $e(\mathcal{S}_{11}) = 5$ and $e(\mathcal{S}_{12}) = e(\mathcal{S}_{13}) = 1$. Therefore, by Corollary 1, $5\mathcal{O}_K = [1^5,1^1,1^1]$.

3. If $7\nu_5(b) < 6\nu_5(c), 6\nu_5(a) > 5\nu_5(b)$ and $\nu_5(b) \in \{1,5\}$, then $\mathcal{N}^-_{\phi_1}(F) = \mathcal{S}_{11}+\mathcal{S}_{12}$ has two distinct sides of degree 1 each joining the points $(0,\nu_5(c)),(1,\nu_5(b))$ and $(7,0)$. Applying Corollary 1, we see that $5\mathcal{O}_K = [1^6,1^1]$.

4. If $7\nu_5(b) < 6\nu_5(c), 6\nu_5(a) > 5\nu_5(b)$ and $\nu_5(b) \in \{2,4\}$, then $\mathcal{N}^-_{\phi_1}(F) = \mathcal{S}_{11}+\mathcal{S}_{12}$ joining $(0,\nu_5(c)),(1,\nu_5(b))$ and $(7,0)$. Thus the degree of the first side is 1 and the degree of the second one is 2. Moreover, we have

$$\mathcal{R}_{12}(F)(y) = y^2 + b_5 = \begin{cases} (y+2)(y+3), & \text{if } b_5 \equiv 1 \pmod{5}. \\ y^2+2, & \text{if } b_5 \equiv 2 \pmod{5}. \\ y^2+3, & \text{if } b_5 \equiv 3 \pmod{5}. \\ (y+1)(y+4), & \text{if } b_5 \equiv 4 \pmod{5}, \end{cases}$$

where $b_5 = \frac{b}{5^{\nu_5(b)}}$. So, $F(x)$ is 5-regular. Therefore, by Theorem 1, $5\mathcal{O}_K = [1^1,1^3,1^3]$ or $5\mathcal{O}_K = [1^1,2^3]$.

5. If $7\nu_5(b) < 6\nu_5(c), 6\nu_5(a) > 5\nu_5(b)$ and $\nu_5(b) = 3$, then $\mathcal{N}^-_{\phi_1}(F) = \mathcal{S}_{11}+\mathcal{S}_{12}$ joining $(0,\nu_5(c)),(1,\nu_5(b))$ and $(7,0)$. This implies that $d(\mathcal{S}_{11}) = 1$ and $d(\mathcal{S}_{12}) = 3$. Furthermore, we have

$$\mathcal{R}_{12}(F)(y) = y^3 + b_5 = \begin{cases} (y+1)(y^2+4y+1), & \text{if } b_5 \equiv 1 \pmod{5}. \\ (y+3)(y^2+2y+4), & \text{if } b_5 \equiv 2 \pmod{5}. \\ (y+2)(y^2+3y+4), & \text{if } b_5 \equiv 3 \pmod{5}. \\ (y+4)(y^2+y+1), & \text{if } b_5 \equiv 4 \pmod{5}. \end{cases}$$

Thus, $F(x)$ is 5-regular. Therefore, $5\mathcal{O}_K = [1^1,2^2,1^2]$.

**A2:** If $(a,b,c) \equiv (5-k,0,0) \pmod 5$, with $k \in \{1,2,3,4\}$, then $F(x) \equiv x^2(x-k)^5 \pmod 5$. Let $\phi_k = x-k$, then the $\phi_k$-adic development of $F(x)$ is

$$\begin{array}{rcl} F(x) & = & \phi_k(x)^7 + 7k\phi_k(x)^6 + 21k^2\phi_k(x)^5 + 35k^3\phi_k(x)^4 + 35k^4\phi_k(x)^3 + (a+21k^5)\phi_k(x)^2 \\ & + & (2ka+b+7k^6)\phi_k(x) + k^2a + kb + c + k^7. \end{array}$$

Let $u_{k0} := \nu_5(k^2a+kb+c+k^7), u_{k1} := \nu_5(2ka+b+7k^6), u_{k2} := \nu_5(a+21k^5)$ and note that $\nu_5(35k^3) = \nu_5(35k^4) = 1$. It follows that $\mathcal{N}^-_{\phi_k}(F)$ is the lower convex hull of the points $(0,u_{k0}), (1,u_{k1}), (2,u_{k2}), (3,1)$ and $(5,0)$. We distinguish two cases:

1. If $u_{k0} = 1$ or $u_{k1} = 1$ or $u_{k2} = 1$, then $\mathcal{N}^-_{\phi_k}(F)$ has a side of degree 1 having ramification index greater than or equals to 3. Therefore $5\mathcal{O}_K$ is divisible by a non-zero ideal of $\mathcal{O}_K$ having one of the forms: $[1^3]$ or $[1^4]$, or $[1^5]$. It follows from Equality (5) that $L_5(1) \leq 5$.

2. If $u_{k0} \geq 2$, $u_{k1} \geq 2$ and $u_{k2} \geq 2$, then $c \equiv 10k^7 \pmod{25}$, so $\nu_5(c) = 1$. Thus, for $\phi_1 := x$, we have $\mathcal{N}^-_{\phi_1}(F)$ has a single side of degree 1 joining the points $(0,1)$ and $(2,0)$. Thus, $\phi_1$ provides one prime ideal of residue degree 1 above 2 with ramification index 2, say $\mathfrak{p}_{111}$. For $\mathcal{N}^-_{\phi_k}(F)$, we discuss three sub-cases.

   (a) If $u_{k0} = 2$, then $\mathcal{N}^-_{\phi_k}(F) = \mathcal{S}_{k1}$ has a single side of degree 1 joining the points $(0,2)$ and $(5,0)$ with ramification index 5. Therefore, by Corollary 1, $5\mathcal{O}_K = [1^2, 1^5]$.

   (b) If $u_{k0} \geq 3$ and $u_{k1} = 2$, $\mathcal{N}^-_{\phi_k}(F) = \mathcal{S}_{k1} + \mathcal{S}_{k2}$ has two distinct sides joining the points $(0,u_{k0}), (1,2), (3,1)$ and $(5,0)$. Thus, $d(\mathcal{S}_{k1}) = 1, e(\mathcal{S}_{k1}) = 1, d(\mathcal{S}_{k2}) = 2$ and $e(\mathcal{S}_{k2}) = 2$. Therefore, $5\mathcal{O}_K = \mathfrak{p}_{111}^2\mathfrak{p}_{k11}\mathfrak{a}^2$, where $\mathfrak{p}_{111}$ and $\mathfrak{p}_{k11}$ are two distinct prime ideals of $\mathcal{O}_K$ having residue degree 1 each and $\mathfrak{a}$ is a non-zero ideal of $\mathcal{O}_K$. By Equality (5), $L_5(1) \leq 4$.

   (c) If $u_{k0} \geq 3$ and $u_{k1} \geq 3$, then the side $\mathcal{S}_k$ jogging the points $(3,1)$ and $(5,0)$ is a side of $\mathcal{N}^-_{\phi_k}(F)$ and having ramification index 2. So, it provides a unique prime ideal above 5 of residue degree 1 with ramification index 2, say $\mathfrak{p}_k$. Thus, $5\mathcal{O}_K = \mathfrak{p}_{111}^2\mathfrak{p}_k^2\mathfrak{a}$, where $\mathfrak{a}$ is a non-zero ideal of $\mathcal{O}_K$. Hence by Equality (5), $L_5(1) \leq 5$.

**A3:** If $(a,b,c) \in \{(1,1,1),(1,1,2),(4,1,3),(4,1,4),(2,2,1),(0,2,2),(0,2,3),(3,2,4),(0,3,1),$ $(1,3,2),(4,3,3),(0,3,4),(2,4,1),(3,4,2),(2,4,3),(3,4,4)\} \pmod 5$, then the factorization of $F(x)$ modulo 5 has an irreducible factor $\phi(x)$ of degree greater or equals 3, This factor provides at least one prime ideal $\mathfrak{p}$ of residual degree greater or equals 3. By Equality (5), $L_5(1) \leq 4$.

In every case, we have $L_5(1) \leq 5$. Consequently, by Lemma 1, $5 \nmid i(K)$. $\square$

By the above theorem, $i(K) = 2^{\nu_2(i(K))} \cdot 3^{\nu_3(i(K))}$ for any septic number field $K$ generated by an irreducible quadrinomial $x^7 + ax^2 + bx + c$. In the following result, we give sufficient conditions on $a, b$ and $c$ so that 2 is a common index divisor of $K$, and we evaluate $\nu_2(i(K))$.

**Theorem 3.** *Let $F(x)$ and $K$ be as above, then we have the following statements:*

1. *If $a \equiv 1 \pmod 2, b \equiv 0 \pmod 2$ and $c \equiv 0 \pmod{2^{2\nu_2(b)+1}}$, then $\nu_2(i(K)) = 1$.*

2. *If $(a,b,c) \in \{(1,15,15),(9,15,7),(5,7,3),(13,7,11)\} \pmod{16}$ and $1+a+b+c \equiv 0 \pmod{64}$, then $\nu_2(i(K)) = 1$.*

3. *If* $\nu_2(a) \in \{1,2,3,4\}$, $b \equiv 0 \pmod{2^{\nu_2(a)+1}}$ *and* $c \equiv 0 \pmod{2^{2\nu_2(b)-\nu_2(a)+1}}$*, then* $\nu_2(i(K)) = 1$.

4. *If* $a \equiv 32 \pmod{64}$, $b \equiv 0 \pmod{2^7}$ *and* $c \equiv 0 \pmod{2^{2\nu_2(b)-4}}$*, then* $\nu_2(i(K)) = 1$.

5. *If* $(a,b,c) \in \{(2,3,6),(2,7,2),(6,3,2),(6,7,6),(2,3,2),(2,7,6),(6,3,6),(6,7,2),(0,3,4),$ $(4,3,0)\} \pmod 8$*, then* $\nu_2(i(K)) = 1$.

6. *If* $a \equiv c \equiv 0,4 \pmod 8$ *and*

   *(a)* $b \equiv 7 \pmod 8$*, then* $\nu_2(i(K)) = 3$.

   *(b)* $b \equiv 3 \pmod 8$*, then* $\nu_2(i(K)) = 2$.

7. *If* $(a,b,c) \in \{(2,1,2),(18,17,18),(10,9,10),(26,25,26),(6,5,6),(22,21,22),(14,13,14),$ $(30,29,30),(2,17,2),(18,1,18),(10,25,10),(26,9,26),(6,21,6),(22,5,22),(14,29,14),$ $(30,13,30)\} \pmod{32}$*, then* $\nu_2(i(K)) = 2$.

8. *If* $(a,b,c) \in \{(0,5,2),(4,5,6),(2,1,4),(6,1,0)\} \pmod 8$ *and* $1+a+b+c \equiv 0 \pmod{16}$*, then* $\nu_2(i(K)) = 1$.

9. *If* $(a,b,c) \in \{(0,1,14),(8,1,6),(4,9,2),(12,9,10),(2,13,0),(10,13,8),(6,5,4),(14,5,12)\}$ $\pmod{16}$ *and* $1+a+b+c \equiv 0 \pmod{64}$*, then* $\nu_2(i(K)) = 1$.

10. *If* $(a,b,c) \in \{(0,9,22),(16,9,6),(8,25,30),(24,25,14),(4,1,26),(20,1,10),(12,17,2),$ $(28,17,18),(2,5,24),(18,5,8),(10,21,0),(26,21,16),(6,29,28),(22,29,12),(14,13,4),$ $(30,13,20)\} \pmod{32}$ *and* $1+a+b+c \equiv 0 \pmod{256}$*, then* $\nu_2(i(K)) = 1$.

*In particular, if any one of the above conditions holds, then* $K$ *cannot be monogenic.*

*Proof.* In each case, we show that 2 is a common index divisor of $K$ and we determine the exact value of $\nu_2(i(K))$. Throughout this proof, let

$$u := \nu_2(1+a+b+c). \tag{6}$$

1. $a \equiv 1 \pmod 2, b \equiv 0 \pmod 2$ and $c \equiv 0 \pmod{2^{2\nu_2(b)+1}}$. Reducing modulo 2, we see that $F(x) \equiv x^2(x-1)(x^4+x^3+x^2+x+1) \pmod 2$. Let $\phi_1(x) = x, \phi_2(x) = x-1$ and $\phi_3(x) = x^4+x^3+x^2+x+1$. By Corollary 1, the factor $\phi_2$ and $\phi_3$ provide two distinct prime ideals $\mathfrak{p}_{211}$ and $\mathfrak{p}_{311}$ of $\mathcal{O}_K$ having residue degrees 1 and 4 respectively. On the other hand, $\mathcal{N}^-_{\phi_1}(F)$ has two distinct sides of degree 1 each joining the points $(0,\nu_2(c)),(1,\nu_2(b))$ and $(2,0)$, because $\nu_2(c) \geq \nu_2(b)+1$. Therefore, by virtue of Theorem 1, $2\mathcal{O}_K = \mathfrak{p}_{111}\mathfrak{p}_{121}\mathfrak{p}_{211}\mathfrak{p}_{311}$, where $f(\mathfrak{p}_{111}/2) = f(\mathfrak{p}_{121}/2) = f(\mathfrak{p}_{211}/2) = 1$ and $f(\mathfrak{p}_{311}/2) = 4$. That is $2\mathcal{O}_K = [1^1,1^1,1^1,4^1]$. By Table 1, we have $\nu_2(i(K)) = 1$.

2. $(a,b,c) \in \{(1,15,15),(9,15,7),(5,7,3),(13,7,11)\} \pmod{16}$ and $1+a+b+c \equiv 0 \pmod{64}$. In this case, $F(x) \equiv (x^4+x^3+1)(x-1)^3 \pmod 2$. Let $\phi_1(x) = x^4+x^3+1$ and $\phi_2(x) = x-1$. The factor $\phi_1$ provides a unique prime ideal $\mathfrak{p}_{211}$ of $\mathcal{O}_K$ above 2 with residue degree 4. The $\phi_2$-adic development of $F(x)$ is

$$F(x) \;=\; \phi_2(x)^7 + 7\phi_2(x)^6 + 21\phi_2(x)^5 + 35\phi_2(x)^4 + 35\phi_2(x)^3 + (a+21)\phi_2(x)^2$$

$$+ \quad (2a+b+7)\phi_2(x) + 1 + a + b + c. \tag{7}$$

It follows by (6) and (7) that $\mathcal{N}^-_{\phi_2}(F) = \mathcal{S}_{21}+\mathcal{S}_{22}+\mathcal{S}_{23}$ joining the points $(0,u),(1,3),(2,1)$ and $(3,0)$ with $u \geq 6$. Thus the degree of $\mathcal{S}_{2j}, j=1,2,3$ equals 1. By Corollary 1, we have $2\mathcal{O}_K = [4^1, 1^1, 1^1, 1^1]$ and by Table 1, we deduce that $\nu_2(i(K)) = 1$.

3. $\nu_2(a) \in \{1,2,3,4\}$, $b \equiv 0 \pmod{2^{\nu_2(a)+1}}$ and $c \equiv 0 \pmod{2^{2\nu_2(b)-\nu_2(a)+1}}$. This condition implies that $\nu_2(b) \geq \nu_2(a)+1$ and $\nu_2(c) \geq 2\nu_2(b) - \nu_(a)+1$. In this case, $F(x) \equiv x^7 \pmod 2$. Let $\phi_1(x) = x$, then $\mathcal{N}^-_{\phi_1}(F)$ has three distinct sides joining $(0,\nu_2(c)),(1,\nu_2(b)),(2,\nu_2(a))$ and $(7,0)$. Thus the degree of each side is 1. Therefore, by Corollary 1(2), $2\mathcal{O}_K = [1^5, 1^1, 1^1]$. In view of Table 1, $\nu_2(i(K)) = 1$.

4. $a \equiv 32 \pmod{64}$, $b \equiv 0 \pmod{2^7}$ and $c \equiv 0 \pmod{2^{2\nu_2(b)-4}}$. Here, we have $\nu_2(a) = 5$, $\nu_2(b) \geq 7$ and $\nu_2(c) \geq 2\nu_2(b) - 4$ and $F(x) \equiv \phi_1(x)^7 \pmod 2$, where $\phi_1(x) = x$. Thus, $\mathcal{N}^-_{\phi_1}(F) = \mathcal{S}_{11} + \mathcal{S}_{12} + \mathcal{S}_{13}$ joining the points $(0,\nu_2(c)),(1,\nu_2(b)),(2,5)$ and $(7,0)$. So, $d(\mathcal{S}_{11}) = d(\mathcal{S}_{12}) = 1$. Moreover, the degree of $\mathcal{S}_{13}$ equals 5 and its attached residual polynomial is $\mathcal{R}_{\lambda_{13}}(F)(y) = 1 + y^5 = (y+1)(y^4+y^3+y^2+y+1)$. So, $F(x)$ is 2-regular. By applying Theorem 1 and using Table 1, we see that $2\mathcal{O}_K = [4^1, 1^1, 1^1, 1^1]$ and $\nu_2(i(K)) = 1$.

   • From case (5), we have $a \equiv c \equiv 0 \pmod 2$ and $b \equiv 1 \pmod 2$. Thus, $F(x) \equiv x(x-1)^2(x^2+x+1)^2 \pmod 2$. Let $\phi_1(x) = x$, $\phi_2(x) = x-1$ and $\phi_3(x) = x^2+x+1$. By Corollary 1(1), $\phi_1$ provides a unique prime ideal $\mathfrak{p}_{111}$ of $\mathcal{O}_K$ of residual degree 1 above 2. To complete the factorization of $2\mathcal{O}_K$, we analyze the contributions of $\phi_2$ and $\phi_3$ in each of the subsequent cases. The $\phi_3$-adic development of $F(x)$ is

$$F(x) = (x-3)\phi_3^3(x) + (3x+5)\phi_3^2(x) + (-4x+a-2)\phi_3(x) + (b-a+1)x + c - a. \tag{8}$$

   So, $\mathcal{N}^-_{\phi_3}(F)$ joins the points $(0, \min(\nu_2(b-a+1), \nu_2(c-a)))$, $(1, \min(2, \nu_2(a-2)))$ and $(2,0)$.

5. $(a,b,c) \in \{(2,3,6),(2,7,2),(6,3,2),(6,7,6),(2,3,2),(2,7,6),(6,3,6),(6,7,2),(0,3,4),$ $(4,3,0)\} \pmod 8$. We distinguish three sub-cases.

   (a) If $(a,b,c) \in \{(2,3,6),(2,7,2),(6,3,2),(6,7,6)\} \pmod 8$, then $\mathcal{N}^-_{\phi_3}(F)$ has a single side of degree 1 with ramification index 2. Also we have, $u = 2$. Thus, by (7), $\mathcal{N}^-_{\phi_2}(F) = \mathcal{S}_{21}$ has a single side of degree 2 joining $(0,2)$, $(1,1)$ and $(2,0)$. Its attached residual polynomial is $\mathcal{R}_{\lambda_{21}}(F)(y) = y^2+y+1$. So, $F(x)$ is 2-regular. Hence, by Theorem 1, $2\mathcal{O}_K = [1^1, 2^1, 2^2]$.

   (b) If $(a,b,c) \in \{(2,3,2),(2,7,6),(6,3,6),(6,7,2)\} \pmod 8$, then $\mathcal{N}^-_{\phi_3}(F)$ is the same as in the above sub-case. Moreover, $\mathcal{N}^-_{\phi_2}(F) = \mathcal{S}_{21} + \mathcal{S}_{21}$ joining $(0,u)$, $(1,1)$, $(2,0)$ with $u \geq 3$. So, the degree of each side is 1. Hence, by Corollary 1, $2\mathcal{O}_K = [1^1, 1^1, 1^1, 2^2]$.

   (c) If $(a,b,c) \in \{(0,3,4),(4,3,0)\} \pmod 8$, then $\mathcal{N}^-_{\phi_3}(F) = S_{31}$ has a single side of degree 2 joining the points $(0,2),(1,1)$ and $(2,0)$. Its attached residual polynomial is

$$R_{_{31}}(F)(y) = \begin{cases} (x+1)y^2 + y + 1, & \text{if } (a,b,c) \equiv (4,3,0) \pmod 8, \\ (x+1)y^2 + y + x + 1, & \text{if } (a,b,c) \equiv (0,3,4) \pmod 8. \end{cases}$$

which is separable in $\mathbb{F}_{\phi_3}[y]$. On the other hand, $\mathcal{N}^-_{\phi_2}(F) = \mathcal{S}_{21} + \mathcal{S}_{21}$ is the same as in the above sub-case. Therefore, $2\mathcal{O}_K = [1^1, 1^1, 1^1, 4^1]$.

In both sub-cases, by Table 1, $\nu_2(i(K)) = 1$ .

6. $a \equiv c \equiv 0, 4 \pmod 8$ and $b \equiv 3, 7 \pmod 8$. Two distinct cases arise.

   (a) $a \equiv c \equiv 0, 4 \pmod 8$ and $b \equiv 3 \pmod 8$. In this case, $\mathcal{N}^-_{\phi_2}(F)$ has a single side of degree 2 and its attached residual polynomial is $\mathcal{R}_{21}(F)(y) = y^2 + y + 1$. For $\phi_3$, we distinguish two sub-cases:

   label=$*$ If $(a, b, c) \equiv (4, 3, 4) \pmod 8$, then $\mathcal{N}^-_{\phi_3}(F) = \mathcal{S}_{31} + \mathcal{S}_{32}$ joining $(0, \mu)$, $(1, 1)$ and $(2, 0)$ with $\mu \geq 3$. So, the degree of each side is 1.

   lbbel=$*$ If $(a, b, c) \equiv (0, 3, 0) \pmod 8$, then $\mathcal{N}^-_{\phi_3}(F)$ has a single side of degree 2. Moreover, $\mathcal{R}_{31}(F)(y) = (y + 1)((x + 1)y + x)$ which is separable.

   In both cases, by Theorem 1 and Table 1, Hence $2\mathcal{O}_K = [2^1, 2^1, 2^1, 1^1]$ and $\nu_2(i(K)) = 2$.

   (b) $a \equiv c \equiv 0, 4 \pmod 8$ and $b \equiv 7 \pmod 8$. Here, we have $\mathcal{N}^-_{\phi_2}(F) = \mathcal{S}_{21} + \mathcal{S}_{21}$ joining $(0, u)$, $(1, 1)$ and $(2, 0)$ with $u \geq 3$. Thus the degree of each side is 1. For $\phi_3$, we have two sub-cases:

   label=$*$ If $(a, b, c) \equiv (4, 7, 4) \pmod 8$, then $\mathcal{N}^-_{\phi_3}(F)$ has a single of degree 2 and its attached residual polynomial is $\mathcal{R}_{31}(F)(y) = (y + 1)((x + 1)y + x) \in \mathbb{F}_{\phi_3}[y]$ which is separable.

   lbbel=$*$ If $(a, b, c) \equiv (0, 7, 0) \pmod 8$, then $\mathcal{N}^-_{\phi_3}(F)$ has two distinct sides of degree 1 each joining $(0, \mu)$, $(1, 1)$ and $(2, 0)$ with $\mu \geq 3$.

   In both cases, by Theorem 1 and Table 1, we have $2\mathcal{O}_K = [2^1, 2^1, 1^1, 1^1, 1^1]$ and $\nu_2(i(K)) = 3$.

7. $(a, b, c) \in \{(2, 1, 2), (18, 17, 18), (10, 9, 10), (26, 25, 26), (6, 5, 6), (22, 21, 22), (14, 13, 14), (30, 29, 30), (2, 17, 2), (18, 1, 18), (10, 25, 10), (26, 9, 26), (6, 21, 6), (22, 5, 22), (14, 29, 14), (30, 13, 30)\} \pmod{32}$. Here, $\mathcal{N}^-_{\phi_2}(F)$ has a single side of height 1 (so of degree 1), because $u = 1$. For $\phi_3$, we distinguish two sub-cases:

   (a) If $(a, b, c) \in \{(2, 1, 2), (18, 17, 18), (10, 9, 10), (26, 25, 26), (6, 5, 6), (22, 21, 22), (14, 13, 14), (30, 29, 30)\} \pmod{32}$, then $\mathcal{N}^-_{\phi_3}(F) = \mathcal{S}_{31} + \mathcal{S}_{32}$ joining $(0, \mu)$, $(1, 2)$ and $(2, 0)$ with $\mu \geq 5$. So, $d(\mathcal{S}_{31}) = d(\mathcal{S}_{32}) = 1$. Hence, $F(x)$ is 2-regular.

   (b) If $(a, b, c) \in \{(2, 17, 2), (18, 1, 18), (10, 25, 10), (26, 9, 26), (6, 21, 6), (22, 5, 22), (14, 29, 14), (30, 13, 30)\} \pmod{32}$, then $\mathcal{N}^-_{\phi_3}(F)$ has a single side of degree 2 and its attached residual polynomial is $\mathcal{R}_{31}(F)(y) = (y+1)((x+1)y+x)$ which is separable in $\mathbb{F}_{\phi_3}[y]$. So, $F(x)$ is 2-regular.

   In both cases, by Theorem 1 and Table 1, we get that $2\mathcal{O}_K = [2^1, 2^1, 1^2, 1^1]$ and $\nu_2(i(K)) = 2$.

8. $(a, b, c) \in \{(0, 5, 2), (4, 5, 6), (2, 1, 4), (6, 1, 0)\} \pmod 8$ and $1 + a + b + c \equiv 0 \pmod{16}$. In this case, $\mathcal{N}^-_{\phi_3}(F)$ has a single side of height 1. For $\phi_2$, we distinguish two sub-cases:

(a) If $1+a+b+c \equiv 0 \pmod{32}$ : $u \geq 5$, then $\mathcal{N}^{-}_{\phi_2}(F) = \mathcal{S}_{21}+\mathcal{S}_{22}$ joining $(0,u)$, $(1,2)$ and $(2,0)$. So, the degree of each side is 1. Thus, $F(x)$ is 2-regular. Hence, by Theorem 1 $2\mathcal{O}_K = [1^1, 1^1, 1^1, 2^2]$.

(b) If $1+a+b+c \equiv 16 \pmod{32}$ : $u = 4$, then $\mathcal{N}^{-}_{\phi_2}(F) = \mathcal{S}_{21}$ has a single side of degree 2 joining $(0,4)$, $(1,2)$ and $(2,0)$. Moreover, $\mathcal{S}_{21} = \mathcal{R}_{21}(F)(y) = y^2+y+1$. Applying Theorem, we get 1 $2\mathcal{O}_K = [2^2, 2^1, 1^1]$.

Consequently, by Table 1, $\nu_2(i(K)) = 1$ in both these sub-cases.

9. If $(a,b,c) \in \{(0,1,14),(8,1,6),(4,9,2),(12,9,10),(2,13,0),(10,13,8),(6,5,4),(14,5,12)\}$ $\pmod{16}$ and $1+a+b+c \equiv 0 \pmod{64}$. Here, $\mathcal{N}^{-}_{\phi_3}(F)$ has a single side of height 1. By Corollary 1(2), $\phi_3$ provides one prime ideal of $\mathcal{O}_K$ of residue degree 2 with ramification index 2 above 2. For $\phi_2$, we distinguish two sub-cases:

   (a) If $1+a+b+c \equiv 64 \pmod{128}$ : $u = 6$, then $\mathcal{N}^{-}_{\phi_2}(F) = \mathcal{S}_{21}$ has a single side of degree 2 joining $(0,6)$, $(1,3)$ and $(2,0)$. This yields $\mathcal{R}_{21}(F)(y) = y^2+y+1$ which is separable. So, $F(x)$ is 2-regular. Using Theorem 1, one gets: $2\mathcal{O}_K = [2^2, 2^1, 1^1]$.

   (b) If $1+a+b+c \equiv 0 \pmod{128}$ : $u \geq 7$, then $\mathcal{N}^{-}_{\phi_2}(F) = \mathcal{S}_{21}+\mathcal{S}_{22}$ has two distinct sides of degree 1 each joining $(0,u)$, $(1,3)$ and $(2,0)$. Therefore, by Theorem 1 $2\mathcal{O}_K = [2^2, 1^1, 1^1, 1^1]$.

   According to Table 1, we conclude that $\nu_2(i(K)) = 1$.

10. The proof of this case is similar to the above one, so it is omitted.

$\square$

The next theorem identifies parametric families of septic number fields whose indices are divisible by 3 and provide the exact value of $\nu_3(i(K))$ for each family.

**Theorem 4.** *With the above notation, we have the following statements:*

1. *If $a \equiv c \equiv 0 \pmod 9$ and $b \equiv -1 \pmod 9$, then $\nu_3(i(K)) = 2$.*

2. *If $(a,b,c) \in \{(0,5,6),(3,2,0),(3,5,3),(6,2,6)\} \pmod 9$, then $\nu_3(i(K)) = 1$.*

3. *If $(a,b,c) \in \{(0,5,3),(3,2,3),(6,2,0),(6,5,6)\} \pmod 9$, then $\nu_3(i(K)) = 1$.*

4. *If $(a,b,c) \in \{(0,74,6),(27,20,33),(54,47,60),(9,56,15),(36,2,42),(63,29,69),(18,38,24),(45,65,51),(72,11,78)\} \pmod{81}$ and $1+a+b+c \equiv 486 \pmod{729}$, or $(a,b,c) \in \{(3,68,9),(30,14,36),(57,41,63),(12,50,18),(39,77,45),(66,23,72),(21,32,27),(48,59,54),(75,5,0)\} \pmod{81}$ and $1+a+b+c \equiv 243 \pmod{729}$, then $\nu_3(i(K)) = 2$.*

5. *If $(a,b,c) \in \{(3,8,6),(3,35,33),(3,62,60),(30,8,60),(30,35,6),(30,62,33),(57,8,33),(57,35,60),(57,62,6),(12,26,15),(12,53,42),(12,80,69),(39,26,69),(39,53,15),(39,80,42),(66,26,42),(66,53,69),(66,80,15),(21,17,78),(21,44,24),(21,71,51),(48,17,51),(48,44,78),(48,71,24),(75,17,24),(75,44,51),(75,71,78)\} \pmod{81}$, then $\nu_3(i(K)) = 3$.*

6. *If* $(a,b,c) \in \{(6,17,57),(6,44,30),(6,71,3),(33,17,30),(33,44,3),(33,71,57),(60,17,3),(60,44,57),(60,71,30),(15,26,39),(15,53,12),(15,80,66),(42,26,12),(42,53,66),(42,80,39),(69,26,66),(69,53,39),(69,80,12),(24,8,48),(24,35,21),(24,62,75),(51,35,75),(51,8,21),(51,62,48),(78,8,75),(78,35,48),(78,62,21)\} \pmod{81}$*, then* $\nu_3(i(K)) = 3$.

7. *If* $(a,b,c) \in \{(0,11,12),(9,2,21),(18,20,3),(6,14,9),(15,5,18),(24,23,0)\} \pmod{27}$ *and* $4a+2b+c+128 \equiv 81, 162 \pmod{243}$*, then* $\nu_3(i(K)) = 1$.

8. *If* $(a,b,c) \in \{(6,8,66),(6,35,39),(33,35,12),(33,62,66),(60,8,12),(60,62,39),(15,17,48),(15,71,75),(42,17,21),(42,44,75),(69,44,48),(69,71,21),(24,53,3),(24,80,57),(51,26,3),(51,80,30),(78,26,57),(78,53,30)\} \pmod{81}$ *and* $1+a+b+c \equiv 0 \pmod{243}$*, then* $\nu_3(i(K)) = 2$.

9. *If* $(a,b,c) \in \{(6,14,63),(33,68,9),(60,41,36),(15,59,18),(42,32,45),(69,5,72),(24,23,54),(51,77,0),(78,50,27)\} \pmod{81}$ *and* $4a+2b+c+128 \equiv 486 \pmod{729}$*, or* $(a,b,c) \in \{(0,38,39),(27,11,66),(54,65,12),(9,2,75),(36,56,21),(63,29,48),(18,47,30),(45,20,57),(72,74,3)\} \pmod{81}$ *and* $4a+2b+c+128 \equiv 243 \pmod{729}$*, then* $\nu_3(i(K)) = 2$.

10. *If* $(a,b,c) \in \{(0,74,6),(27,20,33),(54,47,60),(9,56,15),(36,2,42),(63,29,69),(18,38,24),(45,65,51),(72,11,78)\} \pmod{81}$ *and* $a+b+c+1 \equiv 486 \pmod{729}$*, or* $(a,b,c) \in \{(3,68,9),(30,14,36),(57,41,63),(12,50,18),(39,77,45),(66,23,72),(21,32,27),(48,59,54),(75,5,0)\} \pmod{81}$ *and* $a+b+c+1 \equiv 243 \pmod{729}$*, then* $\nu_3(i(K)) = 2$.

11. *If* $(a,b,c) \in \{(0,20,60),(0,47,33),(27,47,6),(27,74,60),(54,20,6),(54,74,33),(9,2,69),(9,29,42),(36,29,15),(36,56,69),(63,2,15),(63,56,42),(18,11,51),(18,65,78),(45,11,24),(45,38,78),(72,38,51),(72,65,24),(3,14,63),(3,41,36),(30,41,9),(30,68,63),(57,14,9),(57,68,36),(12,23,45),(12,77,72),(39,23,18),(39,50,72),(66,50,45),(66,77,18),(21,5,54),(21,59,0),(48,5,27),(48,32,0),(75,32,54),(75,59,27)\} \pmod{81}$ *and* $a+b+c+1 \equiv 0 \pmod{729}$*, then* $\nu_3(i(K)) = 2$.

12. *If* $(a,b,c) \in \{(15,44,183),(96,125,21),(177,206,102),(42,233,210),(123,71,48),(204,152,129),(69,179,237),(150,17,75),(231,98,156),(24,26,192),(105,107,30),(186,188,111),(51,215,219),(132,53,57),(213,134,138),(78,161,3),(159,242,84),(240,80,165)\} \pmod{243}$ *and* $a+b+c+1 \equiv 0 \pmod{2187}$*, then* $\nu_3(i(K)) = 1$.

13. *If* $(a,b,c) \in \{(15,206,21),(69,44,102),(177,125,183),(42,152,48),(123,233,129),(204,71,210),(69,98,75),(150,179,156),(231,17,237),(24,188,30),(105,26,111),(186,107,192),(51,134,57),(132,215,138),(213,53,219),(78,80,84),(159,161,165),(240,242,3)\} \pmod{243}$ *and* $2a+b+7 \equiv 243, 486 \pmod{729}$ *and* $a+b+c+1 \equiv 0 \pmod{3^9}$*, then* $\nu_3(i(K)) = 3$.

*In particular, if any one of the above conditions holds, then $K$ cannot be monogenic.*

***Proof of Theorem 4.*** First, we note that in all cases, we have $(a,b,c) \equiv (0,-1,0) \pmod 3$. Thus, $F(x) \equiv x(x-1)^3(x-2)^3 \pmod 3$. Let $\phi_1(x) = x, \phi_2(x) = x-1$ and $\phi_3(x) = x-2$. By Corollary 1(1), $\phi_1$ provides a unique prime ideal $\mathfrak{p}_{111}$ of $\mathcal{O}_K$ of residual degree 1 above 3. We consider the $\phi_2$-adic development (7) of $F(x)$ and let $u_0 = \nu_3(a+b+c+1)$, $u_1 = \nu_3(2a+b+7)$ and

$u_2 = \nu_3(a+21)$. It follows that $\mathcal{N}^-_{\phi_2}(F)$ is the lower convex hull of the points $(0, u_0), (1, u_1), (2, u_2)$ and $(3, 0)$. On the other hand, the $\phi_3$-adic development of $F(x)$ is

$$\begin{aligned} F(x) &= \phi_3(x)^7 + 14\phi_3(x)^6 + 84\phi_3(x)^5 + 280\phi_3(x)^4 + 560\phi_3(x)^3 + (a+672)\phi_3(x)^2 \\ &+ (4a+b+448)\phi_3(x) + 4a+2b+c+128 \end{aligned} \tag{9}$$

Set $w_0 = \nu_3(4a+2b+c+128)$, $w_1 = \nu_3(4a+b+448)$ and $w_2 = \nu_3(a+672)$. So, $\mathcal{N}^-_{\phi_3}(F)$ is the lower convex hull of the points $(0, w_0), (1, w_1), (2, w_2)$ and $(3, 0)$.

In what follows, in each case stated in the theorem, we decompose the prime 3 in $\mathcal{O}_K$ and we determine $\nu_3(i(K))$.

1. If $a \equiv c \equiv 0 \pmod 9$ and $b \equiv -1 \pmod 9$, then $\mathcal{N}^-_{\phi_2}(F) = \mathcal{S}_{21} + \mathcal{S}_{22}$ joining $(0, u_0)$, $(1, 1)$ and $(3, 0)$ with $u_0 \geq 2$, thus the degree of each side is 1. On the other hand, $\mathcal{N}^-_{\phi_3}(F) = \mathcal{S}_{31} + \mathcal{S}_{32}$ joining $(0, w_0)$, $(1, 1)$ and $(3, 0)$ with $w_0 \geq 2$, therefore the degree of each side is 1. By Corollary 1(1), $3\mathcal{O}_K = [1^2, 1^2, 1^1, 1^1, 1^1]$. Hence, 3 divides $i(K)$ and according to Table 1, $\nu_3(i(K)) = 2$.

2. If $(a, b, c) \in \{(0, 5, 6), (3, 2, 0), (3, 5, 3), (6, 2, 6)\} \pmod 9$, then $\mathcal{N}^-_{\phi_2}(F)$ has a single side of height 1. On the other hand, $\mathcal{N}^-_{\phi_3}(F) = \mathcal{S}_{31} + \mathcal{S}_{32}$ joining $(0, w_0)$, $(1, 1)$ and $(3, 0)$ with $w_0 \geq 2$, thus the degree of each side is 1. Using Corollary 1, we get, $3\mathcal{O}_K = [1^3, 1^2, 1^1, 1^1]$. Consequently, $\nu_3(i(K)) = 1$.

3. If $(a, b, c) \in \{(0, 5, 3), (3, 2, 3), (6, 2, 0), (6, 5, 6)\} \pmod 9$, then $\mathcal{N}^-_{\phi_3}(F)$ has a single side of height 1. Moreover, $\mathcal{N}^-_{\phi_2}(F) = \mathcal{S}_{21} + \mathcal{S}_{22}$ joining $(0, u_0)$, $(1, 1)$ and $(3, 0)$ with $u_0 \geq 2$, thus the degree of each side is 1. By Theorem, 1, $3\mathcal{O}_K = [1^3, 1^2, 1^1, 1^]$. Hence, $\nu_3(i(K)) = 1$.

• For shortness, from case (4), we will not recall the conditions, the reader can see them in the statement of the theorem.

4. In this case, $\mathcal{N}^-_{\phi_3}(F)$ has a single side of height 1. Also, $\mathcal{N}^-_{\phi_2}(F) = \mathcal{S}_{21} + \mathcal{S}_{22}$ joining $(0, 5), (1, 3), (2, 1)$ and $(3, 0)$. Thus, $d(\mathcal{S}_{22}) = 1$ and $d(\mathcal{S}_{22}) = 2$. Further, we have

$$R_{22}(F)(y) = \begin{cases} (y-1)(y-2), & \text{if } (a,b,c) \equiv (0,2,6) \pmod 9, \\ 2(y-1)(y-2), & \text{if } (a,b,c) \equiv (3,5,0) \pmod 9. \end{cases}$$

which is separable in $\mathbb{F}_{\phi_3}[y] \simeq \mathbb{F}_3[y]$. Therefore, by Theorem 1, $3\mathcal{O}_K = [1^3, 1^1, 1^1, 1^1, 1^1]$. Consequently, by Table 1, $\nu_3(i(K)) = 2$.

5. In this case, $\mathcal{N}^-_{\phi_2}(F) = \mathcal{S}_{21} + \mathcal{S}_{22}$ has two distinct sides of degree 1 each joining $(0, u_0), (1, 1)$ and $(3, 0)$ with $u_0 \geq 2$. On the other hand, $\mathcal{N}^-_{\phi_3}(F) = \mathcal{S}_{31} + \mathcal{S}_{32}$ joining $(0, w_0), (1, 2)$ and $(3, 0)$ with $w_0 \geq 2$, then $d(\mathcal{S}_{31}) = 1$ and the degree of $\mathcal{S}_{32}$ is 2. The attached residual polynomial to the second side is $R_{32}(F)(y) = 2y^2 + 1 = 2(y-1)(y-2)$ which is separable in $\mathbb{F}_{\phi_3}[y]$. In view of Theorem 1, $3\mathcal{O}_K = [1^2, 1^1, 1^1, 1^1, 1^1, 1^1]$. It follows by Table 1 that $\nu_3(i(K)) = 3$.

6. Here, $\mathcal{N}_{\phi_3}^{-}(F) = \mathcal{S}_{31} + \mathcal{S}_{32}$ has two distinct sides of degree 1 each joining $(0, w_0), (1, 1)$ and $(3, 0)$ with $w_0 \geq 2$, and $\mathcal{N}_{\phi_2}^{-}(F) = \mathcal{S}_{21} + \mathcal{S}_{22}$ joining $(0, u_0), (1, 2)$ and $(3, 0)$ with $u_0 \geq 4$. This implies that $d(\mathcal{S}_{21}) = 1$ and $d(\mathcal{S}_{22}) = 2$. Furthermore, $R_{22}(F)(y) = 2y^2 + 1 = 2(y-1)(y-2)$ which is separable in $\mathbb{F}_{\phi_2}[y]$. Therefore, by Theorem 1, $3\mathcal{O}_K = [1^2, 1^1, 1^1, 1^1, 1^1, 1^1]$. Hence, $\nu_3(i(K)) = 3$.

7. Here, we have $\mathcal{N}_{\phi_3}^{-}(F) = \mathcal{S}_{31} + \mathcal{S}_{32}$ has two distinct sides of degree 1 each joining $(0, 4), (2, 1)$ and $(3, 0)$. Also, we have $\mathcal{N}_{\phi_2}^{-}(F) = \mathcal{S}_{21}$ has a single side of height 1. Applying Theorem 1, one has: $3\mathcal{O}_K = [1^3, 1^2, 1^1, 1^1]$, thus in view of Table 1, $\nu_3(i(K)) = 1$.

8. Here, $\mathcal{N}_{\phi_3}^{-}(F) = \mathcal{S}_{31} + \mathcal{S}_{32}$ has two distinct sides of degree 1 each joining $(0, w_0), (1, 1)$ and $(3, 0)$ with $w_0 \geq 2$, and $\mathcal{N}_{\phi_2}^{-}(F) = \mathcal{S}_{21} + \mathcal{S}_{22}$ has two distinct sides of degree 1 each joining $(0, u_0), (1, 3)$ and $(3, 0)$ with $u_0 \geq 5$. By Corollary 1, $3\mathcal{O}_K = [1^2, 1^2, 1^1, 1^1, 1^1]$. So, $\nu_3(i(K)) = 2$.

9. In this case, $\mathcal{N}_{\phi_2}^{-}(F)$ has a single side of height 1. On the other hand, $\mathcal{N}_{\phi_3}^{-}(F) = \mathcal{S}_{31} + \mathcal{S}_{32}$ joining $(0, 5), (2, 1)$ and $(3, 0)$. Thus, $d(\mathcal{S}_{32}) = 1$ and the degree of $\mathcal{S}_{31}$ is 2. Moreover, we have
$$\mathcal{R}_{31}(F)(y) = \begin{cases} (y-1)(y-2), & \text{if } (a,b,c) \equiv (0,2,3) \pmod 9, \\ 2(y-1)(y-2), & \text{if } (a,b,c) \equiv (6,5,0) \pmod 9. \end{cases}$$
It follows that $F(x)$ is 2-regular. Therefore, by Theorem 1, $3\mathcal{O}_K = [1^3, 1^1, 1^1, 1^1, 1^1]$. Consequently, by Table 1, $\nu_3(i(K)) = 2$.

10. This case is similar to Case (4), so $3\mathcal{O}_K = [1^3, 1^1, 1^1, 1^1, 1^1]$ and $\nu_3(i(K)) = 2$.

11. Here, $\mathcal{N}_{\phi_3}^{-}(F) = \mathcal{S}_{31}$ has a single side of height 1 and $\mathcal{N}_{\phi_2}^{-}(F) = \mathcal{S}_{21} + \mathcal{S}_{22} + \mathcal{S}_{23}$ has three distinct sides of degree 1 each joining $(0, u_0), (1, 3), (2, 1)$ and $(3, 0)$ with $u_0 \geq 6$. By Corollary 1, $3\mathcal{O}_K = [1^3, 1^1, 1^1, 1^1, 1^1]$ and by Table 1, $\nu_3(i(K)) = 2$.

12. In this case, $\mathcal{N}_{\phi_3}^{-}(F) = \mathcal{S}_{31} + \mathcal{S}_{32}$ has two distinct sides of degree 1 each joining $(0, w_0), (1, 1)$ and $(3, 0)$ with $w_0 \geq 2$. Also, $\mathcal{N}_{\phi_2}^{-}(F) = \mathcal{S}_{21} + \mathcal{S}_{22}$ joining $(0, u_0), (1, 4)$ and $(3, 0)$ with $u_0 \geq 7$. This yields that $d(\mathcal{S}_{21}) = 1$ and $d(\mathcal{S}_{22}) = 2$. Also, we have $\mathcal{R}_{22}(F)(y) = 2y^2 + y + 1$ which is separable in $\mathbb{F}_{\phi_2}[y] \simeq \mathbb{F}_3[y]$. In view of Theorem 1, $3\mathcal{O}_K = [2^1, 1^2, 1^1, 1^1, 1^1]$. Consequently, $\nu_3(i(K)) = 1$.

13. Here, $\mathcal{N}_{\phi_3}^{-}(F) = \mathcal{S}_{31} + \mathcal{S}_{32}$ has two distinct sides of degree 1 each joining $(0, w_0), (1, 1)$ and $(3, 0)$ with $w_0 \geq 2$. In addition, $\mathcal{N}_{\phi_2}^{-}(F) = \mathcal{S}_{21} + \mathcal{S}_{22} + \mathcal{S}_{23}$ has three distinct sides of degree 1 each joining $(0, u_0), (1, 5), (2, 2)$ and $(3, 0)$ with $u_0 \geq 9$. Therefore, $3\mathcal{O}_K = [1^2, 1^1, 1^1, 1^1, 1^1, 1^1]$ and $\nu_3(i(K)) = 3$. This completes the proof.

□

The next two corollaries follow immediately from the above theorems. We start by the following one, which gives explicitly infinite families of non-monogenic septic number fields whose indices are simultaneously divisible by 2 and 3 with different exponents.

**Corollary 2.** *Let $F(x)$ and $K$ be as above, then we have the following statements:*

1. *If* $(a,b,c) \in \{(66,47,18),(18,59,6),(30,59,66),(6,47,6),(18,59,66),(30,47,66),(24,11,36),(60,59,24)\} \pmod{72}$, *then* $i(K)=6$.
2. *If* $(a,b,c) \equiv (63,62,0) \pmod{72}$, *then* $i(K)=18$.
3. *If* $a \equiv c \equiv 0,36 \pmod{72}$ *and* $b \equiv 35 \pmod{72}$, *then* $i(K)=36$.
4. *If* $(a,b,c) \in \{(2274,1313,2082),(303,233,1866),(1734,1637,678),(258,593,930),(2130,737,210),(2022,2357,966)\} \pmod{2592}$, *then* $i(K)=108$.
5. *If* $(a,b,c) \in \{(408,575,168),(84,359,276),(336,431,96),(12,647,636)\} \pmod{648}$, *then* $i(K)=216$.

*In particular, $K$ is not monogenic.*

The following corollary is about the case when $a=0$. It provides infinite families of non-monogenic number fields defined by trinomials of type $x^7+bx+c$.

**Corollary 3.** *Let $K=\mathbb{Q}(\alpha)$ with $\alpha$ a root of an irreducible trinomial $F(x)=x^7+bx+c$, then the following hold:*

1. *If* $(b,c) \equiv (3,4) \pmod{8}$, *then* $\nu_2(i(K))=1$.
2. *If* $c \equiv 0 \pmod{8}$ *and*
   (a) $b \equiv 3 \pmod{8}$, *then* $\nu_2(i(K))=2$.
   (b) $b \equiv 7 \pmod{8}$, *then* $\nu_2(i(K))=3$.
3. *If* $b \equiv 5 \pmod{9}$ *and* $c \equiv 3,6 \pmod{9}$, *then* $\nu_3(i(K))=1$.

*In particular, $K$ is not monogenic.*

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