

Sediment transport control in a peripheral sluice-gate band system using the nonlinear Saint-Venant–Exner model

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Abstract. Rivers, dams, and lakes undergo constant morphological changes, driven by the dynamic interactions between transported sediments, flow fluctuations, and mobile boundaries vulnerable to erosion. This study proposes an innovative method for controlling sediment transport in water reservoirs (lakes, dams, rivers). The system combines a peripheral band equipped with sluice gates and advanced mathematical modeling. The band channels and regulates flows around the basin before reinjecting them via the sluice gates. A closed-loop control system adjusts their opening in real time to stabilize and minimize upstream sediment transport. Numerical simulations demonstrate the system's effectiveness in protecting reservoirs against siltation.

Keywords: Water reservoir, sediment transport, feedback control, nonlinear Saint-Venant-Exner model.

AMS Subject Classification 2010: 93B30, 93B52, 76M27, 76N25, 76M15, 76F70, 76M12.

1 Introduction

The natural behavior of water reservoirs such as rivers, lakes, and dams is characterized by continuous morphological changes, driven by the complex interactions between transported sediments, flow variations, and mobile boundaries subject to erosion. If the flow and sedimentation in these rivers are not controlled, excessive erosion of riverbeds can jeopardize the stability of infrastructures such as bridges, levees, and other hydraulic control structures. Furthermore, sediment deposits in channels can lead to

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chronic flooding of surrounding floodplains, leading to significant and avoidable reconstruction costs. Similarly, water reservoirs receiving sediment-laden inflows risk gradually losing their storage capacity, thereby reducing their useful lifespan, economic value, and associated benefits.

Remediation techniques, such as periodic dredging, reservoir flushing, or lake cleaning, are often implemented. However, these approaches remain expensive and challenging to sustain over the long term. These limitations underscore the need to develop advanced solutions to control sedimentation before it becomes problematic. An effective approach involves strategically managing hydraulic networks to mitigate the effects of sedimentation.

The one-dimensional form of the Saint-Venant-Exner equations is widely used to model bedload sediment transport on large temporal and spatial scales. This model combines shallow water equations for flow with a morphodynamic continuity equation for bathymetry. In the literature, various methods have been developed to design control systems based on this model. For example, the Lyapunov method [5] has been employed to establish dissipative boundary conditions ensuring exponential stability for one-dimensional linear hyperbolic systems. Other works, such as [6] and [7], have proposed techniques using linear feedback or the backstepping method to stabilize system states. These approaches have primarily focused on stabilization around an equilibrium state, but operational challenges remain for proactive management of sediment deposits.

Over the past decades, optimal control and the stabilization of systems arising from the mathematical modeling of physical phenomena have attracted increasing attention from the scientific community. This growing interest has led to the development of numerous theoretical and numerical approaches aimed at ensuring the stability, robustness, and performance of dynamical systems. For instance, the authors in [17] proposed a novel robust learning-based controller combining sliding mode control with reinforcement learning for a class of uncertain SISO linear systems with unknown relative degree. In [4], a Lyapunov functional approach was developed to establish the exponential stability of the nonlinear Kawahara equation with a time-varying internal delay posed on a bounded domain.

Significant advances have also been achieved in the field of optimal control of sediment transport and deposition. As early as 1995, Carriaga Carlos C. and co-authors [3] developed optimization models for reservoir releases to minimize riverbed aggradation and degradation processes. Subsequently, hybrid approaches combining hydrodynamic models with genetic algorithms were proposed to mitigate excessive sedimentation in multi-reservoir river systems [12]. More recently, variational techniques based on adjoint equations, together with rigorous numerical methods, have enabled the optimization of control strategies for more complex hydraulic configurations [1, 8, 11, 15].

Despite these advances, relatively few studies have focused on exploiting the existing stability and control results available in the literature to design effective control laws for regulating sediment transport in natural lakes and water reservoirs. The development of such control strategies is of considerable practical importance, as it contributes to preserving reservoir storage capacity, reducing sediment accumulation, and promoting the sustainable management of water resources.

This study proposes a control technique based on the use of a hydraulic barrier equipped with controlled sluice openings around a water reservoir. This device acts as a physical and hydraulic management structure that alters water flows and sediment transport mechanisms. By adapting the design and configuration of the sluice gates to the specific characteristics of the site, it is possible to stabilize flow and sedimentation in connected channels. Each sluice gate is controlled to ensure an optimal distribution of discharges, thereby limiting sediment transport into the reservoir and promoting upstream deposition. This innovative device, combined with advanced modeling techniques, constitutes a promising solution

to mitigate sedimentation issues in lakes and reservoirs, while preserving their long-term sustainability and functionality.

The remainder of this paper is organized as follows. Section 2 introduces the sediment transport problem in water reservoirs systems and presents the proposed control strategy. Section 3 develops the mathematical framework used to analyze the problem. Section 4 describes the governing equations for the flow in the lateral channels and the reservoir. Preliminary analytical results are established in Section 5. The implementation of the proposed control strategy is presented in Section 6. Finally, Section 7 is devoted to the numerical simulations and the discussion of the corresponding results.

2 Problem analysis and control technique description

2.1 Analysis of the siltation problem in water reservoirs

Sedimentation is a natural geological phenomenon resulting from erosion. Several types of erosion contribute to this process, including water erosion, wind erosion, and erosion caused by soil tillage activities. Water erosion, in particular, is a natural phenomenon whose intensity depends on factors such as precipitation, the strength of the current, soil type, slope, drainage, and the terrain's exposure to weather conditions.

Under the action of water, sediments are transported from watersheds through various waterways until they reach natural reservoirs such as lakes. This transport, which can cover varying distances, depends on soil characteristics and topography. Coarse sediments, due to their weight, settle quickly, contributing to the silting of water reservoirs. These deposits gradually reduce the storage capacity of natural reservoirs. To prevent this phenomenon, it is essential to implement an effective sediment transport control system upstream of water reservoirs. Such an approach enables the prediction and regulation of sediment transport dynamics to minimize deposits in these reservoirs and preserve their storage capacity.

2.2 Operation of a hydraulic barrier with controlled openings

A hydraulic barrier equipped with controlled sluice openings is a peripheral structure that surrounds or delineates a water reservoir, often equipped with devices such as sluice gates (calibrated openings) to regulate water flow. The sluice gates allow control over the water discharge passing through the band by slowing down the flow, promoting the deposition of heavy sediments, and reducing sediment transport downstream. Thus, a hydraulic barrier equipped with controlled sluice openings can be used to control sediment transport on multiple levels, including:

- **Reduction of water velocity**

By slowing down water flow, the hydraulic barrier reduces its sediment transport capacity. This promotes the sedimentation of suspended particles, particularly coarser particles such as sand and gravel.

- **Promotion of sedimentation**

The sluice gates can be designed to act as filters or temporary retention zones where sediments have time to settle. These deposits reduce the sediment load in downstream waterways.

- **Flood control**

By limiting peak discharges during periods of heavy rainfall, the hydraulic barrier equipped with controlled sluice openings reduces erosion in upstream areas and near watercourses.

- **Protection of downstream ecosystems**

By reducing sediment inflow, it minimizes siltation issues in reservoirs, irrigation channels, and wetlands, thereby protecting infrastructure and biodiversity.

2.3 Description of the control technique

To control sediment transport upstream of water reservoirs such as lakes, we propose a system based on the use of a hydraulic barrier equipped with controlled sluice openings around these reservoirs. The purpose of this band is to redirect lateral flows into various channels, depending on the terrain topography. The collected flows are then directed into the water reservoir through points equipped with control devices such as sluice gates or adjustable spillways.

In parallel, a mathematical modeling of flow and sediment transport in these channels is implemented. The objective is to design control laws that stabilize the flows and sediment fluxes in the lateral channels. This stabilization aims to bring the system to an optimal state, reducing sediment transport into the reservoir and promoting their deposition upstream in specific sections of the channels that can serve as dredging zones.

Finally, the evolution of flows and the reservoir bed is regularly monitored to assess the system's effectiveness.

This system acts as a physical and hydraulic management structure, modifying water flows and sediment transport mechanisms at the reservoir inlets to minimize silting and preserve storage capacity. Figure 1 provides an illustration of the implementation of the proposed technique.

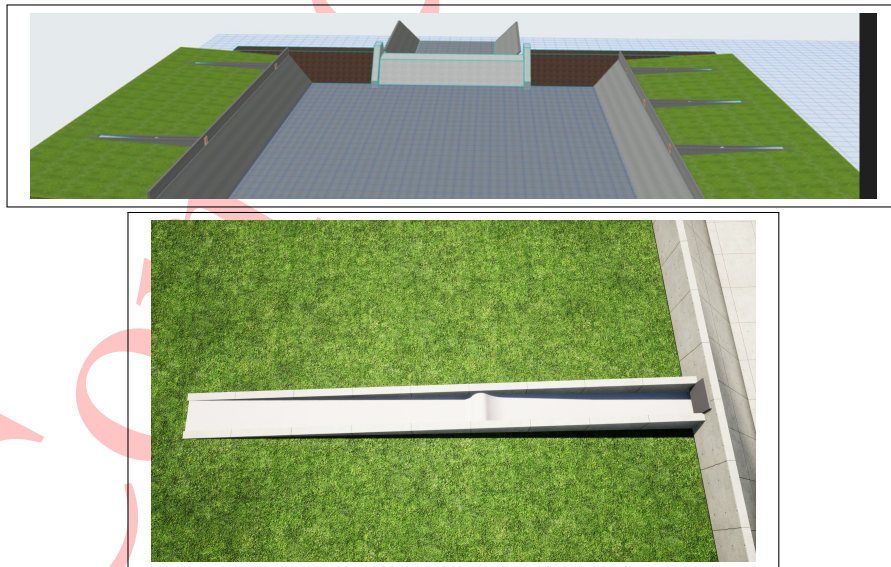


Figure 1: Illustration of the control technique

3 Mathematical approach and Main result

The previously described reservoir protection technique represents an innovative approach to controlling and reducing sediment transport within a watershed. Its implementation is structured around three main steps (see Figure 2):

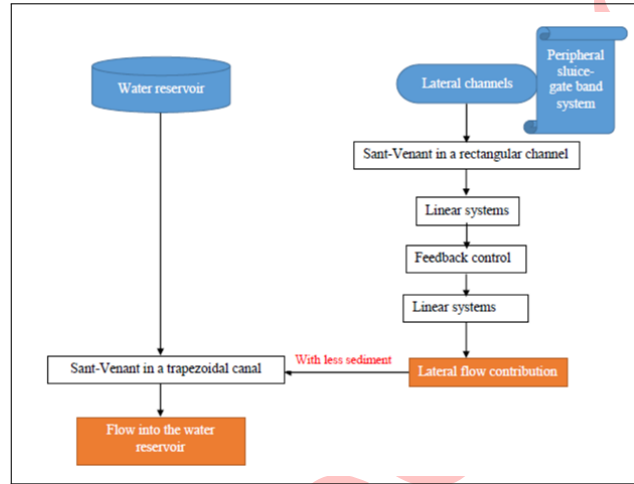


Figure 2: General modeling procedure

- **Step 1: Modeling and stabilizing lateral channels**

First, a model is developed to describe the behavior of water and sediment dynamics in the lateral channels. Once this model is established, a closed-loop feedback control system is designed to stabilize the states of the system. This control allows for precise regulation of discharges at the sluice gates to effectively limit sediment transport into the reservoir.

- **Step 2: Modeling lateral inflows into the reservoir**

Next, the lateral inflows into the reservoir are modeled, taking into account the location of control systems, such as sluice gates, relative to the reservoir inlet. This step optimizes the management of lateral flows to reduce upstream silting.

- **Step 3: Modeling flow within the reservoir**

Finally, the flow within the reservoir is modeled to observe the evolution of discharges and the bed level. This step provides an overview of the impact of the control system on sediment dynamics and the state of the reservoir.

These combined steps enable proactive and sustainable management of sediment transport, thereby preserving the capacity and functionality of water reservoirs.

4 Set of governing equations for flow in lateral channels and the reservoir

Let us consider a water reservoir surrounded by a hydraulic barrier equipped with controlled sluice openings equipped with sluice gates, which serve as entry points for flows through the associated pipelines,

as illustrated in Figure 1. The system is activated and dynamic during rainfall events.

4.1 Set of governing equations for flow in lateral channels

To model the behavior of water and sediments in each channel k , we use the nonlinear Saint-Venant-Exner model. This model couples the one-dimensional nonlinear shallow water equations with the sediment continuity equation (Exner equation) and is expressed as follows:

$$\begin{cases} \partial_t H_k + \partial_x (H_k V_k) = 0, \\ \partial_t (H_k V_k) + \partial_x \left(H_k V_k^2 + g \frac{H_k^2}{2} \right) = -g H_k \partial_x B_k, \\ (1 - \Phi_k) \partial_t B_k + \partial_x Q_{s,k} = 0. \end{cases} \quad (1)$$

In system (1), the variables H_k , V_k , B_k , and $Q_{s,k}$ respectively represent the water column height, velocity, bathymetry, and sediment transport flux in channel k , where the parameter Φ_k , such that $0 \leq \Phi_k < 1$, represents the sediment porosity.

In this work, the sediment flux is expressed as follows (see [6, Section 2.1, P.390], [16, P.98]):

$$Q_{s,k} = A_g H_k V_k - \varepsilon \partial_x B_k, \quad (2)$$

where ε is the positive diffusion coefficient of the sediment, and A_g is a given physical parameter depending on the sediment particle properties.

4.2 Set of governing equations for flow in the water reservoir

We approximate the water flow in the reservoir to that of trapezoidal open channels. This approach is more realistic than that of rectangular channels, as certain parameters of the reservoir, such as the base width and side slope, can vary along the flow. The water flow in such channels is derived from the most general version of the Saint-Venant equations, which applies to channels with non-uniform slopes and cross-sectional areas, given by:

$$\begin{cases} \partial_t A + \partial_x (Q) = q_L, \\ \partial_t Q + \partial_x \left(\frac{Q^2}{A} \right) + g A [\partial_x H - S_b(x) + S_f] = 0, \end{cases} \quad (3)$$

where $A(t, x)$ represents the cross-sectional area of the water in the channel, $Q(t, x)$ is the flow rate, $H(t, x)$ is the water depth, S_f is the friction term, $S_b(x)$ is the bottom slope, q_L represents the lateral flow contribution per unit length, and g is the constant gravitational acceleration.

The friction term S_f , which represents momentum losses due to interactions between the flow and the channel bed, can be modeled using various empirical laws commonly employed in hydraulics, particularly the Manning–Strickler and Chézy formulas. In this work, we adopt the following simplified model:

$$S_f = k \frac{V^2}{A}, \quad (4)$$

where l is the width of the basin at the bottom, m is the inverse of the slope of the basin walls and $k > 0$ is a constant effective friction coefficient.

This choice is motivated by the following considerations. First, in the flow regime under investigation, the flow develops in sections characterized by a large width-to-depth ratio, implying that friction is mainly governed by bottom interactions, while lateral effects remain negligible. Second, under these assumptions, classical Manning–Strickler-type laws can be asymptotically reduced to a quadratic dependence on the velocity, leading to a dissipation term proportional to V^2 and inversely proportional to a characteristic quantity of the wetted cross-section.

Furthermore, the introduction of a constant coefficient k provides a structurally simple formulation that is compatible both with the theoretical analysis (in particular Lyapunov stability analysis) and with the numerical implementation of the coupled hydrodynamic–sedimentary model. This coefficient may be interpreted as an effective parameter incorporating bed roughness and average hydraulic properties of the flow, and can be calibrated using experimental data or reference simulations (see, for example, [14], [13, Section 2], [6, Section 4]).

5 Preliminary computations

5.1 Linearization of the lateral channel flow model

For each lateral channel k , we denote by $W_k^*(x) = (H_k^*(x), V_k^*(x), B_k^*(x))^T$ a reference state associated with system (1). We introduce the residual state $w_k = (h, q, b)$ as the difference between the current state W_k and the stationary state W_k^* . That is,

$$h_k(t, x) = H_k(t, x) - H_k^*(x), \quad v_k(t, x) = V_k(t, x) - V_k^*(x), \quad b_k(t, x) = B_k(t, x) - B_k^*(x). \quad (5)$$

In practice, the reference state W_k^* can be seen as a stationary state corresponding to a stable hydraulic configuration, such as uniform flow or a stable bed configuration. The perturbations modeled by $h_k(t, x)$, $v_k(t, x)$, and $b_k(t, x)$ may represent fluctuations caused by events such as flow variations, localized sediment inputs, or small changes in bed morphology.

Assuming that the perturbations $h_k(t, x)$, $v_k(t, x)$, and $b_k(t, x)$ around the stationary state are sufficiently small ($|h_k| \ll H_k^*$, $|v_k| \ll V_k^*$, $|b_k| \ll B_k^*$), the linearization of system (1) around the reference state W_k^* is obtained as follows:

$$\begin{cases} \partial_t h_k + \partial_x q_k = 0, \\ \partial_t q_k + \beta_{1,k} \partial_x q_k + \gamma_{1,k} \partial_x h_k + \xi_{1,k} \partial_x b_k = 0, \\ \partial_t b_k + \beta_{2,k} \partial_x q_k = \varepsilon \partial_x^2 b_k, \end{cases} \quad (6)$$

where

$$\xi_{1,k} = gH_k^*, \quad \beta_{1,k} = 2 \frac{Q_k^*}{H_k^*}, \quad \beta_{2,k} = \frac{A_g}{1 - \Phi_k}, \quad \text{and} \quad \gamma_{1,k} = gH_k^* - \left(\frac{Q_k^*}{H_k^*} \right)^2. \quad (7)$$

For the boundary conditions, we assume that upstream ($x = 0$), there are no perturbations, meaning that the volumetric flow rate and bathymetry are zero at this point. Downstream ($x = L$), a feedback control $q_{L,k}$ is designed to regularize the model, and it is imposed that the flux is zero.

Thus the boundary conditions of the system are given by

$$q_k(t, 0) = 0, \quad q_k(t, L) = q_{L,k}(t), \quad b_k(t, 0) = 0 \text{ and } \partial_x b_k(t, L) = 0, \quad (8)$$

where $q_{L,k}$ is the control law (feedback control) to be constructed in order to obtain the exponential convergence of the energy associated with system (6) towards zero.

We also consider the following initial conditions:

$$h_k(0, x) = h_k^0(x), \quad q_k(0, x) = q_k^0(x), \quad b_k(0, x) = b_k^0(x). \quad (9)$$

By reorganizing the preceding terms, system (6) can be written as

$$\begin{cases} \partial_t h_k + \partial_x q_k = 0, & \Omega_{1,k}, \quad (\mathbf{L1}) \\ \partial_t q_k + \partial_x (\beta_{1,k} q_k + \gamma_{1,k} h_k + \xi_{1,k} b_k) = 0, & \Omega_{1,k}, \quad (\mathbf{L2}) \\ \partial_t b_k + \partial_x (\beta_{2,k} q_k - \varepsilon \partial_x b_k) = 0, & \Omega_{1,k}, \quad (\mathbf{L3}) \\ h_k(0, x) = h_k^0(x), q_k(0, x) = q_k^0(x), b_k(0, x) = b_k^0(x), \\ q_k(t, 0) = 0, \quad q_k(t, L) = q_{L,k}, \quad b_k(t, 0) = 0 \text{ and } \partial_x b_k(t, L) = 0, \end{cases} \quad (10)$$

where $\Omega_{1,k} = (0, T) \times (0, L)$, L the length of the channel k .

5.2 Function space

We consider

$$\mathcal{S}^0 = \{\phi \in H^1(0, L) : \phi(0) = 0\}, \quad (11)$$

$$H(\text{div}, \Omega_{1,k}) = \{v \in L^2(\Omega_{1,k}), \text{div}(v) \in L^2(\Omega_{1,k})\}, \quad (12)$$

$$X_k(h_k, q_k) = \begin{pmatrix} h_k \\ q_k \end{pmatrix}, \quad Y_k(h_k, q_k, b_k) = \begin{pmatrix} q_k \\ \beta_{1,k} q_k + \gamma_{1,k} h_k + \xi_{1,k} b_k \end{pmatrix}, \quad Z_k(q_k, b_k) = \begin{pmatrix} b_k \\ \beta_{2,k} q_k - \varepsilon \partial_x b_k \end{pmatrix},$$

and

$$W_k = \left\{ (h_k, q_k, b_k) \in L^\infty((0, T); L^2(0, L))^2 \times L^2(0, T; \mathcal{S}^0), \right. \\ \left. X_k(h_k, q_k), Y_k(h_k, q_k, b_k), Z_k(q_k, b_k) \in H(\text{div}, \Omega_{1,k}) \right\}. \quad (13)$$

Note that in a fluvial regime, if the solution of system (10) is in $L^2(\Omega_{1,k})$ then the vectors $X_k(h, q)$, $Y_k(h, q, b)$ and $Z_k(q, b)$ are respectively in $H(\text{div}, \Omega_{1,k})$. Moreover, thanks to [9, p.27] (see also [6]), if $(h_k, q_k, b_k) \in W$, then

$$X_k(h_k, q_k) \cdot n, \quad Y_k(h_k, q_k, b_k) \cdot n, \quad Z_k(q_k, b_k) \cdot n \in H^{-\frac{1}{2}}(\partial\Omega_1),$$

where $\partial\Omega_{1,k}$ is the boundary of $\Omega_{1,k}$ and $\mathbf{n}$ is the unit normal outward to the boundary of $\Omega_{1,k}$. Therefore,

$$\begin{aligned} h_k(0, \cdot), \quad h_k(T, \cdot), \quad b_k(0, \cdot), \quad b_k(T, \cdot) &\in H^{-\frac{1}{2}}([0, L]), \quad q_k \in H^{-\frac{1}{2}}(\partial\Omega_{1,k}), \\ \gamma_{1,k}h_k(\cdot, 0), \quad \gamma_{1,k}h_k(\cdot, L), \quad \partial_x b_k(\cdot, 0), \quad \partial_x b_k(\cdot, L) &\in H^{-\frac{1}{2}}([0, L]), \quad b_k(\cdot, 0), \quad b_k(\cdot, L) \in L^2([0, L]). \end{aligned}$$

So, if $(h_k, q_k, b_k) \in W$ then the trace of (h_k, q_k, b_k) on the boundary of $\Omega_{1,k}$ is defined in $H^{-\frac{1}{2}}$.

5.3 Weak formulation

To obtain the weak formulation of system (10), consider $\varphi \in H^1(0, L)$, $\phi \in \mathcal{S}^0$ and $\pi \in \mathcal{S}^0$. Multiplying (E_1) , (E_2) and (E_3) by $\gamma_{1,k}\varphi$, ϕ and $\xi\pi$ respectively, and integrating each equation obtained with respect to x we obtain the weak formulation of the system as follows:

$$\left\{ \begin{aligned} \int_0^L \gamma_{1,k} \varphi \partial_t h_k dx - \int_0^L q_k \partial_x (\gamma_{1,k} \varphi) dx + \gamma_{1,k} \varphi(L) q_k(t, L) &= 0, \\ \int_0^L \phi \partial_t q_k dx - \int_0^L q_k \partial_x (\beta_{1,k} \phi) dx - \int_0^L h_k \partial_x (\gamma_{1,k} \phi) dx - \int_0^L b_k \partial_x (\xi_{1,k} \phi) dx \\ + \beta_{1,k} \phi(L) q_k(t, L) + \phi(L) (\gamma_{1,k} h_k(t, L) + \xi_{1,k} b_k(t, L)) &= 0, \\ \int_0^L \xi_{1,k} \pi \partial_t b_k dx - \int_0^L q_k \partial_x (\xi_{1,k} \pi) dx + \int_0^L \varepsilon \xi_{1,k} \partial_x \pi(L) \partial_x b_k dx + \xi_{1,k} \pi(L) q_k(t, L) &= 0. \end{aligned} \right. \quad (14)$$

6 Regulation process

We consider a control applied exclusively at the sluice gates (at point $x = L_c$ of each lateral canal). At the downstream boundary, the discharge is prescribed according to the following control law:

$$q_k(t, L) = \mu_k(t) \left(\gamma_{1,k} h_k(t, L) + \xi_{1,k} b_k(t, L) \right). \quad (15)$$

This formulation defines a linear boundary feedback control at the downstream end, designed to regulate the system's behavior.

In the control law (15), $\mu_k(\cdot)$ denotes the control parameter, a positive function such that $\mu_k(\cdot) \in L^\infty(\mathbb{R}^+)$. In practice, the stability result requires the existence of a uniform lower bound, namely $\mu_k(t) \geq \mu_0 > 0$, in order to guarantee a sufficient dissipative effect of the control. A comparative analysis of the decay of the energy $E(t)$, defined in (14), for different values of $\mu_k(t)$ (namely 0.1, 0.3, and 0.8) shows that the decay rate of $E(t)$ increases with $\mu_k(t)$. Therefore, the convergence toward the target state can be accelerated by appropriately tuning this control parameter. In the numerical simulations presented in this work, the parameter is taken to be constant, with the value $\mu_k(t) = 0.82$.

Before stating our stability result, we first provide the well-posedness result of system (10).

Proposition 1. Suppose that $(h_k^0, q_k^0, b_k^0) \in L^2(0, L)^2 \times \mathcal{S}^0$. System (10), together with the control law (15) has a unique solution (h_k, q_k, b_k) in W_k .

The proof of this proposition relies on the Faedo-Galerkin method, adapted to our complex evolution problem. This method transforms the model while preserving the energy properties of the continuous system. Once the decreasing energy of the model is established, Faedo-Galerkin allows us to: construct an explicit sequence of approximate solutions in finite dimension, preserve the negative energy at each approximation level, pass to the limit to obtain an exact weak solution, and guarantee the continuity of the limit solution.

In order to prove this proposition, with the Faedo-Galerkin method, we introduce the following lemma.

Lemma 1. *Let us consider the following energy function*

$$E(t) = \frac{1}{2} \int_0^L (\gamma_{1,k} h_k^2 + q_k^2 + \xi_{1,k} b_k^2) dx. \quad (16)$$

The energy associated with system (10) decreases over time, and its instantaneous variation satisfies the following equation:

$$\partial_t E(t) = -\frac{\beta_{1,k}}{2} q_k^2(t, L) - \mu_k(t) \left(\gamma_{1,k} h_k(t, L) + \xi_{1,k} b_k(t, L) \right)^2 - \varepsilon \int_0^L \xi_{1,k} (\partial_x b_k)^2 dx. \quad (17)$$

Proof. By replacing (φ, ϕ, π) by (h_k, q_k, b_k) in system (14), we obtain

$$\begin{cases} \frac{1}{2} \int_0^L \gamma_{1,k} \partial_t h_k^2 dx - \int_0^L q_k \partial_x (\gamma_{1,k} h_k) dx + \gamma_{1,k} h_k(t, L) q_k(t, L) = 0, \\ \frac{1}{2} \int_0^L \partial_t q_k^2 dx - \int_0^L q_k \partial_x (\beta_{1,k} q_k) dx - \int_0^L h_k \partial_x (\gamma_{1,k} q_k) dx - \int_0^L b_k \partial_x (\xi_{1,k} q_k) dx \\ + \beta_{1,k} q_k^2(t, L) + \gamma_{1,k} q_k(t, L) h_k(t, L) + \xi_{1,k} q_k(t, L) b_k(t, L) = 0, \\ \frac{1}{2} \int_0^L \xi_{1,k} \partial_t b_k^2 dx - \int_0^L q_k \partial_x (\xi_{1,k} b_k) dx + \int_0^L \varepsilon \xi_{1,k} (\partial_x b_k)^2 dx + \xi_{1,k} b_k(t, L) q(t, L) = 0. \end{cases} \quad (18)$$

The sum of the three equations in (18) gives

$$\begin{aligned} & \frac{1}{2} \int_0^L \gamma_{1,k} \partial_t h_k^2 dx - \int_0^L q_k \partial_x (\gamma_{1,k} h_k) dx + \gamma_{1,k} h_k(t, L) q_k(t, L) + \frac{1}{2} \int_0^L \partial_t q_k^2 dx \\ & - \int_0^L q_k \partial_x (\beta_{1,k} q_k) dx - \int_0^L h_k \partial_x (\gamma_{1,k} q_k) dx - \int_0^L b_k \partial_x (\xi_{1,k} q_k) dx + \beta_{1,k} q_k^2(t, L) \\ & + \gamma_{1,k} q_k(t, L) h_k(t, L) + \frac{1}{2} \int_0^L \xi_{1,k} \partial_t b_k^2 dx - \int_0^L q_k \partial_x (\xi_{1,k} b_k) dx \\ & + \int_0^L \varepsilon \xi_{1,k} (\partial_x b_k)^2 dx + 2 \xi_{1,k} b_k(t, L) q_k(t, L) = 0. \end{aligned} \quad (19)$$

By ordering its terms, the equation (19) becomes

$$\frac{1}{2} \int_0^L \gamma_{1,k} \partial_t h_k^2 dx + \frac{1}{2} \int_0^L \partial_t q_k^2 dx + \frac{1}{2} \int_0^L \xi_{1,k} \partial_t b_k^2 dx - \int_0^L q_k \partial_x (\gamma_{1,k} h_k) dx$$

$$\begin{aligned}
& - \int_0^L q_k \partial_x (\beta_{1,k} q_k) dx - \int_0^L h_k \partial_x (\gamma_{1,k} q_k) dx - \int_0^L b_k \partial_x (\xi_{1,k} q_k) dx \\
& - \int_0^L q_k \partial_x (\xi_{1,k} b_k) dx + \varepsilon \int_0^L \xi_{1,k} (\partial_x b_k)^2 dx + \gamma_{1,k} h_k(t, L) q_k(t, L) + \beta_{1,k} q_k^2(t, L) \\
& + \gamma_{1,k} q_k(t, L) h_k(t, L) + 2 \xi_{1,k} q_k(t, L) b_k(t, L) = 0.
\end{aligned} \tag{20}$$

By integrating by part, we obtain

$$\int_0^L h_k \partial_x (\gamma_{1,k} q_k) dx = \gamma_{1,k} q_k(t, L) h_k(t, L) - \int_0^L q_k \partial_x (\gamma_{1,k} h_k) dx. \tag{21}$$

$$\int_0^L b_k \partial_x (\xi_{1,k} q_k) dx = \xi_{1,k} q_k(t, L) b_k(t, L) - \int_0^L q_k \partial_x (\xi_{1,k} b_k) dx. \tag{22}$$

Equation (20) becomes

$$\begin{aligned}
\frac{1}{2} \int_0^L (\gamma_{1,k} \partial_t h_k^2 + \partial_t q_k^2 + \xi_{1,k} \partial_t b_k^2) dx &= \int_0^L q_k \left(\partial_x (\gamma_{1,k} h_k) + \partial_x (\beta_{1,k} q_k) + \partial_x (\xi_{1,k} b_k) \right) dx \\
& - \int_0^L q_k \partial_x (\gamma_{1,k} h_k) dx - \int_0^L q_k \partial_x (\xi_{1,k} b_k) dx - \varepsilon \int_0^L \xi_{1,k} (\partial_x b_k)^2 dx \\
& - \beta_{1,k} q_k^2(t, L) + \xi_{1,k} q_k(t, L) b_k(t, L) - \gamma_{1,k} q_k(t, L) h_k(t, L) - 2 \xi_{1,k} q_k(t, L) b_k(t, L)
\end{aligned} \tag{23}$$

or

$$\beta_{1,k} q_k^2(t, L) = 2 * \frac{\beta_{1,k}}{2} q_k^2(t, L) \quad \text{and} \quad \frac{\beta_{1,k}}{2} q_k^2(t, L) = \int_0^L q_k \partial_x (\beta_{1,k} q_k) dx.$$

Therefore

$$\beta_{1,k} q_k^2(t, L) = \frac{\beta_{1,k}}{2} q_k^2(t, L) + \int_0^L q_k \partial_x (\beta_{1,k} q_k) dx.$$

Consequently

$$\begin{aligned}
\frac{1}{2} \int_0^L (\gamma_{1,k} \partial_t h_k^2 + \partial_t q_k^2 + \xi_{1,k} \partial_t b_k^2) dx &= \int_0^L q_k \left(\partial_x (\gamma_{1,k} h_k) + \partial_x (\beta_{1,k} q_k) + \partial_x (\xi_{1,k} b_k) \right) dx \\
& - \int_0^L q_k \left(\partial_x (\gamma_{1,k} h_k) + \partial_x (\beta_{1,k} q_k) + \partial_x (\xi_{1,k} b_k) \right) dx - \varepsilon \int_0^L \xi_{1,k} (\partial_x b_k)^2 dx - \frac{\beta_{1,k}}{2} q_k^2(t, L) \\
& - \gamma_{1,k} q_k(t, L) h_k(t, L) - \xi_{1,k} q_k(t, L) b_k(t, L).
\end{aligned} \tag{24}$$

By ordering these terms, the equation (24) becomes:

$$\begin{aligned}
\frac{1}{2} \int_0^L (\gamma_{1,k} \partial_t h_k^2 + \partial_t q_k^2 + \xi_{1,k} \partial_t b_k^2) dx &= - \frac{\beta_{1,k}}{2} q_k^2(t, L) \\
& - q_k(t, L) (\gamma_{1,k} h_k(t, L) + \xi_{1,k} b_k(t, L)) - \varepsilon \int_0^L \xi_{1,k} (\partial_x b_k)^2 dx.
\end{aligned} \tag{25}$$

By defining the boundary control as

$$q_k(t, L) = \mu_k(t) \left(\gamma_{1,k} h_k(t, L) + \xi_{1,k} b_k(t, L) \right),$$

as introduced in equation (15), equation (25) becomes

$$\begin{aligned} \frac{1}{2} \int_0^L (\gamma_{1,k} \partial_t h_k^2 + \partial_t q_k^2 + \xi_{1,k} \partial_t b_k^2) dx &= -\frac{\beta_{1,k}}{2} q_k^2(t, L) \\ &- \mu_k(t) \left(\gamma_{1,k} h_k(t, L) + \xi_{1,k} b_k(t, L) \right) \left(\gamma_{1,k} h_k(t, L) + \xi_{1,k} b_k(t, L) \right) - \varepsilon \int_0^L \xi_{1,k} (\partial_x b_k)^2 dx, \end{aligned}$$

from which

$$\begin{aligned} \partial_t \left(\frac{1}{2} \int_0^L (\gamma_{1,k} h_k^2 + q_k^2 + \xi_{1,k} b_k^2) dx \right) &= -\frac{\beta_{1,k}}{2} q_k^2(t, L) - \mu_k(t) \left(\gamma_{1,k} h_k(t, L) + \xi_{1,k} b_k(t, L) \right)^2 \\ &- \varepsilon \int_0^L \xi_{1,k} (\partial_x b_k)^2 dx. \end{aligned} \quad (26)$$

Thus, the energy variation satisfies

$$\partial_t E(t) dt = -\frac{\beta_{1,k}}{2} q_k^2(t, L) - \mu_k(t) \left(\gamma_{1,k} h_k(t, L) + \xi_{1,k} b_k(t, L) \right)^2 - \varepsilon \int_0^L \xi_{1,k} (\partial_x b_k)^2 dx.$$

□

Proof of Proposition 1. Let us consider

$$\{v_i \in H^1(0, L), i \in \mathbb{N}\} \text{ a Hilbertian basis of } L^2(0, L) \quad (27)$$

and

$$\{w_i \in S_0, i \in \mathbb{N}\} \text{ a Hilbertian basis of } S_0. \quad (28)$$

Also, let S_v^n and S_w^n be the finite-dimensional spaces defined as follows:

$$S_v^n = \text{span}\{v_i, 1 \leq i \leq n\}, \quad S_w^n = \text{span}\{w_i, 1 \leq i \leq n\}. \quad (29)$$

Consider $(h_k^{0,n}(x), q_k^{0,n}(x), b_k^{0,n}(x)) \subset S_v^n \times (S_w^n)^2$ be a sequence converging to the initial conditions (h_k^0, q_k^0, b_k^0) in $L^2((0, L))^2 \times S_0$. We now consider the discrete weak formulation (14) in the space $S_v^n \times (S_w^n)^2$: for all $(\psi, \phi, \pi) \in S_v^n \times (S_w^n)^2$,

$$\int_0^L \gamma_{1,k} \psi \partial_t h_k^n dx - \int_0^L \gamma_{1,k} q_k^n \partial_x \psi dx + \gamma_{1,k} \psi(L) \mu_k(t) \left(\gamma_{1,k} h_k^n(t, L) + \xi_{1,k} b_k^n(t, L) \right) = 0, \quad (30)$$

$$\begin{aligned} \int_0^L \phi \partial_t q_k^n dx - \int_0^L q_k^n \partial_x (\beta_{1,k} \phi) dx - \int_0^L h_k^n \partial_x (\gamma_{1,k} \phi) dx - \int_0^L b_k^n \partial_x (\xi_{1,k} \phi) dx \\ + \beta_{1,k} \phi(L) q_k^n(t, L) + \gamma_{1,k} \phi(L) h_k^n(t, L) + \xi_{1,k} \phi(L) b_k^n(t, L) = 0, \end{aligned} \quad (31)$$

$$\begin{aligned} \int_0^L \xi_{1,k} \pi \partial_t b_k^n dx - \int_0^L q_k^n \partial_x (\xi_{1,k} \pi) dx + \int_0^L \varepsilon \xi_{1,k} \partial_x b_k^n \partial_x \pi dx \\ + \xi_{1,k} \pi(L) \mu_k(t) \left(\gamma_{1,k} h_k^n(t, L) + \xi_{1,k} b_k^n(t, L) \right) = 0. \end{aligned} \quad (32)$$

Using the a priori energy estimate (17), for all $t > 0$, we have

$$E^n(T) + \varepsilon \xi_{1,k} \int_0^T \int_0^L (\partial_x b_k^n)^2 dx dt + \frac{\beta_{1,k}}{2} \int_0^T (q_k^n(t, L))^2 dt + \gamma_{1,k} \int_0^T \mu_k(t) h_k^n(t, L) dt = E^n(0), \quad (33)$$

where

$$E^n(t) = \|\sqrt{\gamma_{1,k}} h_k^n(t, \cdot)\|_{L^2((0,L))}^2 + \|q_k^n(t, \cdot)\|_{L^2((0,L))}^2 + \|\sqrt{\xi_{1,k}} b_k^n(t, \cdot)\|_{L^2((0,L))}^2.$$

Since $L^2(0, T; H^1(0, L))$ and $L^\infty(0, T; L^2(0, L))$ are Banach spaces, the energy estimate allows us to conclude that there exists a subsequence of (h_k^n, q_k^n, b_k^n) that converges weakly-* to (h, q, b) in

$$L^\infty(0, T; L^2(0, L))^2 \times (L^\infty(0, T; L^2(0, L)) \cap L^2(0, T; H^1(0, L))).$$

Furthermore, due to the continuity of the trace operator in $H^1(0, L)$, $b_k^n(\cdot, L)$ also has a subsequence that converges weakly in $L^2(0, T)$.

Moreover, it follows directly from (33) that the sequence of boundary traces $(h_k^n(\cdot, L), q_k^n(\cdot, L), b_k^n(\cdot, L))$ is bounded in $L^2(0, T)^3$. Consequently, it admits a subsequence that converges weakly to $(h_{k,L}, q_{k,L}, b_{k,L})$ in $L^2(0, T)^3$.

To pass to the limit in equation (30), we choose

$$\psi(t, x) = g(t)v(x), \quad g \in H_T^1(0, T), \quad v \in S_v.$$

By integrating by parts and letting n tend to infinity, we obtain

$$-\int_0^T (h_k, v)_{(0,L)} \partial_t g dt - (h_k^0, v)_{(0,L)} g(0) - \int_0^T (q_k, \partial_x v)_{(0,L)} g dt + v(L) \int_0^T g(t) \mu_k(t) h_{k,L}(t) dt = 0.$$

Let us define the set

$$V_T = \{g v \mid (g, v) \in H^1(0, T) \times H^1(0, L), g(T) = 0\}. \quad (34)$$

By density, $\forall (g, v) \in V_T$, we have

$$\begin{aligned} & -\int_0^T (h_k, v)_{(0,L)} \partial_t g dt - (h_k^0, v)_{(0,L)} g(0) - \int_0^T (q_k, \partial_x v)_{(0,L)} g dt \\ & + v(L) \int_0^T g(t) \mu_k(t) (\gamma_{1,k} h_{k,L}(t) + \xi_{1,k} b_{k,L}(t)) dt = 0. \end{aligned} \quad (35)$$

So, taking $(g, v) \in \mathcal{D}(0, T) \times \mathcal{D}(0, L)$, we get

$$\partial_t h_k + \partial_x q_k = 0. \quad (36)$$

Then, multiplying (36) again by $\psi \in V_T$, we get

$$\begin{aligned} & -\int_0^T (h_k, v)_{(0,L)} \partial_t g dt - g(0) (h_k(0, \cdot), v)_{H^{-1/2}(0,L), H^{1/2}(0,L)} - \int_0^T (q_k, \partial_x v)_{(0,L)} g dt \\ & - v(0) (q_k(\cdot, 0), g)_{H^{-1/2}(0,T), H^{1/2}(0,T)} + v(L) (q_k(\cdot, L), g)_{H^{-1/2}(0,T), H^{1/2}(0,T)} = 0. \end{aligned} \quad (37)$$

By comparing equations (35) and (37), we derive the initial and boundary conditions

$$h_k(0, \cdot) = h_k^0, \quad q_k(\cdot, 0) = 0, \quad q_k(\cdot, L) = \mu_k(\cdot)(\gamma_{1,k}h_{k,L}(\cdot) + \xi_{1,k}b_k(\cdot, L)). \quad (38)$$

Similarly, we show that the weak limit of (h_k^n, q_k^n, b_k^n) satisfies the variational equation of the momentum conservation law, as follows :

$$\begin{aligned} & - \int_0^T (q_k, v)_{(0,L)} \partial_t g \, dt - (q_k^0, v)_{(0,L)} g(0) - \int_0^T (q_k, \partial_x(\beta_{1,k}v))_{(0,L)} g \, dt \\ & - \int_0^T (h_k, \partial_x(\gamma_{1,k}v))_{(0,L)} g \, dt - \int_0^T (b_k, \partial_x(\xi_{1,k}v))_{(0,L)} g(t) \, dt \\ & + \beta_{1,k}v(L) \int_0^T g(t)q_{k,L}(t) \, dt + v(L) \int_0^T g(t)(\gamma_{1,k}h_{k,L}(t) + \xi_{1,k}b_k(t, L)) \, dt = 0, \end{aligned}$$

which corresponds to $\partial_t q_k + \beta_{1,k} \partial_x q_k + \gamma_{1,k} \partial_x h_k + \xi_{1,k} \partial_x b_k = 0$, and to the initial and boundary conditions

$$q_k(0, \cdot) = q_k^0, \quad h_k(\cdot, L) = h_{k,L}(\cdot), \quad q_k(\cdot, L) = q_{k,L}(\cdot). \quad (39)$$

The same method applied to the law of continuity of sediments gives

$$\begin{aligned} & - \int_0^T (b_k, v)_{(0,L)} \partial_t g \, dt - (b_k^0, v)_{(0,L)} g(0) - \int_0^T (q_k, \partial_x(\beta_{2,k}v))_{(0,L)} g \, dt \\ & + \varepsilon \int_0^T (\partial_x b_k, \partial_x v)_{(0,L)} g \, dt - \beta_{2,k}v(L) \int_0^T g(t)q_k(t, L) \, dt = 0, \end{aligned}$$

which corresponds to

$$\partial_t b_k + \beta_{2,k} \partial_x q_k = \varepsilon \partial_x^2 b_k, \quad (40)$$

and to the initial and boundary conditions $b_k(0, \cdot) = b_k^0$, $\partial_x b_k(\cdot, L) = 0$. $\square$

Using the control (15) and the decay of energy (17), we obtain the following stability result.

Theorem 1. *Let $(h_k^0, q_k^0, b_k^0) \in L^2(0, L)^2 \times \mathcal{S}^0$. The unique solution (h_k, q_k, b_k) of system (10) in W_k satisfies*

$$\lim_{t \rightarrow +\infty} h_k(t, x) = 0, \quad \lim_{t \rightarrow +\infty} q_k(t, x) = 0, \quad \lim_{t \rightarrow +\infty} b_k(t, x) = 0, \quad \forall x \in (0, L). \quad (41)$$

Proof. The proof is divided into four steps.

Step 1. Characterization of the invariant set.

From the energy identity (33), we obtain

$$E(T) = E(0) - \varepsilon \xi_{1,k} \int_0^T \int_0^L (\partial_x b_k)^2 \, dx \, ds - \int_0^T \left(\frac{\beta_{1,k}}{2} q_{L,k}^2 + \mu_k(s)(\gamma_{1,k}h_k(s, L) + \xi_{1,k}b_k(s, L))^2 \right) \, ds. \quad (42)$$

Let $0 < t_1 < t_2 < \infty$. Then

$$\begin{aligned} E(t_2) - E(t_1) &= -\varepsilon \xi_{1,k} \int_{t_1}^{t_2} \int_0^L (\partial_x b_k)^2 \, dx \, ds \\ &\quad - \int_{t_1}^{t_2} \left(\frac{\beta_{1,k}}{2} q_{L,k}^2 + \mu_k(t)(\gamma_{1,k}h_k(t, L) + \xi_{1,k}b_k(t, L))^2 \right) \, dt. \end{aligned} \quad (43)$$

Hence, $E(t_2) = E(t_1)$ if and only if

$$\varepsilon \xi_{1,k} \int_{t_1}^{t_2} \int_0^L (\partial_x b_k)^2 dx ds + \int_{t_1}^{t_2} \left(\frac{\beta_{1,k}}{2} q_{L,k}^2 + \mu_k(t) (\gamma_{1,k} h_k(t, L) + \xi_{1,k} b_k(t, L))^2 \right) dt = 0. \quad (44)$$

Since every term in (44) is nonnegative, equality holds if and only if

$$\partial_x b_k = 0, \quad q_{L,k} = 0, \quad \gamma_{1,k} h_k(t, L) + \xi_{1,k} b_k(t, L) = 0,$$

almost everywhere on $(t_1, t_2) \times (0, L)$.

Since $\partial_x b_k = 0$, the bathymetry is spatially constant. Using the boundary condition $b_k(t, 0) = 0$, we conclude that

$$b_k \equiv 0 \quad \text{in } (t_1, t_2) \times (0, L).$$

Substituting this result into (40) and using the boundary condition $q_k(t, 0) = 0$, we deduce that

$$q_k \equiv 0 \quad \text{in } (t_1, t_2) \times (0, L).$$

Finally, the first equation of system (10) implies that $\partial_x h_k = 0$, so that h_k is spatially constant. Moreover,

$$\gamma_{1,k} h_k(t, L) + \xi_{1,k} b_k(t, L) = 0.$$

Since $b_k \equiv 0$, it follows that $h_k(t, L) = 0$.

Therefore

$$h_k \equiv 0 \quad \text{in } (t_1, t_2) \times (0, L).$$

Consequently,

$$b_k \equiv 0 \implies q_k \equiv 0 \implies h_k \equiv 0.$$

Hence, the largest invariant set associated with the energy dissipation is reduced to the origin. Therefore, either there exists a finite time T such that $(h_k, q_k, b_k)(T) = 0$, which corresponds to exact controllability, or the energy $E(t)$ is strictly decreasing.

Step 2. Boundary dissipation and asymptotic estimates.

Assume that exact controllability does not occur. From the energy identity (42), we obtain

$$E(t) \geq 0, \quad E'(t) = -\varepsilon \xi_{1,k} \|\partial_x b_k(t, \cdot)\|_{L^2(0,L)}^2 - \frac{\beta_{1,k}}{2} q_{L,k}^2(t) - \mu_k(t) (\gamma_{1,k} h_k(t, L) + \xi_{1,k} b_k(t, L))^2.$$

Hence,

$$\int_0^{+\infty} \|\partial_x b_k(t, \cdot)\|_{L^2(0,L)}^2 dt < +\infty,$$

and

$$\int_0^{+\infty} q_{L,k}^2(t) dt < +\infty, \quad \int_0^{+\infty} (\gamma_{1,k} h_k(t, L) + \xi_{1,k} b_k(t, L))^2 dt < +\infty.$$

Since the solution is sufficiently regular, the functions

$$t \mapsto \|\partial_x b_k(t, \cdot)\|_{L^2(0,L)}^2, \quad t \mapsto q_{L,k}(t), \quad t \mapsto \gamma_{1,k} h_k(t, L) + \xi_{1,k} b_k(t, L)$$

are uniformly continuous on $[0, +\infty)$. Applying Barbalat's lemma (see, e.g., [2, 10]), we conclude that

$$\partial_x b_k(t, \cdot) \longrightarrow 0 \quad \text{in } L^2(0, L),$$

and

$$q_{L,k}(t) \longrightarrow 0, \quad \gamma_{1,k} h_k(t, L) + \xi_{1,k} b_k(t, L) \longrightarrow 0, \quad t \rightarrow +\infty.$$

Step 3. Analysis along the characteristic curves.

We now analyze the asymptotic behavior of the first two equations of system (10). By introducing the vector

$$\chi_{1,k} = \begin{pmatrix} \gamma_{1,k} h_k \\ q_k \end{pmatrix},$$

the system can be rewritten as the symmetric hyperbolic system

$$\partial_t \chi_{1,k} + A_k \partial_x \chi_{1,k} = F_k, \quad (45)$$

where

$$A_k = \begin{pmatrix} 0 & \gamma_{1,k} \\ \gamma_{1,k} & \beta_{1,k} \end{pmatrix}, \quad F_k = \begin{pmatrix} 0 \\ -\xi_{1,k} \partial_x b_k \end{pmatrix}.$$

The matrix A_k is diagonalizable. More precisely,

$$A_k = P_k \Lambda_k P_k^{-1},$$

where

$$\Lambda_k = \begin{pmatrix} \lambda_{1,k} & 0 \\ 0 & \lambda_{2,k} \end{pmatrix},$$

with

$$\delta_k^* = \sqrt{\beta_{1,k}^2 + 4\gamma_{1,k}^2} \quad \text{and} \quad \lambda_{1,k} = -\frac{\delta_k^* + \beta_{1,k}}{2}, \quad \lambda_{2,k} = \frac{\delta_k^* - \beta_{1,k}}{2}.$$

Since $\delta_k^* > \beta_{1,k} > 0$, it follows that $\lambda_{1,k} < 0 < \lambda_{2,k}$.

By defining the characteristic variables $\tilde{\xi}_k = P_k^{-1} \chi_{1,k}$ and $\tilde{F}_k = P_k^{-1} F_k$, system (45) becomes

$$\partial_t \tilde{\xi}_k + \Lambda_k \partial_x \tilde{\xi}_k = \tilde{F}_k. \quad (46)$$

Since $\lambda_{1,k} < 0 < \lambda_{2,k}$, one family of characteristics enters the domain through $x = L$, while the other enters through $x = 0$.

Let $(\tilde{t}, \tilde{x}) \in]0, \infty[\times (0, L)$. By integrating over the characteristic lines

$$\{(t, x) \in (0, T) \times (0, L), \quad x + \lambda_{i,k}(\tilde{t} - t) = \tilde{x}\},$$

it stems from (46) the following identities:

$$\tilde{\xi}_{i,k}(\tilde{t}, \tilde{x}) = \tilde{\xi}_{i,k} \left(\frac{L - \tilde{x}}{\lambda_{i,k}} + \tilde{t}, L \right) - \int_{\tilde{t}}^{\frac{L - \tilde{x}}{\lambda_{i,k}} + \tilde{t}} \tilde{F}_{i,k}(t, \tilde{x} + \lambda_{i,k}(t - \tilde{t})) dt,$$

$$\begin{aligned}
&= \tilde{\xi}_{i,k} \left(\frac{L-\tilde{x}}{\lambda_{i,k}} + \tilde{t}, L \right) + \frac{\lambda_{i,k} \xi_{1,k}}{\gamma_{1,k}} \int_{\tilde{t}}^{\frac{L-\tilde{x}}{\lambda_{i,k}} + \tilde{t}} \partial_x b(t, \tilde{x} + \lambda_{i,k}(t - \tilde{t})) dt, \\
&= \tilde{\xi}_{i,k} \left(\frac{L-\tilde{x}}{\lambda_{i,k}} + \tilde{t}, L \right) + \frac{\xi_{1,k}}{\gamma_{1,k}} \int_{\tilde{x}}^L \partial_x b_k \left(\frac{s-\tilde{x}}{\lambda_{i,k}} + \tilde{t}, s \right) ds.
\end{aligned} \tag{47}$$

Now, we prove that

$$\int_a^\infty \tilde{\xi}_{i,k}(\tilde{t}, x)^2 d\tilde{t} < \infty,$$

where $a \geq \max \left(\frac{L}{|\lambda_{1,k}|}, \frac{L}{|\lambda_{2,k}|} \right)$. To this end, we deduce from (47) that

$$\begin{aligned}
\frac{1}{2} \int_a^T \tilde{\xi}_{i,k}(\tilde{t}, \tilde{x})^2 d\tilde{t} &\leq \int_a^T \tilde{\xi}_{i,k} \left(\frac{L-\tilde{x}}{\lambda_{i,k}} + \tilde{t}, L \right)^2 d\tilde{t} + \frac{\xi_{1,k}^2}{\gamma_{1,k}^2} \int_a^T \left(\int_{\tilde{x}}^L \partial_x b_k \left(\frac{s-\tilde{x}}{\lambda_{i,k}} + \tilde{t}, s \right) ds \right)^2 d\tilde{t}, \\
&\leq \int_a^T \tilde{\xi}_{i,k} \left(\frac{L-\tilde{x}}{\lambda_{i,k}} + \tilde{t}, L \right)^2 d\tilde{t} + \frac{L\xi_{1,k}^2}{\gamma_{1,k}^2} \int_a^T \int_0^L \left(\partial_x b_k \left(\frac{s-\tilde{x}}{\lambda_{i,k}} + \tilde{t}, s \right) \right)^2 ds d\tilde{t}, \\
&\leq \int_a^T \tilde{\xi}_{i,k} \left(\frac{L-\tilde{x}}{\lambda_{i,k}} + \tilde{t}, L \right)^2 d\tilde{t} + \frac{L\xi_{1,k}^2}{\gamma_{1,k}^2} \int_0^L \int_{\frac{s-\tilde{x}}{\lambda_{i,k}} + a}^{\frac{s-\tilde{x}}{\lambda_{i,k}} + T} (\partial_x b_k(t, s))^2 dt ds, \\
&\leq \int_a^T \tilde{\xi}_{i,k} \left(\frac{L-\tilde{x}}{\lambda_{i,k}} + \tilde{t}, L \right)^2 d\tilde{t} + \frac{L\xi_{1,k}^2}{\gamma_{1,k}^2} \int_0^L \int_0^{a+T} (\partial_x b_k(t, s))^2 dt ds.
\end{aligned} \tag{48}$$

Then, one deduces from (48) that

$$\frac{1}{2} \int_a^\infty \tilde{\xi}_{i,k}(\tilde{t}, x)^2 d\tilde{t} \leq \int_a^\infty \tilde{\xi}_{i,k} \left(\frac{L-\tilde{x}}{\lambda_{i,k}} + \tilde{t}, L \right)^2 d\tilde{t} + \frac{L\xi_{1,k}^2}{\gamma_{1,k}^2} \int_0^\infty \int_0^L (\partial_x b_k(t, s))^2 ds dt. \tag{49}$$

From Step 2, we know that

$$q_{L,k}(t) \longrightarrow 0, \quad \gamma_{1,k} h_k(t, L) + \xi_{1,k} b_k(t, L) \longrightarrow 0, \quad t \longrightarrow +\infty.$$

Consequently, the boundary traces of $\tilde{\xi}_{i,k}$ vanish as $t \longrightarrow +\infty$.

Moreover,

$$\int_0^{+\infty} \int_0^L (\partial_x b_k)^2 dx dt < +\infty.$$

Hence, (49) implies that

$$\tilde{\xi}_{i,k}(\cdot, \tilde{x}) \in L^2(a, +\infty).$$

Therefore

$$\tilde{\xi}_{i,k}(t, \tilde{x}) \longrightarrow 0, \quad t \longrightarrow +\infty.$$

Finally, since $\chi_{1,k} = P_k \tilde{\xi}_k$ and the matrix P_k is invertible, we conclude that

$$(h_k(t, x), q_k(t, x)) \longrightarrow (0, 0), \quad t \longrightarrow +\infty,$$

for every $x \in (0, L)$.

Step 4. Convergence of the bathymetry and conclusion.

From Step 2, we have

$$\partial_x b_k(t, \cdot) \longrightarrow 0 \quad \text{in } L^2(0, L).$$

Since the boundary condition $b_k(t, 0) = 0$ holds for all $t \geq 0$, Poincaré's inequality yields

$$\|b_k(t, \cdot)\|_{L^2(0, L)} \leq C \|\partial_x b_k(t, \cdot)\|_{L^2(0, L)}.$$

Hence,

$$b_k(t, \cdot) \longrightarrow 0 \quad \text{in } L^2(0, L).$$

Moreover, since $b_k(t, \cdot) \in H^1(0, L)$, the one-dimensional Sobolev embedding implies

$$b_k(t, \cdot) \longrightarrow 0 \quad \text{uniformly on } [0, L].$$

Combining this result with Step 3, we conclude that

$$(h_k(t, x), q_k(t, x), b_k(t, x)) \longrightarrow (0, 0, 0), \quad t \rightarrow +\infty,$$

for every $x \in (0, L)$. □

6.1 Modeling q_L

We denote by L_c the length of the water reservoir, by d_k the distance between the openings, and by l_k their widths, as illustrated in Figure 3.

For an accurate modeling of the water inflows, it is necessary to account for the positions of each lateral inflow in the reservoir. We propose the following model:

$$q_L(t) = \sum_{k=1}^{k_{max}^d} q_{L,k}^d(t, x) + \sum_{k=1}^{k_{max}^g} q_{L,k}^g(t, x), \quad (50)$$

$$q_{L,k}^{d,g}(t, x) = \begin{cases} Q_k(L, t) 1_{\Omega_k}(x), \\ \Omega_k = \left[\sum_{s=1}^k (d_s + L_{s-1}), \sum_{s=1}^k (d_s + L_s) \right), \\ 1_{\Omega_k}(x) = \begin{cases} 1 & \text{if } x \in \Omega_k, \\ 0 & \text{otherwise,} \end{cases} \end{cases} \quad (51)$$

where, k_{max}^d and k_{max}^g denote the total number of lateral channels on the right and left of the basin, respectively. Similarly, $q_k^d(L, t)$ and $q_k^g(L, t)$ denote the inflows on the right and left of the basin through the sluice k , respectively.

In the numerical simulations, the linearized system (10) is solved to obtain the controlled discharge perturbations $q_k(t, L)$. These perturbations are converted into physical discharges through

$$Q_k(t, L) = Q_k^* + q_k(t, L).$$

The resulting physical discharge is then used to compute the lateral inflow q_L via Equation (50), which serves as a source term in the reservoir model (3). Finally, the reservoir model is solved using the nonlinear Saint-Venant equations.

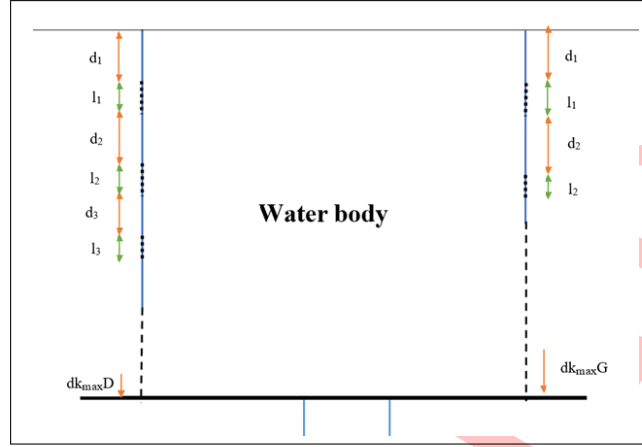


Figure 3: Illustration of the location of the openings

7 Numerical simulation

7.1 Description of the numerical schemes used

The systems governing water flow and sediment transport in the channels (10) are solved using a finite volume scheme based on the Rusanov method, while the system governing water flow in the reservoir (3) is solved using a finite difference approach.

System (10) can be expressed as

$$\partial_t w_k + \partial_x F(w_k) = S(w_k), \quad (52)$$

where

$$w_k = \begin{pmatrix} h_k \\ q_k \\ b_k \end{pmatrix}, \quad F(w_k) = \begin{pmatrix} q_k \\ \beta_{1,k} q_k + \gamma_{1,k} h_k + \xi_{1,k} b_k \\ \beta_{2,k} q_k \end{pmatrix} \text{ and } S(b_k) = \begin{pmatrix} 0 \\ 0 \\ \varepsilon \frac{\partial^2 b_k}{\partial x^2} \end{pmatrix}. \quad (53)$$

The system in its quasilinear version takes the form

$$\partial_t w + A \partial_x w_k = S(w_k), \quad (54)$$

where $A(W_k^*)$ is the Jacobian matrix of $F(w_k)$ with respect to w_k . Thus, system (10) is solved by the finite volume formula

$$w_k^{n+1} = w_k^n - \frac{\Delta t}{\Delta x} (F_{i+1/2} - F_{i-1/2}) + \Delta t S_i^n(b_k), \quad (55)$$

where $F_{i+1/2}$ is a numerical flux, in this case, the Rusanov flux is implemented, which corresponds to

$$F_{i+1/2} = \frac{1}{2} \left(F(w_{i+1,k}^n) - F(w_{i,k}^n) \right) - \frac{\lambda^{i+1/2}}{2} \left(w_{i+1,k}^n - w_{i,k}^n \right).$$

Here $\lambda^{i+\frac{1}{2}} = \frac{Q_k^*}{H_k^*} + \sqrt{gH_k^*(\varepsilon_1 + 1)}$, with $\varepsilon_1 = \frac{A_g}{1-\Phi_k}$. This expression follows from the definition of the Rusanov flux, which uses the largest characteristic wave speed, namely the spectral radius of the Jacobian matrix

$$A = \begin{pmatrix} 0 & 1 & 0 \\ \gamma_{1,k} & \beta_{1,k} & \xi_{1,k} \\ 0 & \beta_{2,k} & 0 \end{pmatrix}.$$

The eigenvalues of A are

$$\lambda_0 = 0, \quad \lambda_{\pm} = \frac{\beta_{1,k} \pm \sqrt{\beta_{1,k}^2 + 4(\gamma_{1,k} + \xi_{1,k}\beta_{2,k})}}{2}.$$

Using the relations (7), these eigenvalues can be rewritten as

$$\lambda_0 = 0, \quad \lambda_{\pm} = \frac{Q_k^*}{H_k^*} \pm \sqrt{gH_k^*(\varepsilon_1 + 1)}.$$

Since

$$\lambda_+ > |\lambda_-|,$$

the maximum characteristic speed is given by

$$\lambda^{i+\frac{1}{2}} = \max(|\lambda_0|, |\lambda_-|, |\lambda_+|) = \frac{Q_k^*}{H_k^*} + \sqrt{gH_k^*(\varepsilon_1 + 1)}. \quad (56)$$

For the source term, we use a centered finite difference scheme for the spatial approximation, i.e.

$$S_i^n(b^k) = \left(0, 0, \frac{b_{i+1,k}^n - 2b_{i,k}^n + b_{i-1,k}^n}{\Delta x^2} \right)^T. \quad (57)$$

The stability of the numerical scheme (55) requires the satisfaction of a Courant–Friedrichs–Lewy (CFL) condition. The Courant number is defined by

$$\text{CFL} = \frac{\Delta t}{\Delta x} \lambda_{\max},$$

where $\lambda_{\max}$ denotes the maximum characteristic wave speed of the hyperbolic part of the system, given by (56).

Accordingly, the CFL stability condition is expressed as

$$\Delta t \leq \text{CFL} \frac{\Delta x}{\lambda_{\max}}.$$

In all the numerical simulations presented in this work, we adopt the value $\text{CFL} = 0.5$.

Furthermore, since the diffusive term is discretized using an explicit Euler scheme, the stability of the diffusion operator also requires the classical restriction

$$\frac{\varepsilon \Delta t}{\Delta x^2} \leq \frac{1}{2}.$$

System (3) is reformulated and discretized as follows:

$$A_i^{j+1} = A_i^j - \frac{\Delta t}{\Delta x} (Q_i^j - Q_{i-1}^j) + \Delta t \cdot (q_L)_i^j, \quad (58)$$

$$Q_i^{j+1} = Q_i^j - \frac{\Delta t}{\Delta x} \alpha_i^j (Q_i^j - Q_{i-1}^j) - \theta_i^j \Delta t, \quad (59)$$

with

$$\alpha_j^j = 2 \frac{Q_i^j}{A_i^j} + \frac{\frac{gA_i^j}{(b+2mH_{c,i}^j)} - \frac{(Q_i^j)^2}{(A_i^j)^2}}{\frac{Q_i^j}{A_i^j} \left(\frac{5}{3} - \frac{4R_i^j \sqrt{1+z^2}}{3(b+2mH_{c,i}^j)} \right)} \quad \text{and} \quad \theta_i^j = gA_i^j (S_f - S_0).$$

The following algorithm summarizes the general procedure for stabilizing the flow in lateral channels and iteratively observing the flow in the water reservoir.

Algorithm 1: Global resolution algorithm.

- 1: **Input:** $H_k^*, Q_k^*, B_k^*, d_k, L_k, \mu_k, k = 1, \dots, k_{\max}$.
 - 2: **Step 1:** Solve (10) under control (15) using (55).
 - 3: Generate $\{w_i^n\}_{n=1}^{N_T}$ and $\{q_{L,k}^{d,g}\}_{n=1}^{N_T}, i = 1, \dots, N_x$.
 - 4: **Step 2:** Solve (50).
 - 5: Generate $\{q_L\}_{n=1}^{N_T}$.
 - 6: **Final Step:** Solve (3) using (58)–(59).
 - 7: Generate $\{W_i^n\}_{n=1}^{N_T}$.
-

7.2 Numerical results

To illustrate the operation of the proposed device during a 3-hour rainfall event (i.e., 10,800 seconds), let us consider a trapezoidal water reservoir with a width of 120m, a length of 25km, a bottom slope of 0.00061, and a channel side slope of 45 degrees. We assume that the water flow is uniform, with an initial discharge at $t = 0$ of $100 \text{ m}^3/\text{s}$ and an initial water level of 4m. For the boundary conditions, the upstream discharge is given as a function:

$$Q(0,t) = \begin{cases} \frac{Q_p}{2} \sin\left(\left(\frac{t}{t_p} - \frac{1}{2}\right)\pi\right) + \frac{Q_p}{2} + Q_b & \text{if } 0 < t < t_p \\ \frac{Q_p}{2} \cos\left(\frac{t-t_p}{t_b-t_p}\pi\right) + \frac{Q_p}{2} + Q_b & \text{if } t_p < t < t_b \\ Q_b & \text{if } t > t_b, \end{cases} \quad (60)$$

where, the peak discharge $Q_p = 200 \text{ m}^3/\text{s}$, the base discharge $Q_b = 100 \text{ m}^3/\text{s}$, the time to peak $t_p = 5$ minutes, end the time of hydrograph $t_b = 15$ minutes.

Table 1: Physical parameters and dimensionless numbers

Parameters	$\epsilon \text{ (m}^3/\text{s)}$	μ_k	$A_g \text{ (m)}$	Φ	P_1	P_2	P_3	P_4	P_5
Values	0.1	0.82	0.008	0.002	5	12.5	2.5	15	10

$Q_3^*(\text{m}^3/\text{s})$	$Q_4^*(\text{m}^3/\text{s})$	$Q_5^*(\text{m}^3/\text{s})$	$B_1^*(\text{m})$	$B_2^*(\text{m})$	$B_3^*(\text{m})$	$B_4^*(\text{m})$	$B_5^*(\text{m})$
0.25	1.5	1	0.2	0.5	0.1	0.6	0.4

$L_1 \text{ (m)}$	$L_2 \text{ (m)}$	$L_3 \text{ (m)}$	$L_4 \text{ (m)}$	$L_5 \text{ (m)}$	$d_1 \text{ (m)}$	$d_2 \text{ (m)}$	$d_3 \text{ (m)}$	$d_4 \text{ (m)}$	N_x
2	1	3	1.5	2.5	1000	1500	2000	7000	500

$H_1^*(\text{m})$	$H_2^*(\text{m})$	$H_3^*(\text{m})$	$H_4^*(\text{m})$	$H_5^*(\text{m})$	$Q_1^*(\text{m}^3/\text{s})$	$Q_2^*(\text{m}^3/\text{s})$	$d_5 \text{ (m)}$	CFL
0.5	1.25	0.25	1.5	1	0.5	1.25	1500	0.5

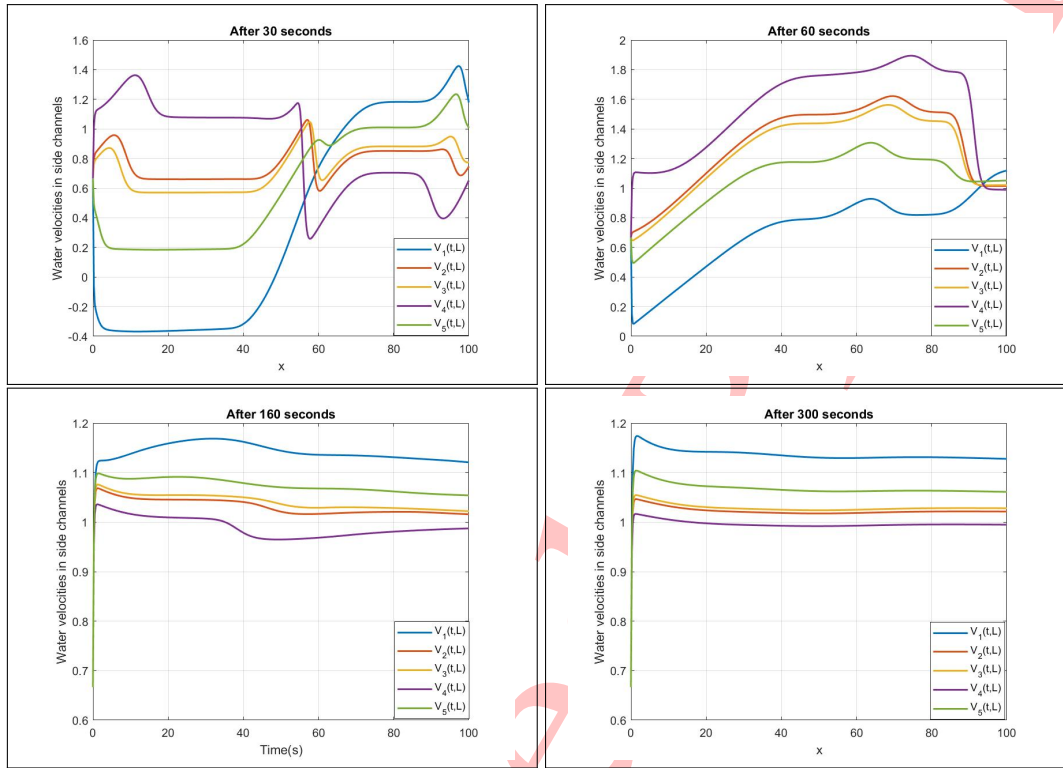


Figure 5: Velocity convergence

Figures 6 and 7 shows the evolution of the bed and velocity in Channel 5 at four specific time points. These figures once again and more clearly illustrate the stability of the flow and bathymetry. As for Figure 8, it depicts the evolution of the flow along the reservoir at five specific time points. It can be observed that, although the flow is highly non-uniform at the inlet of the reservoir, it becomes uniform downstream of this section.

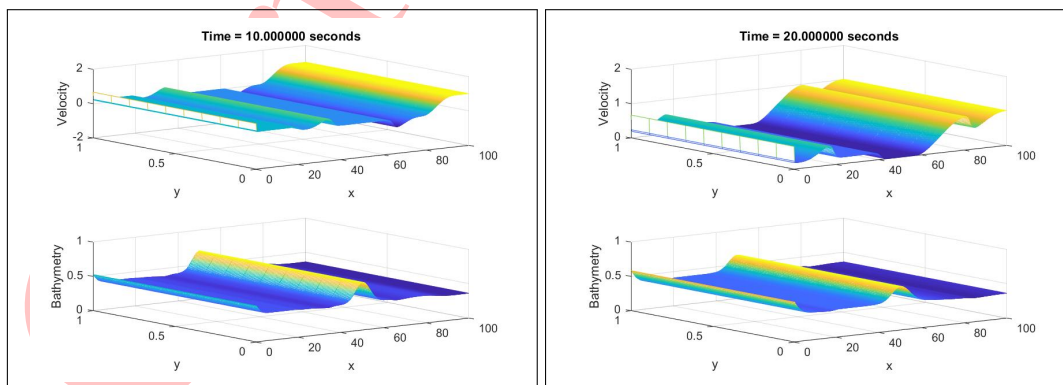


Figure 6: Convergence of velocity and bathymetry

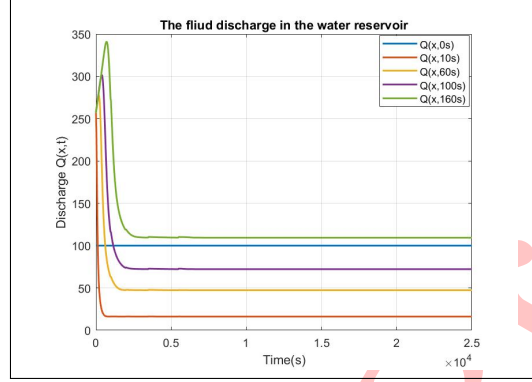


Figure 8: Discharge evolution in the water reservoir

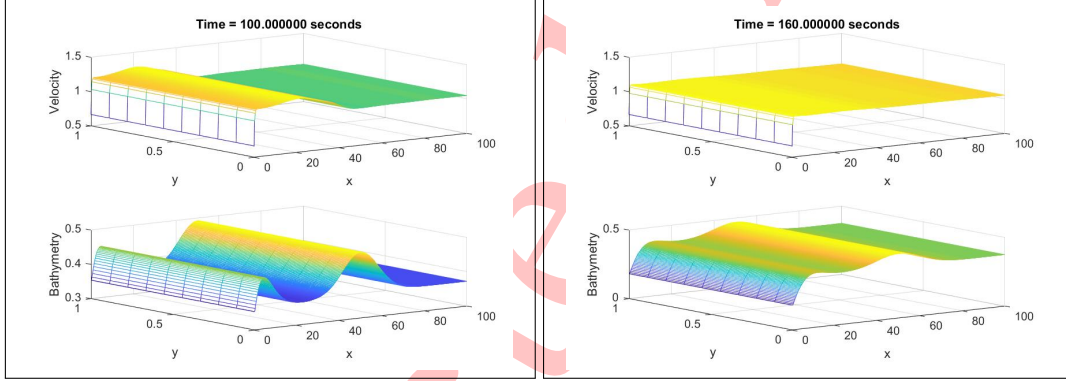


Figure 7: Convergence of velocity and bathymetry

To further demonstrate the effectiveness of our control system, in addition to qualitative results, we provide quantitative indicators such as the behavior of the solid discharge, the total mass of sediment trapped by the system, and the reduction in sediment input compared to an uncontrolled scenario.

In Figure 9, the comparison of sediment fluxes in the lateral channels with and without control clearly highlights the effect of the control mechanism on sediment transport dynamics. In the absence of control, the obtained profiles show a progressive increase in sediment discharge throughout the spatio-temporal domain. This behavior reflects an intense downstream sediment transport toward the reservoir. In contrast, when the control mechanism is activated, the sediment flux becomes progressively attenuated and constrained to decrease toward the downstream direction. This indicates a significant reduction in sediment transport toward the outlet. Therefore, the proposed control strategy effectively limits sediment propagation and contributes to retaining sediments outside the hydraulic reservoir.

In Figure 10, the comparison between the contour plots of $Q_{s,k}$ with and without control clearly highlights the effect of the control mechanism on sediment transport dynamics.

In the uncontrolled case, the contour plot shows a progressive increase of the sediment discharge throughout the space-time domain. The regions associated with high values of $Q_{s,k}$ (green to yellow areas) occupy a large portion of the domain and become more pronounced as time evolves. This behavior

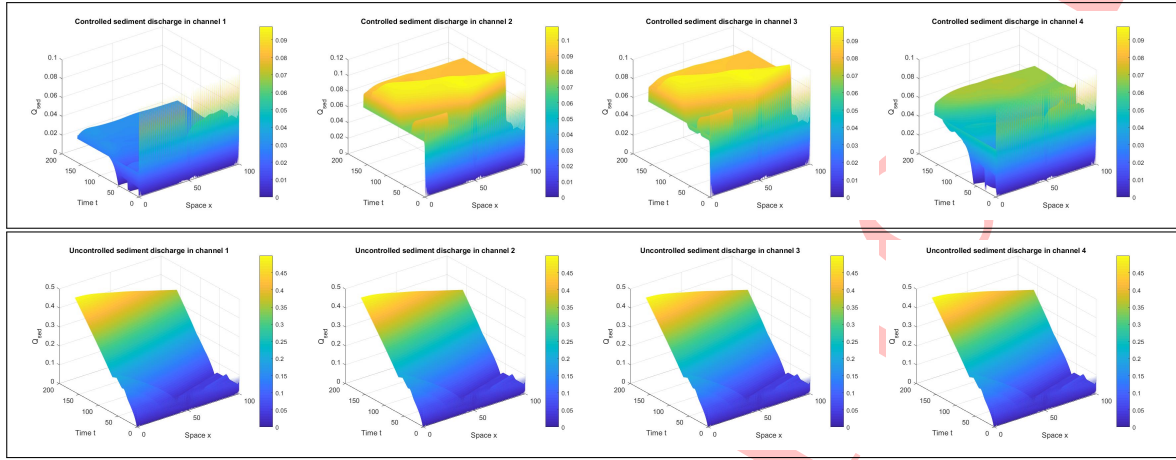


Figure 9: Evolution of sediment flux in the lateral channels

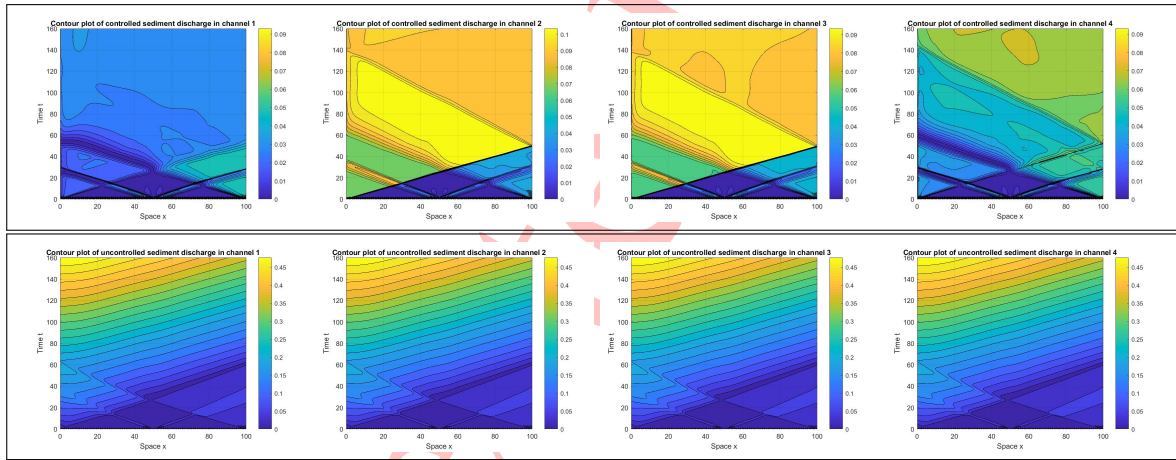


Figure 10: Evolution of sediment flux contours in the lateral channels

indicates that sediment perturbations propagate freely without sufficient dissipative effects to attenuate the transport process.

The diagonal structures observed in the contour plot correspond to the propagation of waves along the characteristic directions of the hyperbolic system. In the absence of control, these structures persist over long times, revealing weak attenuation of perturbations and progressive amplification of sediment transport.

In contrast, in the controlled case, the amplitudes of $Q_{s,k}$ are significantly reduced. The regions associated with high sediment discharge become more localized, while low-amplitude regions (blue and purple areas) progressively dominate the domain. This behavior shows that the control mechanism acts as a dissipative process capable of attenuating the propagation of morphodynamic perturbations.

Moreover, the comparison of the color scales confirms this reduction: $Q_{s,2}^{\max} \approx 0.45$ (without control), whereas $Q_{s,2}^{\max} \approx 0.10$ (with control).

Hence, the proposed control strategy significantly reduces sediment transport intensity and promotes

the progressive stabilization of the hydrodynamic–sedimentary system. These results suggest that the control mechanism efficiently limits erosion phenomena, excessive sediment transport, and sediment accumulation within the computational domain.

Figure 11 presents the evolution of the total sediment mass at the outlet of channel number 5 over time, with and without application of the control. It can be observed that the control considerably reduces the solid input to the water reservoir. The analysis of the total sediment mass deposited during the first 160 seconds highlights the effectiveness of the proposed control strategy. Specifically, a 75.5% reduction in the exported sediment mass is achieved in Channel 5. The corresponding reduction rates for all five channels are reported in Table 2. It can be observed that Channel 4 exhibits the lowest reduction rate. This behavior is attributed to the large initial flow amplitude ($P = 15$), which induces a significant initial disturbance of the system and consequently limits the control performance during the transient phase.

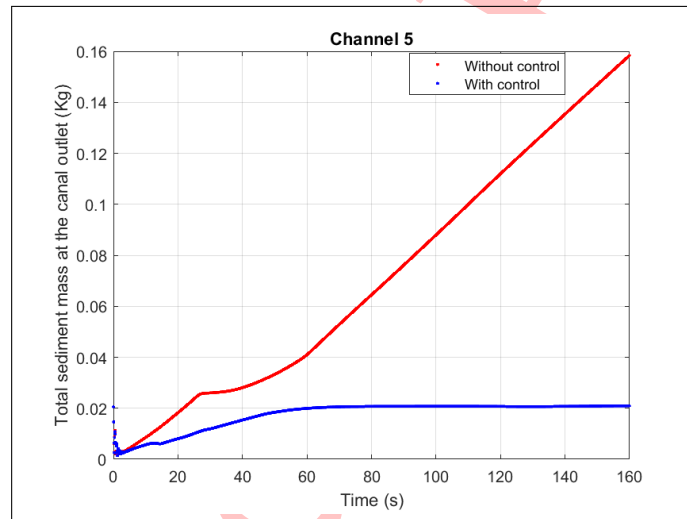


Figure 11: Evolution of the total sediment mass at the outlet of channel number 5 over time

Table 2: Relative reduction of solid input into the reservoir through the five channels

Channels	1	2	3	4	5
Without control	126.4980 kg	126.4980 kg	126.4980 kg	126.4980 kg	126.4980 kg
With control	16.8499 kg	52.4218 kg	47.7093 kg	79.1587 kg	30.9962 kg
Reduction	86.68 %	58.56 %	62.28 %	37.42 %	75.5 %

To assess the sensitivity of the proposed method to variations in the sediment diffusion coefficient, we varied the parameter ε from 0.01 to (0.15), encompassing the nominal value of 0.1 adopted in our numerical simulations.

The results for Channel 1, presented in Figure 12, illustrate the temporal evolution of the deposited sediment mass for the different values of ε . It can be observed that the proposed method is moderately

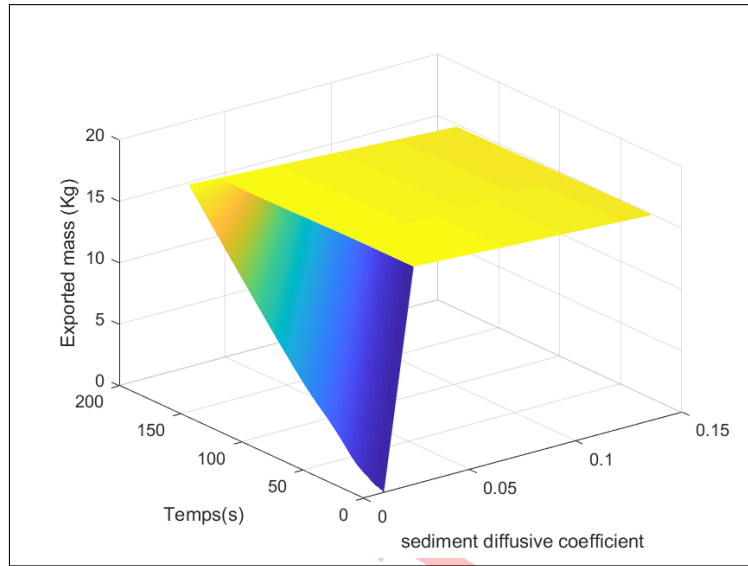


Figure 12: Sensitivity of the controlled solution to variations in the sediment diffusion coefficient

sensitive to variations in the diffusion coefficient for values below (0.03). Within this range, the deposited sediment mass increases with increasing ϵ , while exhibiting only a weak dependence on time. In contrast, for $\epsilon > 0.03$, the method becomes practically insensitive to further changes in the diffusion coefficient, indicating that the influence of this parameter reaches a saturation level and has only a negligible effect on the computed results.

8 Conclusion

In this article, we propose an innovative method for controlling sediment transport in water reservoirs by implementing a peripheral band equipped with sluice gates around the basin, accompanied by advanced mathematical modeling. This modeling is used to design a closed-loop control system that adjusts the opening rate of each sluice gate in real-time to stabilize and reduce sediment transport upstream. Numerical simulations were also conducted to illustrate the efficiency and relevance of this approach in protecting reservoirs from siltation. In the proposed control system, each lateral channel retains an average of 30 kg of sediment over 160 seconds, representing a retention capacity exceeding 64%. This contribution could play a key role in integrated sediment management, promoting the sustainability of soils, hydraulic infrastructure, and ecosystems within water reservoirs.

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- Data availability : Not applicable
- Materials availability : Not applicable
- Code availability : Not applicable
- Author contribution : The authors contributed equally to this work

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