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On intuitionistic L-fuzzy PMS-ideals in PMS-algebras

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Abstract. In this paper, we apply the concept of an intuitionistic L-fuzzy set to PMS-ideals of PMS-algebras. The notion of the intuitionistic L-fuzzy PMS-ideal of a PMS-algebra is introduced and several associated properties are investigated. A condition for an intuitionistic L-fuzzy set in a PMS-algebra to be an intuitionistic L-fuzzy PMS-ideal is established. Characterizations of intuitionistic L-fuzzy PMS-ideals of a PMS-algebra in terms of their level subsets are given. The algebraic nature of intuitionistic L-fuzzy PMS-ideals of a PMS-algebra under homomorphism is discussed. Moreover, the Cartesian product of the intuitionistic L-fuzzy PMS-ideals of a PMS-algebra is also studied and some interesting results are obtained. Finally, the relationship between the strongest intuitionistic L-fuzzy relations in a PMS-algebra and intuitionistic L-fuzzy PMS-ideals of a PMS-algebra are explored.

Keywords: PMS-algebra, PMS-ideal, Intuitionistic L-fuzzy PMS-ideal, Homomorphism and Cartesian product.
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1 Introduction

In 1965, Zadeh [27] introduced the concept of fuzzy sets as a generalization of classical sets to deal with vagueness and imprecision in real-life situations. Since then this concept has been applied to various algebraic structures. Rosenfield [22] applied the notion of fuzzy sets in group theory to develop fuzzy groups and studied them under homomorphism. Choudhury [7] also introduced the notion of a fuzzy homomorphism and studied its effect on fuzzy subgroups. Priya and Ramachandran [21] discussed fuzzy PS-ideals and fuzzy PS-subalgebras of PS-algebra under homomorphism and Cartesian products as well as the strongest fuzzy relations on PS-algebra.

After introducing the concept of fuzzy sets by Zadeh, many generalizations of this fundamental concept have been given by several researchers. Atanassov [3–5] extended the concept

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of a fuzzy set to an intuitionistic fuzzy set to better deal with uncertainties and imprecision by introducing an additional function, which he called a nonmembership function with some natural conditions on membership and nonmembership functions to each element of a set. Zarandi and Saeid [28] studied the intuitionistic fuzzy BG-ideals of the BG-algebra. Panigrahi and Nanda [20] studied the idea of intuitionistic fuzzy relations over intuitionistic fuzzy subsets and found several interesting properties of intuitionistic fuzzy relations in intuitionistic fuzzy subsets. Anitha and Arjunan [2] studied the strongest intuitionistic fuzzy relations on intuitionistic fuzzy ideals of Hemirings and obtained some interesting results

In 1967, Goguen [13] introduced the notion of L-fuzzy sets as another extension of zadeh's fuzzy set where L is a complete lattice with least element 0 and greatest element 1. Jefferson et al. [8,9] introduced the notion of L-Fuzzy BP-Algebras and explored some properties of L-Fuzzy BP-subalgebras and L-Fuzzy BP ideals. In 1984, Atanassov and Stoeva [5] further generalized the notion of L-fuzzy subset and intuitionistic fuzzy subset to L-intuitionistic fuzzy subset. Atanassov and Stoeva have defined both membership and nonmembership functions from the nonempty set X to L, where L is any complete lattice with a complete order reversing involution N. Basing with this motivation, many mathematicians started to review various concepts in intuitionistic L-fuzzy structures. Naganathan et al. [18,19] introduced the intuitionistic L-fuzzy subgroups and Level subsets of intuitionistic L-fuzzy subgroups of a group and discussed their properties in detail. Muralikrishna and Chandramouleeswaran [16, 17] discussed Intuitionistic L-fuzzy BF-ideals and the Cartesian product on Intuitionistic L-fuzzy ideals of BF-algebras. Chandramouleeswaran and Muralikrishna [6] also discussed the notions of Intuitionistic L-fuzzy subalgebras of a BG-algebra and some of their basic properties. Sowmiya and Jeyalakshmi [25] introduced the notions of intuitionistic L-fuzzy Z-ideals in Z-algebras and discussed some of their properties. Moreover, they explored the Cartesian product of intuitionistic L-fuzzy Z-ideals in Z-algebras.

The notion of BCK-algebras was introduced by Iseki and Tanaka [14] and Iseki [15] also introduced the notion of a BCI-algebra as a generalization of a BCK-algebra. In 2016, Sithar Selvam and Nagalakshmi [24] introduced a new class of algebra called PMS-algebra as a generalization of several algebras such as BCK, BCI, TM and PS-algebras. In our earlier works, we studied intuitionistic fuzzy PMS-subalgebras and intuitionistic fuzzy PMS-ideals [11, 26], and we looked at a number of findings. Moreover, Alaba et al. [1, 12] discussed the notion of intuitionistic fuzzy PMS-subalgebras and intuitionistic fuzzy PMS-ideals under homomorphism and Cartesian product in PMS-algebras, and addressed several results. Several researchers have applied the concept of intuitionistic L-fuzzy set to different algebraic structures. However, as far as we know there is no any study on intuitionistic L-fuzzy PMS-ideals of a PMS-algebra. With this motivation we develop the intuitionistic L-fuzzy PMS-ideals in a PMS-subalgebra.

In this article, we apply the concept of intuitionistic L-fuzzy set to PMS-ideals of PMS-algebras. We introduce the notion of the intuitionistic L-fuzzy PMS-ideal of a PMS-algebra and investigate several results. We establish a condition for an intuitionistic L-fuzzy set in a PMS-algebra to be an intuitionistic L-fuzzy PMS-ideal. We provide some characterizations of intuitionistic L-fuzzy PMS-ideals of a PMS-algebra in terms of their level subsets. We discuss the algebraic nature of intuitionistic L-fuzzy PMS-ideals of a PMS-algebra under homomorphism and obtain some important results. Moreover, we study the Cartesian product of the intuitionistic L-fuzzy PMS-ideals of a PMS-algebra and investigate some interesting results. Finally, we discuss the relationship between the strongest intuitionistic L-fuzzy relations in a PMS-algebra and intuitionistic L-fuzzy PMS-ideals of a PMS-algebra.

2 Preliminaries

In this section, we recall some basic definitions that are required for the development of this work.

Definition 1 ([24]). *A nonempty set X with a constant element 0 and a binary operation $*$ is called a PMS-algebra if it satisfies the following axioms;*

(i). $0 * x = x$,

(ii). $(y * x) * (z * x) = z * y$, *for all* $x, y, z \in X$.

A binary relation "$\leq$" on X can be defined as $x \leq y$ if and only if $x * y = 0$.

Definition 2 ([24]). *A nonempty subset S of a PMS-algebra X is said to be a PMS-subalgebra if $x * y \in S$, for all $x, y \in S$.*

Definition 3 ([24]). *A nonempty subset I of a PMS-algebra X is said to be a PMS-ideal of X if it satisfies the following conditions:*

(i). $0 \in I$,

(ii). $z * y, z * x \in I \Rightarrow y * x \in I$, *for all* $x, y, z \in X$.

Proposition 1 ([24]). *In any PMS-algebra X the following properties hold for all $x, y, z \in X$.*

(i). $x * x = 0$,

(ii). $(y * x) * x = y$,

(iii). $x * (y * x) = y * 0$,

(iv). $(y * x) * z = (z * x) * y$,

(v). $(x * y) * 0 = y * x = (0 * y) * (0 * x)$.

Definition 4 ([24]). *Let X and Y be two PMS-algebras. A mapping $f : X \to Y$ is said to be a homomorphism of PMS-algebras if $f(x * y) = f(x) * f(y)$ for all $x, y \in X$. f is called an epimorphism of PMS-algebras if it is onto. If $f : X \to Y$ is a homomorphism , then $f(0) = 0$.*

Definition 5 ([13]). *Let $(L, \leq, \wedge, \vee)$ be a complete lattice with least element 0 and greatest element 1. An L-fuzzy subset A of any non-empty set X is defined as a membership function* $\zeta_A : X \to \mathrm{L}$.

Definition 6 ([5]). *Let (L,≤,∧,∨) be a complete lattice with least element 0 and greatest element 1 and an involutive order reversing operation N: L→ L. Then Intuitionistic L-Fuzzy Subset (ILFS) A of a nonempty set X is an object having the form* $A = \{\langle x, \zeta_A(x), \xi_A(x)\rangle | x \in X\}$, *where* $\zeta_A : X \to \mathrm{L}$ *is the degree of membership and* $\xi_A : X \to \mathrm{L}$ *is the degree of nonmembership of the element* $x \in X$ *satisfying* $\zeta_A(x) \leq \mathrm{N}(\xi_A(x))$. *For simplicity we write* $A = (\zeta_A, \xi_A)$ *for* $A = \{\langle x, \zeta_A(x), \xi_A(x)\rangle | x \in X\}$.

Lemma 1 ([10]). *Let* $A = (\zeta_A, \xi_A)$ *be an intutionistic L-fuzzy set in X. Then the following statements hold for any* $x, y \in X$.

(1). $1 - [\zeta_A(x) \vee \zeta_A(y)] = [1 - \zeta_A(x)] \wedge [1 - \zeta_A(y)]$.

(2). $1 - [\zeta_A(x) \wedge \zeta_A(y)] = [1 - \zeta_A(x)] \vee [1 - \zeta A(y)]$.

(3). $1 - [\xi_A(x) \vee \xi_A(y)] = [1 - \xi_A(x)] \wedge [1 - \xi_A(y)]$.

(4). $1 - [\xi_A(x) \wedge \xi_A(y)] = [1 - \xi_A(x)] \vee [1 - \xi_A(y)]$.

Definition 7 ([10]). *An ILFS* $A = (\zeta_A, \xi_A)$ *of a PMS-algebra X is called an Intuitionistic L-fuzzy PMS-subalgebra (ILF PMS-SA) of X if it satisfies the following conditions;*

(i). $\zeta_A(x * y) \geq \zeta_A(x) \wedge \zeta_A(y)$,

(ii). $\xi_A(x * y) \leq \xi_A(x) \vee \xi_A(y)$, *for all* $x, y \in X$.

Definition 8 ([5]). *Let* $A = (\zeta_A, \xi_A)$*and* $B = (\zeta_B, \xi_B)$ *be any two intuitionistic L-fuzzy subset of a nonempty set X. Then we have*

(1). $\bar{A} = \{\langle x, \xi_A(x), \zeta_A(x)\rangle | x \in X\}$,

(2). $A \cap B = \{\langle x, \zeta_A(x) \wedge \zeta_B(x), \xi_A(x) \vee \xi_A(x)\rangle | x \in X\}$,

(3). $A \cup B = \{\langle x, \zeta_A(x) \vee \zeta_B(x), \xi_A(x) \wedge \xi_A(x)\rangle | x \in X\}$,

(4). $\Box A = \{\langle x, \zeta_A(x), 1 - \zeta_A(x)\rangle | x \in X\}$,

(5). $\Diamond A = \{\langle x, 1 - \xi_A(x), \xi_A(x)\rangle | x \in X\}$,

(6). $\cap A_i = \{\langle x, \wedge\zeta_(A_i)(x), \vee\xi_(A_i)(x)\rangle | x \in X\}$, *where* $\{A_i | i \in I\}$ *is an arbitrary family of Intuitionistic L-fuzzy subsets of a nonempty set X.*

Definition 9 ([2, 23]). *Let* $A = (\zeta_A, \xi_A)$ *be ILFS of Y and* $f : X \to Y$ *be a mapping. Then the inverse image of A under f, denoted by* $f^{-1}(A)(x) = \left\{\left\langle x, \zeta_{f^{-1}(B)}(x), \xi_{f^{-1}(A)}(x)\right\rangle | x \in X\right\}$, *is an intuitionistic L-fuzzy set in Y defined by* $\zeta_{f^{-1}(A)}(x) = \zeta_A(f(x))$ *and* $\xi_{f^{-1}(A)})(x) = \xi_A(f(x))$, *for all* $x \in X$.

Definition 10 ([5]). *Let* $A = (\zeta_A, \xi_A)$ *and* $B = (\zeta_B, \xi_B)$ *be any two intuitionistic L-fuzzy subsets (ILFSs) of a set X. The Cartesian product* $A \times B = (\zeta_{A\times B}, \xi_{A\times B})$ *whose membership function* $\zeta_{A\times B} : X \times X \to \mathrm{L}$ *and nonmembership function* $\xi_{A\times B} : X \times X \to \mathrm{L}$ *are defined by* $\zeta_{A\times B}(x, y) = \zeta_A(x) \wedge \zeta_B(y)$ *and* $\xi_{A\times B}(x, y) = \xi_A(x) \vee \xi_B(y)$, *for all* $x, y \in X$.

Definition 11 ([2, 20]). *Let $B = (\zeta_B, \xi_B)$ be an intuitionistic L-fuzzy set on a set X and $A = (\zeta_A, \xi_A)$ is an intuitionistic L-fuzzy relation on a set X. Then the strongest intuitionistic L-fuzzy relation A_B on X, whose membership function $\zeta_{A_B} : X \times X \to \mathrm{L}$ and whose nonmembership function $\zeta_{A_B} : X \times X \to \mathrm{L}$ are given by $\zeta_{A_B}(x, y) = \zeta_B(x) \wedge \zeta_B(y)$ and $\xi_{A_B}(x, y) = \xi_B(x) \vee \xi_B(y)$.*

3 Intuitionistic L-fuzzy PMS-ideals in PMS-algebras

In this section, we introduce the notion of intuitionistic L-fuzzy PMS-ideal of a PMS-algebra and find some interesting results. Throughout this and the subsequent sections, X and Y denote the PMS-algebras.

Definition 12. *An ILFS $A = (\zeta_A, \xi_A)$ of a PMS-algebra X is called an intuitionistic L-fuzzy PMS-ideal (ILF PMS-Id) of X if it satisfies the following conditions;*

(i). $\zeta_A(0) \geq \zeta_A(x)$ *and* $\xi_A(0) \leq \xi_A(x)$,

(ii). $\zeta_A(y * x) \geq \zeta_A(z * y) \wedge \zeta_A(z * x)$,

(iii). $\xi_A(y * x) \leq \xi_A(z * y) \vee \xi_A(z * x)$, *for all* $x, y, z \in X$.

Example 1. Let L=[0,1] be a complete lattice and $\mathbb{Z}$ be the ring of integers. For all $x, y \in \mathbb{Z}$, the binary operation $*$ is defined by $x * y = y - x$, where $'-'$ is the usual subtraction of integers gives that $(\mathbb{Z}, *, 0)$ is a PMS-algebra. Then one can easily verify that the ILFS $A = (\zeta_A, \xi_A)$ of $\mathbb{Z}$ defined by

$$\zeta_A(x) = \begin{cases} 0.6 & if\ x \in \langle 4 \rangle \\ 0.4 & if\ x \in \langle 2 \rangle - \langle 4 \rangle \\ 0 & otherwise \end{cases} \text{ and } \xi_A(x) = \begin{cases} 0.2 & if\ x \in \langle 4 \rangle \\ 0.5 & if\ x \in \langle 2 \rangle - \langle 4 \rangle \\ 1 & otherwise \end{cases}$$

is an ILF PMS-ideal of $\mathbb{Z}$.

Theorem 1. *Let $A = (\zeta_A, \xi_A)$ be an ILF PMS-Id of a PMS-algebra X with $x * y = 0$, for all $x, y \in X$. Then $\zeta_A(x) = \zeta_A(y)$ and $\xi_A(x) = (y)$.*

Proof. Let $x, y \in X$ such that $x * y = 0$. Since A is an ILF PMS-Id of X, by Proposition 1 (ii) and Definition 1(i), $\zeta_A(x) = \zeta_A((x * y) * y) = \zeta_A(0 * y) = \zeta_A(y)$ and $\xi_A(x) = \xi_A((x * y) * y) = \xi_A(0 * y) = \xi_A(y)$. Hence, $\zeta_A(x) = \zeta_A(y)$ and $\zeta_A(x) = (y)$, for all $x, y \in X$. □

Theorem 2. *Every ILF PMS-Id $A = (\zeta_A, \xi_A)$ of X is an ILF PMS-SA of X.*

Theorem 3. *Let $A = (\zeta_A, \xi_A)$ be an ILF PMS-Id of X such that $\zeta_A(x * y) \geq \zeta_A(0)$ and $\xi_A(x * y) \leq \xi_A(0)$. Then $\zeta_A(x) = \zeta_A(y)$ and $\xi_A(x) = \xi_A(y)$ for all $x, y \in X$.*

Proof. Since $\zeta_A(0) \geq \zeta_A(x*y)$ and $\xi_A(0) \leq \xi_A(x*y)$ by Definition 12(i), we get $\zeta_A(0) = \zeta_A(x*y)$ and $\xi_A(0) = \xi_A(x * y)$. Again by Proposition 1(ii) and Theorem 2, we have

$$\zeta_A(x) = \zeta_A\left((x * y) * y\right) \geq \zeta_A(x * y) \wedge \zeta_A(y) = \zeta_A(0) \wedge \zeta_A(y) = \zeta_A(y),$$

and

$$\xi_A(x) = \xi_A((x*y)*y) \leq \xi_A(x*y) \wedge \xi_A(y) = \xi_A(0) \wedge \xi_A(y) = \xi_A(y).$$

Therefore, $\zeta_A(x) \geq \zeta_A(y)$ and $\xi_A(x) \leq \xi_A(y)$.

Conversely, $\zeta_A(y) = \zeta_A((y*x)*x) \geq \zeta_A(y*x) \wedge \zeta_A(x) = \zeta_A(0) \wedge \zeta_A(x) = \zeta_A(x)$ and $\xi_A(y) = \xi_A((y*x)*x) \leq \xi_A(y*x) \wedge \xi_A(x) = \xi_A(0) \wedge \xi_A(x) = \xi_A(x)$ Hence, $\zeta_A(y) \geq \zeta_A(x)$ and $\xi_A(y) \leq \xi_A(x)$. Thus, $\zeta_A(x) = \zeta_A(y)$ and $\xi_A(x) = \xi_A(y)$ for all $x, y \in X$. □

Corollary 1. *Let $A = (\zeta_A, \xi_A)$ be an ILF PMS-Id of X such that $\zeta_A(x*y) = 1$ and $\xi_A(x*y) = 0$, for all $x, y \in X$. Then $\zeta_A(y) = \zeta_A(x)$ and $\xi_A(x) = \xi_A(y)$.*

Proof. By Proposition 1(ii, v) and Theorem 2, it follows that $\zeta_A(x) = \zeta_A(y)$ and $\xi_A(x) = \xi_A(y)$. □

Theorem 4. *If $A = (\zeta_A, \xi_A)$ and $B = (\zeta_B, \xi_B)$ are two ILF PMS-Ids of a PMS-algebra X, then $A \cap B$ is again an ILF PMS-Id of X.*

Proof. Let A and B be ILF PMS-Ids of X. Then for all $x, y, z \in$ X, $\zeta_{A\cap B}(0) = \zeta_A(0) \wedge \zeta_B(0) \geq \zeta_A(x) \wedge \zeta_B(x) = \zeta_{A\cap B}(x)$ and $\xi_{A\cap B}(0) = \xi_A(0) \vee \xi_B(0) \leq \xi_A(x) \vee \xi_B(x) = \xi_{A\cap B}(x)$. Therefore, $\zeta_{A\cap B}(0) \geq \zeta_{A\cap B}(x)$ and $\xi_{A\cap B}(0) \leq \xi_{A\cap B}(x)$.

$$\begin{aligned} \zeta_{A\cap B}(y*x) &= \zeta_A(y*x) \wedge \zeta_B(y*x) \\ &\geq [\zeta_A(z*y) \wedge \zeta_A(z*x)] \wedge [\zeta_B(z*y) \wedge \zeta_B(z*x)] \\ &= [\zeta_A(z*y) \wedge \zeta_B(z*y)] \wedge [\zeta_A(z*x) \wedge \zeta_B(z*x)] \\ &= \zeta_{A\cap B}(z*y) \wedge \zeta_{A\cap B}(z*x) \end{aligned}$$

and also,

$$\begin{aligned} \xi_{A\cap B}(y*x) &= \xi_A(y*x) \vee \xi_B(y*x) \\ &\leq [\xi_A(z*\mathrm{y}) \vee \xi_A(z*x)] \vee [\xi_B(z*y) \vee \xi_B(z*x)] \\ &= [\xi_A(z*y) \vee \xi_B(\mathrm{z}*y)] \vee [\xi_A(z*x) \vee \xi_B(z*x)] \\ &= \xi_{A\cap B}(z*y) \vee \xi_{A\cap B}(z*x) \end{aligned}$$

Therefore, $A \cap B$ is an ILF PMS-Id of X. □

We can generalize the above theorem to any family of ILF PMS-Ids in a PMS-algebra of as follows.

Corollary 2. *Let $\{A_i |\ i \in \mathbb{N}\}$ be a family of ILF PMS-Ids of X. Then $\bigcap_{i\in\mathbb{N}} A_i$ is an ILF PMS-Id of X.*

Theorem 5. *An ILFS $A = (\zeta_A, \xi_A)$ of X is an ILF PMS-Id of X if and only if the L-fuzzy subsets ζ_A and $\overline{\xi}_A$ are L-fuzzy PMS-ideals of X.*

Proof. Let $A = (\zeta_A, \xi_A)$ be an ILF PMS-Id of X. Since A is an ILF PMS-Id of X, it follows that the L-fuzzy subset ζ_A is an L-fuzzy PMS-ideal of X. Now, it remains to show that $\overline{\xi}_A$ is a fuzzy PMS-ideal of X. Let $x, y, z \in X$. Then we have $\overline{\xi}_A(0) = 1 - \xi_A(0) \geq 1 - \xi_A(x) = \overline{\xi}_A(x)$ and also

$$\begin{aligned}\overline{\xi}_A(y * x) &= 1 - \xi_A(y * x) \geq 1 - [\xi_A(z * y) \vee \xi_A(z * x)] \\ &= [1 - \xi_A(z * y)] \wedge [1 - \xi_A(z * x)] \text{ (By lemma 1(3))} \\ &= \overline{\xi}_A(z * y) \wedge \overline{\xi}_A(z * x)\end{aligned}$$

Hence, $\overline{\xi}_A$ is an L-fuzzy PMS-ideal of X.

Conversely, assume that ζ_A and $\overline{\xi}_A$ are L-fuzzy PMS-ideals of X. Then for every $x, y, z \in X$, we get $\zeta_A\ (0) \geq \zeta_A(x)$ and $\overline{\xi}_A(0) \geq \overline{\xi}_A(x)$. Now,

$$\overline{\xi}_A(0) \geq\ \overline{\xi}_A(x) \Longrightarrow 1 - \xi_A(0) \geq 1 - \xi_A(x) \Longrightarrow \xi_A(0) \leq\ \xi_A(x).$$

Also, $\zeta_A(y * x) \geq \zeta_A(z * y) \wedge \zeta_A(z * y)$ and $\overline{\xi}_A(y * x) \geq \overline{\xi}_A(z * y) \wedge \overline{\xi}_A(z * y)$, and

$$\begin{aligned}1 - \xi_A(y * x) = \overline{\xi}_A(y * x) &\geq \overline{\xi}_A(z * y) \wedge \overline{\xi}_A(z * y) \\ &= [1 - \xi_A(z * y)] \wedge [1 - \xi_A(z * x)] \\ &= 1 - [\xi_A(z * y) \vee \xi_A(z * x)] \text{ (By lemma 1 (3)).}\end{aligned}$$

This implies that $\xi_A(y * x) \leq \xi_A(z * y) \vee \xi_A(z * x)$. Hence, $A = (\zeta_A, \xi_A)$ is an ILF PMS-Id of X. □

Theorem 6. *If $A = (\zeta_A, \xi_A)$ is an ILF PMS-ideal of a PMS-algebra X, then $\Box A = (\zeta_A, \overline{\zeta}_A)$ is an ILF PMS-Id of X.*

Proof. Let $A = (\zeta_A, \xi_A)$ be an ILF PMS-Id of X. So, by Definition 12, $\zeta_A(\ 0) \geq \zeta_A(x)$ and $\zeta_A(\ y * x) \geq \zeta_A(z * y) \wedge \zeta_A(z * x)$, for all $x,\ y, z \in X$. Now, we have to show that $\overline{\zeta}_A(0) \leq \overline{\zeta}_A(x)$ and $\overline{\zeta}_A(y * x) \leq \overline{\zeta}_A(z * y) \wedge \overline{\zeta}_A(z * x)$, for all $x, y, z \in X$.Now,

$$\overline{\zeta}_A(0) = 1 - \zeta_A(0) \leq 1 - \zeta_A(x) = \overline{\zeta}_A(x) \Longrightarrow \overline{\zeta}_A(0) \leq \overline{\zeta}_A(x).$$

Also,

$$\begin{aligned}\overline{\zeta}_A(y * x) &= 1 - \zeta_A(y * x) \leq 1 - [\zeta_A(z * y) \wedge \zeta_A(z * x)] \\ &= [1 - \zeta_A(z * y)] \vee [1 - \zeta_A(z * x)] \quad \text{(By lemma 1 (2))} \\ &= \overline{\zeta}_A(z * y) \vee \overline{\zeta}_A(z * x)\end{aligned}$$

Therfore, $\Box A = (\zeta_A, \overline{\zeta}_A)$ is an ILF PMS-Id of X. □

Remark 1. *The converse of Theorem 6 is not necessarily true. This fact is shown by the following example.*

Example 2. Let $\mathbb{Z}$ be the set of integers. Suppose $*$ is a binary operation in $\mathbb{Z}$ as defined in Example 1. Clearly, $(\mathbb{Z}, *, 0)$ is a PMS-algebra. Let $A = (\zeta_A, \xi_A)$ be an ILFS defined by

$$\zeta_A(x) = \begin{cases} 0.6 & if\ x\ is\ even \\ 0.4 & if\ x\ is\ odd \end{cases} \quad and\ \xi_A(x) = \begin{cases} 0.4 & if\ x\ is\ even \\ 0.2 & if\ x\ is\ odd \end{cases}.$$

Observe that $\xi_A(0) = 0.4 > 0.2 = \xi_A(1)$. This contradicts Definition 12(i). Therefore, $A = (\zeta_A, \xi_A)$ is not an ILF PMS-Id of $\mathbb{Z}$. Now,

$$\overline{\zeta}_A(x) = 1 - \zeta_A(x) = \begin{cases} 0.4 & if\ x\ is\ even \\ 0.6 & if\ x\ is\ odd \end{cases}.$$

By routine calculation we can show that $\Box A = (\zeta_A, \overline{\zeta}_A)$ is an ILF PMS-Id of $\mathbb{Z}$. So, this example verifies that the converse of the above Theorem is not necessarily true.

Theorem 7. *If $A = (\zeta_A, \xi_A)$ is an ILF PMS-Id of X, then $\Diamond A$=$(\overline{\xi}_A, \xi_A)$ is an ILF PMS-Id of X.*

Proof. Suppose $A = (\zeta_A, \xi_A)$ is an ILF PMS-Id of X. Then by Definition 12, we have, $\xi_A(0) \leq \xi_A(x)$ and $\xi_A(y * x) \leq \xi_A(z * y) \vee \xi_A(z * x)$, for all $x, y, z \in X$. So it suffices to show that $\overline{\xi}_A(0) \geq \overline{\xi}_A(x)$ and $\overline{\xi}_A(y*x) \geq \overline{\xi}_A(z*y) \wedge \overline{\xi}_A(z*x)$, for all $x, y, z \in X$. Now, $\overline{\xi}_A(0) = 1 - \zeta_A(0) \geq 1 - \xi_A(x) = \overline{\xi}_A(x) \Longrightarrow \overline{\xi}_A(0) \geq \overline{\xi}_A(x)$, and

$$\begin{aligned} \overline{\xi}_A(y * x) &= 1 - \zeta_A(y * x) \\ &\geq 1 - [\xi_A(z * y) \vee \xi_A(z * x)] \\ &= [1 - \xi_A(z * y)] \wedge [1 - \xi_A(z * x)] \quad \text{(By lemma 1(3))} \\ &= \overline{\xi}_A(z * y) \wedge \overline{\xi}_A(z * x) \end{aligned}$$

Hence, $\Diamond A = (\overline{\xi}_A, \xi_A)$ is an ILF PMS-Id of X. □

Remark 2. *The converse of the Theorem 7 is not true. It is shown by the following example.*

Example 3. Consider the PMS-algebra $(\mathbb{Z}, *, 0)$ as defined in Example 2 and define an ILFS $A = (\zeta_A, \xi_A)$ by

$$\zeta_A(x) = \begin{cases} 0.3 & if\ x\ is\ even \\ 0.5 & if\ x\ is\ odd \end{cases} \quad and\ \xi_A(x) = \begin{cases} 0.4 & if\ x\ is\ even \\ 0.5 & if\ x\ is\ odd \end{cases}.$$

We observe that $\zeta_A(0) = 0.3 < 0.5 = \zeta_A(3)$.This contradicts Definition 12. Therefore, $A = (\zeta_A, \xi_A)$ is not an ILF PMS-Id of $\mathbb{Z}$. Also,

$$\overline{\xi}_A(x) = 1 - \xi_A(x) = \begin{cases} 0.6 & if\ x\ is\ even \\ 0.5 & if\ x\ is\ odd \end{cases}.$$

Clearly, $\Diamond A = (\overline{\xi}_A, \xi_A)$ is an ILF PMS-Id of $\mathbb{Z}$. So, this example also verifies that the converse of the above Theorem is not necessarily true.

The next corollary gives the generalization of Theorem 6 and Theorem 7.

Corollary 3. *An ILFS $A = (\zeta_A, \xi_A)$ of a PMS-algebra X is an ILF PMS-Id of X if and only if $\Box A = (\zeta_A, \overline{\zeta}_A)$ and $\Diamond A = (\overline{\xi}_A, \xi_A)$ are ILF PMS-Ids of X.*

Theorem 8. *If $A = (\zeta_A, \xi_A)$ is an ILF PMS-Id of X, then $\overline{A}^c$ is also an ILF PMS-Id of X.*

Proof. Let A be an ILF PMS-Id of a PMS-algebra X. Then $A^c = \{\langle x, \zeta_{A^c}(x), \xi_{A^c}(x)\rangle | x \in X\}$, where $\zeta_{A^c}(x)$ $(x) = 1 - \zeta_A(x)$ and $\xi_{A^c}(x) = 1 - \xi_A(x)$. Therefore, $\overline{A}^c = \{\langle x, \xi_{A^c}(x), \zeta_{A^c}(x)\rangle | x \in X\}$. Let $x, y, z \in X$, then we have $\xi_{A^c}(0) = 1 - \xi_A(0) \geq 1 - \xi_A(x) = \xi_{A^c}(x)$ and $\zeta_{A^c}(0) = 1 - \zeta_A(0) \leq 1 - \zeta_A(x) = \zeta_{A^c}(x)$.
Also,

$$\begin{aligned}\xi_{A^c}(y * x) &= 1 - \xi_A(y * x) \\ &\geq 1 - [\xi_A(z * y) \vee \xi_A(z * x)] \\ &= [1 - \xi_A(z * y)] \wedge [1 - \xi_A(z * x)] \text{ (By Lemma 1(1))} \\ &= \xi_{A^c}(z * y) \wedge \xi_{A^c}(z * x),\end{aligned}$$

and,

$$\begin{aligned}\zeta_{A^c}(y * x) &= 1 - \zeta_A(y * x) \\ &\leq 1 - [\zeta_A(z * y) \wedge \zeta_A(z * x)] \\ &= [1 - \zeta_A(z * y)] \vee [1 - \zeta_A(z * x)] \text{ (By lemma 1(2))} \\ &= \zeta_{A^c}(z * y) \vee \zeta_{A^c}(z * x).\end{aligned}$$

Therefore, $\overline{A}^c$ is an ILF PMS-Id of X. □

Definition 13. *Let $A = (\zeta_A, \xi_A)$ be an ILFS of X. Then the intuitionistic L-fuzzy level subset of A is the subset $A_{s,t}$ of X given by $A_{s,t} = \{x \in X : \zeta_A(x) \geq s$ and $\xi_A(x) \leq t\}$, for $s, t \in L$.*

Theorem 9. *Let $A = (\zeta_A, \xi_A)$ be an ILFS of X. Then $A = (\zeta_A, \xi_A)$ is an ILF PMS-Id of X if and only if the nonempty subset $A_{s,t}$ of X is a PMS-Id of X for all $s, t \in L$.*

Proof. Suppose that A is an ILF PMS-Id of X. As $A_{s,t} \neq \varnothing$, there exist $x \in X$ such that $x \in A_{s,t}$. Then$\zeta_A(x) \geq s$ and $\xi_A(x) \leq t$. Since $A = (\zeta_A, \xi_A)$ is an ILF PMS-Id of X, $\zeta_A(0) \geq \zeta_A(x) \geq s$ and $\xi_A(0) \leq \xi_A(x) \leq t$ for all $x \in X$. Therefore, $0 \in A_{s,t}$.

Let $x, y, z \in$X such that $z * y, z * x \in A_{s,t}$.Then $\zeta_A(z * y) \geq s$, $\zeta_A(z * x) \geq s$ and $\xi_A(z * y) \leq t$, $\xi_A(z * x) \leq t$. Since A $= (\zeta_A, \xi_A)$ is an ILF PMS-Id of X, $\zeta_A(y * x) \geq \zeta_A(z * y) \wedge \zeta_A(z * x) \geq s$ and $\xi_A(y * x) \leq \xi_A(z * y) \vee \xi_A(z * x) \leq t$. Thus, $y * x \in A_{s,t}$.Therefore, $A_{s,t}$ is a PMS-Id of X.

Conversely, $A_{s,t}$ is a PMS-Id of X. Let $x \in X$ such that $\zeta_A(x) = s$ and $\xi_A(x) = t$. As, $A_{s,t}$ is a PMS-Id of X, we have $0 \in A_{s,t}$. This implies that $\zeta_A(0) \geq s = \zeta_A(x)$ and $\xi_A(0) \leq t = \xi_A(x)$. Therefore, $\zeta_A(0) \geq \zeta_A(x)$ and $\xi_A(0) \leq \xi_A(x)$, for all $x \in X$. Let $x, y, z \in$X such that $s = \zeta_A(z * y) \wedge \zeta_A(z * x)$ and $t = \xi_A(z * y) \vee \xi_A(z * x)$. Then $\zeta_A(z * y) \geq s$, $\zeta_A(z * x) \geq s$ and $\xi_A(z * y) \leq t, \xi_A(z * x) \leq t$.Thus, $z * y, z * x \in A_{s,t}$. Since $A_{s,t}$ is a PMS-Id of X, it implies that $y * x \in A_{s,t}$. Therefore, $\zeta_A(y * x) \geq s = \zeta_A(z * y) \wedge \zeta_A(z * x)$ and $\xi_A(y * x) \leq t = \xi_A(z * y) \vee \xi_A(z * x)$. Therefore, $A = (\zeta_A, \xi_A)$ is an ILF PMS-Id of X. □

As a consequence Theorem 9, we have the following Theorem.

Theorem 10. *Any PMS-Id of X can be realized as level PMS-ideal of some ILF PMS -Id of X.*

Proof. Let I be a PMS-Id of X and $A = (\zeta_A, \xi_A)$ be an ILFS in X defined by

$$\zeta_A(x) = \begin{cases} s & \text{if} \quad x \in I \\ s_0 & \text{otherwise} \end{cases}, \text{ and} \qquad \xi_A(x) = \begin{cases} t & \text{if} \quad x \in I \\ t_0 & \text{otherwise} \end{cases}$$

for all $x \in X$, where $s, s_0, t, t_0 \in$ L with $s \geq s_0$ and $t \leq t_0$. Clearly, $A_{s,t} = I$. Since $0 \in$I, $\zeta_A(0) = s \geq \zeta_A(x)$ and $\xi_A(0) = t \leq \xi_A(x)$. Hence, $\zeta_A(0) \geq \zeta_A(x)$ and $\xi_A(0) \leq \xi_A(x)$, for all $x \in X$. Also, consider the following cases for all $x,\ y, z \in X$.
Case(i). If $z * x, z * y \in I$, then $y * x \in I$. So, $\zeta_A(y * x) = s = \zeta_A(z * y) \wedge \zeta_A(z * x)$ and $\xi_A(y * x) = t = \xi_A(z * y) \vee \xi_A(z * x)$.
Case(ii). If $z * x \notin I$ *or* $z * y \notin I$, then $\zeta_A(y * x) \geq s_0 = \zeta_A(z * y) \wedge \zeta_A(z * x)$ and $\xi_A(y * x) \leq t_0 = \xi_A(z * y) \vee \xi_A(z * x)$.
Hence, $A = (\zeta_A, \xi_A)$ is an ILF PMS-ideal of X. Therefore, I is the level PMS-ideal an ILF PMS-ideal A of X. □

4 ILF PMS-ideals of a PMS-algebra under homomorphism

In this section, we prove that the effect of homomorphism on ILF PMS-ideals and obtain interesting results.

Theorem 11. *Let $f : X \to Y$ be a homomorphism of PMS-algebras. If $A = (\zeta_A, \xi_A)$ is an ILF PMS-Id of Y, then the inverse image $f^{-1}(A) = (\zeta_{f^{-1}(A)},\ \xi_{f^{-1}(A)})$ of A under f in X is an ILF PMS-Id of X.*

Proof. For all $x \in X$, $\zeta_{f^{-1}(A)}(0) = \zeta_A(f(0)) \geq \zeta_A(f(x)) = \zeta_{f^{-1}(A)}(x)$ and $\zeta_{f^{-1}(A)}(0) = \zeta_A(f(0)) \leq \zeta_A(f(x)) = \zeta_{f^{-1}(A)}(x)$. Also, Let $x, y, z \in X$. Then,

$$\begin{aligned} \zeta_{f^{-1}(A)}(y * x) = \zeta_A(f(y * x)) &= \zeta_A(f(y) * f(x)) \\ &\geq \zeta_A(f(z) * f(y)) \wedge \zeta_A(f(z) * f(x)) \\ &= \zeta_A(f(z * y)) \wedge \zeta_A(f(z * x)) \\ &= \zeta_{f^{-1}(A)}(z * y) \wedge \zeta_{f^{-1}(A)}(z * x), \end{aligned}$$

and,

$$\begin{aligned} \xi_{f^{-1}(A)}(y * x) = \xi_A(f(y * x)) &= \xi_A(f(y) * f(x)) \\ &\leq \xi_A(f(z) * f(y)) \vee \xi_A(f(z) * f(x)) \\ &= \xi_A(f(z * y)) \vee \xi_A(f(z * x)) \\ &= \xi_{f^{-1}(A)}(z * y) \vee \xi_{f^{-1}(A)}(z * x). \end{aligned}$$

Therefore, $f^{-1}(A) = \left(\zeta_{f^{-1}(A)}, \xi_{f^{-1}(A)}\right)$ is an ILF PMS-Id of X. □

Theorem 12. *Let $f : X \to Y$ be an epimorphism of PMS-algebras. If $f^{-1}(A) = \left(\zeta_{f^{-1}(A)}, \xi_{f^{-1}(A)}\right)$ is an ILF PMS-Id of X. Then A is an ILF PMS-Id of Y.*

Proof. Since f is an epimorphism of PMS-algebras for any $x \in Y$, there exists $a \in X$ such that $f(a) = x$. Then, $\zeta_A(0) = \zeta_A(f(0)) = \zeta_{f^{-1}(A)}(0) \geq \zeta_{f^{-1}(A)}(a) = \zeta_A(f(a)) = \zeta_A(x)$ and $\xi_A(0) = \xi_A(f(0)) = \xi_{f^{-1}(A)}(0) \leq \xi_{f^{-1}(A)}(a) = \xi_A(f(a)) = \xi_A(x)$. Also, let $x, y, z \in Y$. Then, $f(a) = x,\ f(b) = y$ and $f(c) = z$, for some $a,\ b,\ c \in$X. Thus,

$$\begin{aligned}
\zeta_A(y * x) &= \zeta_A(f(b) * f(a)) = \zeta_A(f(b * a)) \\
&= \zeta_{f^{-1}(A)}(b * a) \\
&\geq \zeta_{f^{-1}(A)}(c * b) \wedge \zeta_{f^{-1}(A)}(c * a) \\
&= \zeta_A(f(c * b)) \wedge \zeta_A(f(c * a)) \\
&= \zeta_A(f(c) * f(b)) \wedge \zeta_A(f(c) * f(a)) \\
&= \zeta_A(z * y) \wedge \zeta_A(z * x),
\end{aligned}$$

and,

$$\begin{aligned}
\xi_A(y * x) &= \xi_A(f(b) * f(a)) = \xi_A(f(b * a)) \\
&= \xi_{f^{-1}(A)}(b * a) \\
&\leq \xi_{f^{-1}(A)}(c * b) \vee \xi_{f^{-1}(A)}(c * a) \\
&= \xi_A(f(c * b)) \vee \xi_A(f(c * a)) \\
&= \xi_A(f(c) * f(b)) \vee \xi_A(f(c) * f(a)) \\
&= \xi_A(z * y) \vee \xi_A(z * x).
\end{aligned}$$

Therefore, $A = (\zeta_A, \xi_A)$ is an ILF PMS-Id of Y. □

Definition 14. *Let $f : X \to Y$ be a homomorphism of PMS-algebras. Then, for any ILFS $A = (\zeta_A, \xi_A)$ of Y, we define an ILFS $A^f = (\zeta_A^f, \xi_A^f)$ of X by $\zeta_A^f(x) = \zeta_A(f(x))$ and $\zeta_A^f(x) = \zeta_A(f(x))$, for all $x \in X$.*

Theorem 13. *Let $f : X \to Y$ be a homomorphism of PMS-algebras. If an ILFS $A = (\zeta_A, \xi_A)$ of Y is an ILF PMS-ideal of Y, then an IFS $A^f = (\zeta_A^f, \xi_A^f)$ in X is an ILF PMS-ideal of X.*

Proof. Suppose $f : X \to Y$ be a homomorphism of PMS-algebras and $A = (\zeta_A, \xi_A)$ is an ILF PMS-ideal of Y. Then we have $\zeta_A^f(0) = \zeta_A(f(0)) = \zeta_A(0) \geq \zeta_A(f(x)) = \zeta_A^f(x)$ and $\xi_A^f(0) = \xi_A(f(0)) = \xi_A(0) \leq \xi_A(f(x)) = \xi_A^f(x)$, for all $x \in X$. Also, let $x,\ y,\ z \in X$. Then,

$$\begin{aligned}
\zeta_A^f(y * x) = \zeta_A(f(y * x)) &= \zeta_A(f(y) * f(x)) \\
&\geq \ \zeta_A(f(z) * f(y)) \wedge \zeta_A(f(z) * f(x)) \\
&= \zeta_A(f(z * y) \wedge \zeta_A(f(z * x) \\
&= \zeta_A^f(z * y) \wedge \zeta_A^f(z * x),
\end{aligned}$$

and,

$$\begin{aligned}\xi_A^f(y*x) = \xi_A\left(f(y*x)\right) &= \xi_A\left(f(y)*f(x)\right)\\ &\leq\ \xi_A\left(f(z)*f(y)\right)\vee\xi_A\left(f(z)*f(x)\right)\\ &= \xi_A\left(f(z*y\right)\vee\xi_A\left(f(z*x\right)\\ &= \xi_A^f(z*y)\vee\xi_A^f(z*x).\end{aligned}$$

Hence, $A^f = (\zeta_A^f, \xi_A^f)$ is an ILF PMS-ideal of X. □

Theorem 14. *Let $f : X \to Y$ be an epimorphism of PMS-algebras and let $A = (\zeta_A, \xi_A)$ be an ILFS in Y. If $A^f = (\zeta_A^f, \xi_A^f)$ is an ILF PMS-Id of X, then $A = (\zeta_A, \xi_A)$ is an ILF PMS-Id of Y.*

Proof. Let $A^f = (\zeta_A^f, \xi_A^f)$ be an ILF PMS-Id of X. Since f is an epimorphism of PMS-algebras, for any $x, y, z \in Y$, there exist $a, b, c \in X$ such that $f(a) = x, f(b) = y$ and $f(c) = z$. Then we have, $\zeta_A(0) = \zeta_A\left(f(0)\right) = \zeta_A^f(0) \geq \zeta_A^f(a) = \zeta_A\left(f(a)\right) = \zeta_A(x)$ and $\xi_A(0) = \xi_A\left(f(0)\right) = \xi_A^f(0) \leq \xi_A^f(a) = \xi_A\left(f(a)\right) = \xi_A(x)$. Also,

$$\begin{aligned}\zeta_A(y*x) = \zeta_A(f(b)*f(a)) = \zeta_A\left(f(b*a)\right) &= \zeta_A^f(b*a)\\ &\geq \zeta_A^f(c*b)\wedge\zeta_A^f(c*a)\\ &= \zeta_A\left(f(c*b)\right)\wedge\zeta_A\left(f(c*a)\right)\\ &= \zeta_A\left(f(c)*f(b)\right)\wedge\zeta_A\left(f(c)*f(a)\right)\\ &= \zeta_A(z*y)\wedge\zeta_A(z*x),\end{aligned}$$

and,

$$\begin{aligned}\xi_A(y*x) = \xi_A(f(b)*f(a)) = \xi_A\left(f(b*a)\right) &= \xi_A^f(b*a)\\ &\leq \xi_A^f(c*b)\vee\xi_A^f(c*a)\\ &= \xi_A\left(f(c*b)\right)\vee\xi_A\left(f(c*a)\right)\\ &= \xi_A\left(f(c)*f(b)\right)\vee\xi_A\left(f(c)*f(a)\right)\\ &= \xi_A(z*y)\vee\xi_A(z*x).\end{aligned}$$

Therefore, $A = (\zeta_A, \xi_A)$ is an ILF PMS-Id of Y. □

Theorem 15. *Let A be an ILF PMS-Id of Y and $f : X \to Y$ be an isomorphism of PMS-algebras. Then $A \circ f$ is an ILF PMS-Id of X, where $\circ$ is the operation of composition of functions.*

Proof. Let $x, y, z \in X$ and $A = (\zeta_A, \xi_A)$ be an ILF PMS-Id of Y. Then $(\zeta_A \circ f)(0) = \zeta_A(f(0)) \geq \zeta_A(f(x)) = (\zeta_A \circ f)(x)$ and $(\xi_A \circ f)(0) = \xi_A(f(0)) \leq \xi_A(f(x)) = (\xi_A \circ f)(x)$.

$$(\zeta_A \circ f)(y*x) = \zeta_A(f(y*x)) = \zeta_A(f(y)*f(x))$$

$$\begin{aligned} &\geq \zeta_A\left(f(z) * f(y)\right) \wedge \zeta_A\left(f(z) * f(x)\right) \\ &= (\zeta_A \circ f)(z * y) \wedge (\zeta_A \circ f)(z * x), \end{aligned}$$

and,

$$\begin{aligned} (\xi_A \circ f)(y * x) = \xi_A(f(y * x)) &= \xi_A(f(y) * f(x)) \\ &\leq \xi_A\left(f(z) * f(y)\right) \vee \xi_A\left(f(z) * f(x)\right) \\ &= (\xi_A \circ f)(z * y) \vee (\xi_A \circ f)(z * x). \end{aligned}$$

Hence, $A \circ f$ is an ILF PMS-Id of X. □

5 ILF PMS-ideals of a PMS-algebra under cartesian products

In this section, we discuss the Cartesian products and the strongest fuzzy relation on ILF PMS-ideals of a PMS-algebra and obtain some interesting results.

Theorem 16. *Let $A = (\zeta_A, \xi_A)$ and $B = (\zeta_B, \xi_B)$ be any two ILF PMS-Ids of X. Then $A \times B$ is an ILF PMS-Id of $X \times X$.*

Proof. Suppose $A = (\zeta_A, \xi_A)$ and $B = (\zeta_B, \xi_B)$ be any two ILF PMS-Ids of X. Then for each $(x, y) \in X \times X$, we have $\zeta_{A\times B}(0,\ 0) = \zeta_A(0\ ) \wedge \zeta_B(0) \geq \zeta_A(x\ ) \wedge \zeta_B(y) = \zeta_{A\times B}(x,\ y)$ and $\xi_{A\times B}(0,\ 0) = \xi_A(0\ ) \vee \xi_B(\ 0) \geq \xi_A(x\ ) \vee \xi_B(y) =\ \xi_{A\times B}(x,\ y)$. Also, let $(x_1, x_2), (y_1,\ y_2), (z_1, z_2) \in X \times X$. Then,

$$\begin{aligned} \zeta_{A\times B}[(y_1, y_2) * (x_1, x_2)] &= \zeta_{A\times B}[(y_1 * x_1,\ y_2 * x_2)] \\ &= \zeta_A(y_1 * x_1) \wedge \zeta_B(y_2 * x_2) \\ &\geq [\zeta_A(z_1 * y_1) \wedge \zeta_A(z_1 * x_1)] \wedge [\zeta_B(z_2 * y_2) \wedge \zeta_B(z_2 * x_2)] \\ &= [\zeta_A(z_1 * y_1) \wedge \zeta_B(z_2 * y_2)] \wedge \zeta_A(z_1 * x_1) \wedge \zeta_B(z_2 * x_2)] \\ &= \zeta_{A\times B}(z_1 * y_1, z_2 * y_2) \wedge \zeta_{A\times B}(z_1 * x_1, z_2 * x_2) \\ &= \zeta_{A\times B}[(z_1,\ z_2) * (y_1, y_2)] \wedge \zeta_{A\times B}[(z_1, z_2) * (x_1, x_2)] \end{aligned}$$

and,

$$\begin{aligned} \xi_{A\times B}[(y_1, y_2) * (x_1, x_2)] &= \xi_{A\times B}[(y_1 * x_1, y_2 * x_2)] \\ &= \xi_A(y_1 * x_1) \vee \xi_B(y_2 * (y_2) \\ &\leq [\xi_A(z_1 * y_1) \vee \xi_A(z_1 * x_1)] \vee [\xi_B(z_2 * y_2) \vee \xi_B(z_2 * x_2)] \\ &= [\xi_A(z_1 * y_1) \vee \xi_B(z_2 * y_2] \vee [\xi_A z_1 * x_1) \vee \xi_B(z_2 * x_2)] \\ &= \xi_{A\times B}(z_1 * y_1, z_2 * y_2) \vee \xi_{A\times B}(z_1 * x_1, z_2 * x_2) \\ &= \xi_{A\times B}[(z_1, z_2) * (y_1, y_2)] \vee \xi_{A\times B}[(z_1, z_2) * (x_1, x_2)]. \end{aligned}$$

Therefore, $A \times B$ is an ILF PMS-Id of $X \times X$. □

Theorem 17. *Let $A = (\zeta_A, \xi_A)$ and $B = (\zeta_B, \xi_B)$ be any two ILFSs of X such that $A \times B$ is an ILF PMS-Id of $X \times X$. Then either A or B is an ILF PMS-Id of X.*

Proof. Let $A = (\zeta_A, \xi_A)$ and $B = (\zeta_B, \xi_B)$ be ILFSs of X such that $A \times B$ is an ILF PMS-Id of $X \times X$. Then, $\zeta_{A\times B}(0,0) \geq \zeta_{A\times B}(x,y)$ and $\xi_{A\times B}(0,0) \leq \xi_{A\times B}(x,y)$, for all $(x,y) \in X \times X$. Assume that $\zeta_A(x) > \zeta_A(0)$ and $\zeta_B(y) > \zeta_B(0)$, for some $x, y \in X$. Then $\zeta_{A\times B}(x,y) = \zeta_A(x) \wedge \zeta_B(y) > \zeta_A(0) \wedge \zeta_B(0) = \zeta_{A\times B}(0,0)$, which is a contradiction. Therefore, $\zeta_A(0) \geq \zeta_A(x)$ or $\zeta_B(0) \geq \zeta_B(y)$, for all $x, y \in X$.
Similarly, Assume that $\xi_A(x) < \xi_A(0)$ and $\xi_B(y) < \xi_B(0)$, for some $x, y \in X$. Then. $\xi_{A\times B}(x,y) = \xi_A(x) \wedge \xi_B(y) < \xi_A(0) \wedge \xi_B(0) = \xi_{A\times B}(0,0)$, which is a contradiction. Therefore, $\xi_A(0) \leq \xi_A(x)$ or $\xi_B(0) \leq \xi_B(y)$, for all $x, y \in X$. Let $(x_1, x_2), (y_1, y_2), (z_1, z_2) \in X \times X$. Then By Theorem 16, we have

$$\begin{aligned}\zeta_{A\times B}((y_1, y_2) * (x_1, x_2)) &\geq \zeta_{A\times B}((z_1, z_2) * (y_1, y_2)) \wedge \zeta_{A\times B}((z_1, z_2) * (x_1, x_2))\\ &= \zeta_{A\times B}(z_1 * y_1, z_2 * y_2) \wedge \zeta_{A\times B}(z_1 * x_1, z_2 * x_2)\\ &= (\zeta_A(z_1 * y_1) \wedge \zeta_B(z_2 * y_2)) \wedge (\zeta_A(z_1 * x_1) \wedge \zeta_B(z_2 * x_2)).\end{aligned}$$

As $\zeta_{A\times B}((y_1, y_2)*(x_1, x_2)) = \zeta_{A\times B}(y_1 * x_1, y_2 * x_2)$, we have $\zeta_{A\times B}(y_1 * x_1, y_2 * x_2) \geq (\zeta_A(z_1 * y_1) \wedge \zeta_B(z_2 * y_2)) \wedge (\zeta_A(z_1 * x_1) \wedge \zeta_B(z_2 * x_2))$. So, $\zeta_A(y_1 * x_1) \wedge \zeta_B(y_2 * x_2) \geq (\zeta_A(z_1 * y_1) \wedge \zeta_B(z_2 * y_2)) \wedge ((\zeta_A(z_1 * x_1) \wedge \zeta_B(z_2 * x_2)) = (\zeta_A(z_1 * y_1) \wedge \zeta_A(z_1 * x_1)) \wedge (\zeta_B(z_2 * y_2) \wedge \zeta_B(z_2 * x_2))$. Hence, either $\zeta_A(y_1 * x_1) \geq \zeta_A(z_1 * y_1) \wedge \zeta_A(z_1 * y_1)$ or $\zeta_B(y_2 * x_2) \geq \zeta_B(z_2 * y_2) \wedge \zeta_B(z_2 * x_2)$.
Similarly, we can also show that either $\xi_A(y_1 * x_1) \leq \xi_A(z_1 * y_1) \vee \xi_A(z_1 * y_1)$ or $\xi_B(y_2 * x_2) \leq \xi_B(z_2 * y_2) \vee \xi_B(z_2 * x_2)$. Therefore, either A or B is an ILF PMS-Id of X. □

Theorem 18. *If $A = (\zeta_A, \xi_A)$ and $B = (\zeta_B, \xi_B)$ are ILF PMS-Ids of X, then $\square(A \times B) = (\zeta_{A\times B}, \overline{\zeta}_{A\times B})$ is an ILF PMS-Id of $X \times X$.*

Proof. Since $A = (\zeta_A, \xi_A)$ and $B = (\zeta_B, \xi_B)$ are ILF PMS-Ids of X, By Theorem 16, $A \times B$ is an ILF PMS-Id of $X \times X$. Then for any $(x_1, x_2), (y_1, y_2), (z_1, z_2) \in X \times X$. We have $\zeta_{A\times B}(0,0) \geq \zeta_{A\times B}(x_1, x_2)$ and $\zeta_{A\times B}((y_1, y_2) * (x_1, x_2)) \geq \zeta_{A\times B}((z_1, z_2) * (y_1, y_2)) \wedge \zeta_{A\times B}((z_1, z_2) * (x_1, x_2))$.
So it suffices to show that $\overline{\zeta}_{A\times B}$ satisfies the condition $\overline{\zeta}_{A\times B}(0,0) \leq \overline{\zeta}_{A\times B}(x_1, x_2)$ and $\overline{\zeta}_{A\times B}((y_1, y_2) * (x_1, x_2)) \leq \overline{\zeta}_{A\times B}((z_1, z_2) * (y_1, y_2)) \vee \overline{\zeta}_{A\times B}(z_1, z_2) * (x_1, x_2))$. Thus, for any $(x_1, x_2), (y_1, y_2), (z_1, z_2) \in X \times X$, we have $\overline{\zeta}_{A\times B}(0,0) = 1 - \zeta_{A\times B}(0.0) \leq 1 - \zeta_{A\times B}(x_1, x_2) = \overline{\zeta}_{A\times B}(x_1, x_2)$ and,

$$\begin{aligned}\overline{\zeta}_{A\times B}((y_1, y_2) * (x_1, x_2)) &= 1 - \zeta_{A\times B}((y_1, y_2) * (x_1, x_2))\\ &\leq 1 - [\zeta_{A\times B}((z_1, z_2) * (y_1, y_2)) \wedge \zeta_{A\times B}((z_1, z_2) * (x_1, x_2))]\\ &= [1 - \zeta_{A\times B}((z_1, z_2) * (y_1, y_2))] \vee [1 - \zeta_{A\times B}((z_1, z_2) * (x_1, x_2))]\\ &= \overline{\zeta}_{A\times B}((z_1, z_2) * (y_1, y_2)) \vee \overline{\zeta}_{A\times B}((z_1, z_2) * (x_1, x_2)).\end{aligned}$$

Hence, $\square(A \times B) = (\zeta_{A\times B}, \overline{\zeta}_{A\times B})$ is an ILF PMS-Id of $X \times X$. □

Theorem 19. *If $A = (\zeta_A, \xi_A)$ and $B = (\zeta_B, \xi_B)$ are ILF PMS-Ids of X, then $\diamond(A \times B)$ is an ILF PMS-Id of $X \times X$.*

Proof. Since A and B are ILF PMS-Ids of X, By Theorem 16, $A \times B$ is an ILF PMS-Id of $X \times X$. Then for any $(x_1, x_2), (y_1, y_2), (z_1, z_2) \in X \times X$, we have $\xi_{A\times B}(0,0) \leq \xi_{A\times B}(x_1, x_2)$ and $\xi_{A\times B}((y_1, y_2) * (x_1, x_2)) \leq \xi_{A\times B}((z_1, z_2) * (y_1, y_2)) \wedge \xi_{A\times B}((z_1, z_2) * (x_1, x_2))$. So it is sufficient to show that $\overline{\zeta}_{A\times B}$ satisfies the conditions $\overline{\xi}_{A\times B}(0,0) \leq \overline{\xi}_{A\times B}(x_1, x_2)$ and $\overline{\xi}_{A\times B}((y_1, y_2) * (x_1, x_2)) \leq \overline{\xi}_{A\times B}((z_1, z_2) * (y_1, y_2)) \vee \overline{\xi}_{A\times B}(z_1, z_2) * (x_1, x_2))$. Thus for any $(x_1, x_2), (y_1, y_2), (z_1, z_2) \in X \times X$, we have $\overline{\xi}_{A\times B}(0,\ 0) = 1 - \xi_{A\times B}(0.0) \geq\ 1 - \xi_{A\times B}(x_1, x_2) = \overline{\xi}_{A\times B}(x_1, x_2)$ and,

$$\begin{aligned}
\overline{\xi}_{A\times B}((y_1, y_2) * (x_1, x_2)) &= 1 - \xi_{A\times B}((y_1, y_2) * (x_1, x_2)) \\
&\geq 1 - [\xi_{A\times B}((z_1, z_2) * (y_1, y_2)) \vee \xi_{A\times B}((z_1, z_2) * (x_1, x_2))] \\
&= [1 - \xi_{A\times B}((z_1, z_2) * (y_1, y_2))] \wedge [1 - \xi_{A\times B}((z_1, z_2) * (x_1, x_2))] \\
&= \ \overline{\zeta}_{A\times B}((z_1, z_2) * (y_1, y_2)) \wedge \ \overline{\zeta}_{A\times B}((z_1, z_2) * (x_1, x_2)).
\end{aligned}$$

Hence, $\Diamond(A \times B) = (\ \overline{\xi}_{A\times B}, \xi_{A\times B})$ is an ILF PMS-Id of $X \times X$. □

The following Theorem fallows from the above two theorems

Theorem 20. *The ILFSs $A = (\zeta_A, \xi_A)$ and $B = (\zeta_B, \xi_B)$ are ILF PMS-Ids of X if and only if $\square(A\ \times B) = (\zeta_{A\times B}, \overline{\zeta}_{A\times B})$ and $\Diamond(A\ \times\ B) = (\overline{\xi}_{A\times B}, \xi_{A\times B})$ are an ILF PMS-Id of $X \times X$.*

Proof. Follows from Theorems 18 and 19. □

Definition 15. *Let $A = (\zeta_A, \xi_A)$ and $B = (\zeta_B, \xi_B)$ be ILFSs of X. Then the intuitionistic L-fuzzy level subset of $A \times B$ is a subset $(A \times B)_{s,t}$ of $X \times X$ given by*

$$(A \times B)_{s,t} = \{(x, y) \in X \times X : \zeta_A(x, y) \geq s \text{ and } \xi_A(x, y) \leq t\},$$

for $s, t \in L$.

Theorem 21. *Let $A = (\zeta_A, \xi_A)$ and $B = (\zeta_B, \xi_B)$ be ILFSs of X. Then $A \times B$ is a ILF PMS-Id of $X \times X$ if and only if the nonempty subset $(A \times B)_{s,t}$ of $X \times X$ is a PMS-Id of $X \times X$ for all $s, t \in L$.*

Proof. Since $(A \times B)_{s,t} \neq \varnothing$, there exist $(x, y) \in X \times X$ such that $(x, y) \in (A \times B)_{s,t}$. Then $\zeta_A(x, y) \geq s$ and $\xi_A(x, y) \leq t$. As $A \times B$ is a ILF PMS-Id of $X \times X$, $\zeta_{A\times B}(0, 0) \geq \zeta_{A\times B}(x, y) \geq s$ and $\xi_{A\times B}(0, 0) \leq \xi_{A\times B}(x, y) \leq t$. So that $(0, 0) \in (A \times B)_{s,t}$. Let $(x_1, x_2), (y_1, y_2), (z_1, z_2) \in X \times X$ such that $(z_1, z_2) * (y_1, y_2),\ (z_1, z_2) * (x_1, x_2) \in (A \times B)_{s,t}$ for $s, t \in L$. Then $\zeta_{A\times B}((z_1, z_2) * (y_1, y_2) \geq s, \zeta_{A\times B}((z_1, z_2) * (x_1, x_2)) \geq s$, $(\ \xi_{A\times B}((z_1, z_2) * (y_1, y_2)) \leq t$, and $\xi_{A\times B}((z_1, z_2) * (x_1, x_2)) \leq t$. Since $A \times B$ is an ILF PMS-Id of $X \times X$, it follows that

$$\zeta_{A\times B}((y_1, y_2) * (x_1, x_2)) \geq \zeta_{A\times B}((z_1, z_2) * (y_1, y_2)) \wedge \zeta_{A\times B}((z_1, z_2) * (x_1, x_2)) \geq s$$

and,

$$\xi_{A\times B}((y_1, y_2) * (x_1, x_2)) \leq \xi_{A\times B}((z_1, z_2) * (y_1, y_2)) \wedge \xi_{A\times B}((z_1, z_2) * (x_1, x_2)) \leq t.$$

Then, $\zeta_{A\times B}((y_1, y_2) * (x_1, x_2)) \geq s$ and $\xi_{A\times B}((y_1, y_2) * (x_1, x_2)) \leq t$. Hence, $(y_1, y_2) * (x_1, x_2) \in (A \times B)_{s,t}$ and thus,$(A \times B)_{s,t}$ is a PMS-algebra of $X \times X$.

Conversely, assume that $(A\times B)_{s,t}$ is a PMS-algebra of $X\times X$, Let $(x,y)\in X\times X$ such that $\zeta_A(x,y)=s$ and $\xi_A(x,y)=t$. As, $(A\times B)_{s,t}$ is a PMS-Id of X, we have $(0,0)\in(A\times B)_{s,t}$.This implies that $\zeta_A(0,0)\geq s=\zeta_A(x,y)$ and $\xi_A(0,0)\leq t=\xi_A(x,y)$. Therefore, $\zeta_A(0,0)\geq\zeta_A(x,y)$ and $\xi_A(0,0)\leq\xi_A(x,y)$, for all $(x,y)\in X\times X$. Also, let $(x_1,x_2),(y_1,y_2),(z_1,z_2)\in X\times X$ such that $s=\zeta_A((z_1,z_2)*(y_1,y_2))\wedge\ \zeta_A((z_1,z_2)*(x_1,x_2))$ and $t=\xi_A((z_1,z_2)*(y_1,y_2))\vee\xi_A((z_1,z_2)*(x_1,x_2))$. Then $\zeta_A((z_1,z_2)*(y_1,y_2))\geq s,\ \ \zeta_A((z_1,z_2)*(x_1,x_2))\geq s$ and $\xi_A((z_1,z_2)*(y_1,y_2))\leq t,\ \ \xi_A((z_1,z_2)*x)\leq t$. Thus, $(z_1,z_2)*(y_1,y_2),(z_1,z_2)*(x_1,x_2)\in A_{s,t}$. Since $A_{s,t}$ is a PMS-Id of X, it implies that $(y_1,y_2)*(x_1,x_2)\in A_{s,t}$. Therefore, we have $\zeta_A((y_1,y_2)*(x_1,x_2))\geq s=\zeta_A((z_1,z_2)*(y_1,y_2))\wedge\ \zeta_A((z_1,z_2)*(x_1,x_2))$ and $\xi_A((y_1,y_2)*(x_1,x_2))\leq t=\xi_A((z_1,z_2)*(y_1,y_2))\vee\xi_A((z_1,z_2)*(x_1,x_2))$. Therefore, $A\times B$ is an ILF PMS-Id of $\in X\times X$. □

Theorem 22. *Let A_B be the strongest intuitionistic L-fuzzy relation on X, where $B=(\zeta_B,\xi_B)$ be an ILFS of a PMS-algebra X. If B is an ILF PMS-Id of X, then A_B is an ILF PMS-Id of X $\times$ X.*

Proof. Assume that B is an ILF PMS-Id of X. Let $x_1,x_2\in X$. Then, $\zeta_{A_B}(0,0)=\zeta_B(0)\wedge\zeta_B(0)\geq\zeta_B(x_1)\wedge\zeta_B(x_2)=\zeta_{A_B}(x_1,x_2)$ and $\xi_{A_B}(0,0)=\xi_B(0)\vee\xi_B(0)\leq\xi_B(x_1)\vee\xi_B(x_2)=\xi_{A_B}(x_1,x_2)$ for $x_1,x_2\in X$. Hence, $\zeta_{A_B}(0,0)\geq\zeta_{A_B}(x_1,x_2)$ and $\xi_{A_B}(0,0)\leq\xi_{A_B}(x_1,x_2)$. Also, let $(x_1,x_2),(y_1,y_2)$, $z=(z_1,z_2)\in X\times X$. Then

$$\begin{aligned}\zeta_{A_B}((y_1,y_2)*(x_1,x_2)) &= \zeta_{A_B}(y_1*x_1,y_2*x_2)\\ &= \zeta_B(y_1*x_1)\wedge\zeta_B(y_2*x_2)\\ &\geq (\zeta_B(z_1*y_1)\wedge\zeta_B(z_1*x_1))\wedge(\zeta_B(z_2*y_2)\wedge\zeta_B(z_2*x_2))\\ &= (\zeta_B(z_1*y_1)\wedge\zeta_B(z_2*y_2))\wedge(\zeta_B(z_1*x_1)\wedge\zeta_B(z_2*x_2))\\ &= \zeta_{A_B}(z_1*y_1,z_2*y_2)\wedge\zeta_{A_B}(z_1*x_1,z_2*x_2)\\ &= \zeta_{A_B}((z_1,z_2)*(y_1,y_2))\wedge\zeta_{A_B}((z_1,z_2)*(x_1,x_2)),\end{aligned}$$

and,

$$\begin{aligned}\xi_{A_B}((y_1,y_2)*(x_1,x_2)) &= \xi_{A_B}(y_1*x_1,y_2*x_2)\\ &= \xi_B(y_1*x_1)\vee\xi_B(y_2*x_2)\\ &\leq (\xi_B(z_1*y_1)\vee\xi_B(z_1*x_1))\vee(\xi_B(z_2*y_2)\vee\xi_B(z_2*x_2))\\ &= (\xi_B(z_1*y_1)\vee\xi_B(z_2*y_2))\vee(\xi_B(z_1*x_1)\vee\xi_B(z_2*x_2))\\ &= \xi_{A_B}(z_1*y_1,z_2*y_2)\vee\xi_{A_B}(z_1*x_1,z_2*x_2)\\ &= \xi_{A_B}((z_1,z_2)*(y_1,y_2))\wedge\xi_{A_B}((z_1,z_2)*(x_1,x_2)).\end{aligned}$$

Therefore, A_B is an ILF PMS-Id of $X\times X$. □

The converse of the Theorem 22 is also valid as it has been proved in Theorem 23.

Theorem 23. *If A_B is an ILF PMS-Id of $X\times X$, then $B=(\zeta_B,\xi_B)$ is an ILF PMS-Id of X.*

Proof. Let A_B is an ILF PMS-Id of $X \times X$. Then for $(x_1, x_2), (y_1, y_2), (z_1, z_2) \in X \times X$, we have

$$\begin{aligned}
&\text{(i)}.\zeta_B(0) \wedge \mu_A(0) = \zeta_{A_B}(0,0) \geq \zeta_{A_B}(x_1, x_2) = \zeta_B(x_1) \wedge \zeta_B(x_2) \\
&\qquad \Rightarrow \zeta_B(0) \wedge \zeta_B(0) \geq \zeta_B(x_1) \wedge \zeta_B(x_2) \\
&\qquad \Rightarrow \zeta_B(0) \geq \zeta_B(x_1) \text{ or } \zeta_B(0) \geq \zeta_B(x_2), \text{ and} \\
&\xi_B(0) \vee \mu_A(0) = \xi_{A_B}(0,0) \leq \xi_{A_B}(x_1, x_2) = \xi_B(x_1) \vee \zeta_B(x_2) \\
&\qquad \Rightarrow \xi_B(0) \vee \xi_B(0) \leq \xi_B(x_1) \vee \xi_B(x_2) \\
&\qquad \Rightarrow \xi_B(0) \geq \xi_B(x_1) \text{ or } \xi_B(0) \geq \xi_B(x_2).
\end{aligned}$$

$$\begin{aligned}
(ii).\ \zeta_B(y_1 * x_1) \wedge \zeta_B(y_2 * x_2) &= \zeta_{A_B}(y_1 * x_1, y_2 * x_2) \\
&= \zeta_{A_B}((y_1, y_2) * (x_1, x_2)) \\
&\geq \zeta_{A_B}((z_1, z_2) * (y_1, y_2)) \wedge \zeta_{A_B}((z_1, z_2) * (x_1, x_2)) \\
&= \zeta_{A_B}((z_1 * y_1), (z_2 * y_2)) \wedge \zeta_{A_B}((z_1 * x_1), (z_2 * x_2)) \\
&= (\zeta_B(z_1 * y_1) \wedge \zeta_B(z_2 * y_2)) \wedge (\zeta_B(z_1 * x_1) \wedge \zeta_B(z_2 * x_2)) \\
&= (\zeta_B(z_1 * y_1) \wedge \zeta_B(z_1 * x_1)) \wedge (\zeta_B(z_2 * y_2) \wedge \zeta_B(z_2 * x_2).
\end{aligned}$$

If we put $x_2 = y_2 = z_2 = 0$, then $\zeta_B(y_1 * x_1) \geq \zeta_B(z_1 * y_1) \wedge \zeta_B(z_1 * x_1)$.

$$\begin{aligned}
(iii).\ \xi_B(y_1 * x_1) \vee \xi_B(y_2 * x_2) &= \xi_{A_B}(y_1 * x_1, y_2 * x_2) \\
&= \xi_{A_B}((y_1, y_2) * (x_1, x_2)) \\
&\leq \xi_{A_B}((z_1, z_2) * (y_1, y_2)) \vee \xi_{A_B}((z_1, z_2) * (x_1, x_2)) \\
&= \xi_{A_B}((z_1 * y_1), (z_2 * y_2)) \vee \xi_{A_B}((z_1 * x_1), (z_2 * x_2)) \\
&= (\xi_B(z_1 * y_1) \vee \xi_B(z_2 * y_2)) \vee (\xi_B(z_1 * x_1) \vee \xi_B(z_2 * x_2)) \\
&= (\xi_B(z_1 * y_1) \vee \xi_B(z_1 * x_1)) \vee (\xi_B(z_2 * y_2) \vee \xi_B(z_2 * x_2)).
\end{aligned}$$

Similarly, if $x_2 = y_2 = z_2 = 0$, we get $\xi_B(y_1 * x_1) \leq \xi_B(z_1 * y_1) \vee \xi_B(z_1 * x_1)$. Therefore, B is an ILF PMS-Id of X. □

6 Conclusion

In this paper, we used the concept of intutionistic L-fuzzy set to PMS-algebras. We introduced the notion of the intuitionistic L-fuzzy PMS-ideal of a PMS-algebra and investigated several associated properties. We provide a condition for an intuitionistic L-fuzzy set in a PMS-algebra to be an intuitionistic L-fuzzy PMS-ideal. We characterized intuitionistic L-fuzzy PMS-ideals of a PMS-algebra in terms of their level subsets. We also discussed the algebraic nature of intuitionistic L-fuzzy PMS-ideals of a PMS-algebra under homomorphism and obtained incredible results. Moreover, we studied the Cartesian product of the intuitionistic L-fuzzy PMS-ideals of a PMS-algebra and investigated some intersting results. Finally, we explored the relationship between the strongest intuitionistic L-fuzzy relations in a PMS-algebra and intuitionistic L-fuzzy PMS-ideals of a PMS-algebra. We believe that the results of this study will add new dimensions to the structures of PMS-algebras based on intuitionistic L-fuzzy sets and serve as a foundation for future studies in the intuitionistic L-fuzzification of PMS-algebra. In our future work, we will extend this concept to interval valued intuitionistic L-fuzzy PMS-ideals of a PMS-algebra. Furthermore, we will develop the Neuro-fuzzy algebraic structures for PMS-subalgebras of PMS-algebras.

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