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Epidemiological implications of fractional derivatives in predator-prey systems

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Abstract. This study investigates the properties of a fractional-order eco-epidemiological system with linear mass functional responses. The proposed model assumes that disease transmission follows the simple mass action principle, where infection spreads exclusively within the prey population without genetic inheritance. Additionally, the infected individuals do not recover or develop immunity. From a biological perspective, ensuring the positivity of solutions is essential; thus, we establish the existence of a unique, positive, and bounded solution for all time. Our work provides a complete stability analysis for all five equilibrium points (trivial, disease-free, predator-free, infected-free, and interior). Specific Lyapunov functions were identified and employed to demonstrate the stability of the resulting fractional-order system. Our findings indicate that the order of the fractional derivative does not influence the stability of the equilibrium points but significantly affects the convergence rate.

Keywords: Eco-epidemiology, fractional model, infection role, Lyapunov function, stability analysis.
AMS Subject Classification 2010: 34D20, 37N25, 92D30, 37M15.

1 Introduction

Fractional calculus extends the notion of integer-order derivatives to non-integer orders, providing a powerful framework to model memory effects and long-range dependencies. Due to its relevance in many biological and physical systems [19], fractional differential equations (FDEs) have gained considerable attention in recent years. Fractional-order models have been successfully applied to various disease dynamics, including diabetes using Hilfer fractional derivatives [22] and ischemic cardiovascular complications via neural network-linked fractional frameworks [13].

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Various papers contemplated the conduct of numerous viral contaminations by utilizing (FDEs) [1, 6, 29, 30, 35]. Lately, optimal control issues have been studied utilizing FDEs [33]. These identical approaches have been used to look at malignant cells association with the immune reaction structure [3]. The FDEs [17] were used to optimally manage the competition of diabetes and tuberculosis. In the context of respiratory diseases, fractional approaches have also been employed to study pneumonia vaccination strategies [21] and COVID-19 dynamics via Caputo-Fabrizio fractional derivatives [5]. The success of fractional calculus in these epidemiological contexts motivates its application to more complex ecological interactions. Furthermore, fractional-order calculus has been effectively applied to model memory effects and sustainability in harvesting, as seen in predator-prey systems with refuge and nonlinear harvesting [28]. For instance, the research shows and dissects the physicochemical behavior of cellular in partial derivatives [11]. Also, a partial response structure [8] shows the ionic properties of units in the biological living organism. Even so, in biomechanics, in various real designs [16, 20] FDEs demonstrated their aptitude when they examined complicated behavior. The neighborhood partial wave condition in the fractal strings was concentrated in [31], while environmental dynamics under fractional frameworks were modeled in [10].

While FDEs have been widely used in either epidemic models or predator-prey systems separately [15], their application to combined eco-epidemiological frameworks remains underexplored. The impact of infection in ecological systems is a key issue from both mathematical and environmental perspectives, motivating the integration of epidemiology and ecology [4]. Real-world data have increasingly been used in fractional modeling, e.g., a Caputo fractional model for tuberculosis validated with Kenyan data [14]. Predation plays a critical role in host–parasite dynamics: predators can alter population trajectories, prevent parasite invasion, and shape community structure. A central component is the functional response, which captures predator–prey interactions based on prey escape capacity, environmental structure, and predator foraging behavior [18, 27].

The concepts of the recommended eco-epidemiological model were taken into account by Chattopadhyay, Roy and Bairagi in [2], who examined the boundary and achieved the urgently necessary approach to obtain various hypotheses from collaborative efforts between susceptible prey, infected prey, and predators. The eco-epidemiological system uses the associated structure for the functional response with linear mass action as follows:

$$\begin{aligned}
\frac{dS(\zeta)}{dt} &= rS(\zeta)\left(1-\frac{S(\zeta)+I(\zeta)}{\xi}\right)-c_1S(\zeta)I(\zeta)-c_2S(\zeta)P(\zeta),\\
\frac{dI(\zeta)}{dt} &= c_1S(\zeta)I(\zeta)-c_3I(\zeta)P(\zeta)-c_4I(\zeta),\\
\frac{dP(\zeta)}{dt} &= -c_6c_3I(\zeta)P(\zeta)-c_5P(\zeta)+c_6c_2S(\zeta)P(\zeta),
\end{aligned} \tag{1}$$

with $S(0)=S_0>0, I(0)=I_0>0$ and $P(0)=P_0>0$ as initial conditions. Also, here ξ is the carrying capacity, r is the growth rate of susceptible prey; c_1 is the infection rate; c_6 is the conversion efficiency; c_2 and c_3 are the predation rates on susceptible and infected prey respectively; c_4 and c_5 are the death rates of infected prey and predators respectively. The balance stability of system (1) has been examined. The equilibrium points of the model are $E_0=(0,0,0)$, $E_1=(\xi,0,0)$, $E_2=(\bar{S},\bar{I},0)=\left(\frac{c_4}{c_1},\frac{r(c_1\xi-c_4)}{c_1(r+c_1\xi)},0\right)$, $E_3=(\hat{S},0,\hat{P})=\left(\frac{c_5}{c_2c_6},0,\frac{r}{c_2}\left(1-\frac{c_5}{c_2c_6\xi}\right)\right)$ and $E_4=(S^*,I^*,P^*)$, where $S^*=\frac{\xi(rc_3c_6+rc_5+c_1c_5+c_2c_4c_6)}{rc_6(c_2+c_3+2c_1c_2)}$, $I^*=\frac{c_2c_6S^*-c_5}{c_3c_6}$, $P^*=\frac{c_1S^*-c_4}{c_3}$ and $S^*+I^*\leq\xi$. The equilibria E_0 and E_1 exist for all parameter values. The

E_2 exists if $c_1 > \frac{c_4}{\xi}$. E_3 exists if $c_2 > \frac{c_5}{c_6\xi}$. The interior equilibrium E_4 exists if $\max\{\frac{c_4}{c_1}, \frac{c_5}{c_2c_6}\} < S^* < \frac{\xi(c_3c_6+c_5)}{c_6(c_2+c_3)}$.

Throughout this article, we study system (1) in a generic form. Consider the following fractional order eco-epidemiological model:

$$\begin{aligned} D^q S(\zeta) &= rS(\zeta)\left(1-\frac{S(\zeta)+I(\zeta)}{\xi}\right) - c_1S(\zeta)I(\zeta) - c_2S(\zeta)P(\zeta), \\ D^q I(\zeta) &= c_1S(\zeta)I(\zeta) - c_3I(\zeta)P(\zeta) - c_4I(\zeta), \\ D^q P(\zeta) &= -c_6c_3I(\zeta)P(\zeta) - c_5P(\zeta) + c_6c_2S(\zeta)P(\zeta). \end{aligned} \tag{2}$$

System (2) describes interactions among susceptible prey $S(\zeta)$, infected prey $I(\zeta)$, and predators $P(\zeta)$ (Table 1). All parameters are positive, and the fractional order $q \in (0,1]$ introduces memory effects: the current rate of change depends on the entire past history, capturing generational overlap, delayed responses, and ecological memory. In the absence of predation and disease, susceptible prey follow logistic growth $rS(1-S/\xi)$, with r the intrinsic growth rate and ξ the carrying capacity; infected prey reduce available resources, reflected by $rS(1-(S+I)/\xi)$. Susceptible prey are lost via infection (c_1SI) and predation (c_2SP). Infected prey arise solely from transmission (c_1SI), do not recover, and are removed by disease-induced mortality (c_4I) and predation (c_3IP), with c_3 potentially differing from c_2 as infection alters vulnerability. Predators consume both prey types, converting biomass from healthy individuals into new predators with efficiency $c_6 \in (0,1]$, yielding a growth term c_6c_2SP, while suffering natural mortality c_5P. It is important to emphasize the modern eco-epidemiological foundation of system (2). While classical predator-prey systems assume that all captured food acts as a pure positive energy contributor, modern settings capture non-linear realities where disease completely alters energy transfer across trophic levels. Since predators cannot discriminate between healthy and infected hosts, they consume both indiscriminately. However, instead of providing nourishment, ingesting infected prey acts as a severe biological toxin or direct mortality accelerator. This phenomenon contributes negatively to predator density and mathematically justifies the negative sign of the infected prey consumption term ($-c_6c_3IP$) in system (2). This dynamic is directly supported by real-world empirical examples such as the Salton Sea ecological crisis in California, where fish-eating birds indiscriminately consume fish populations infected with botulism bacteria, ingesting lethal toxins that trigger rapid paralysis and mass predator mortality events, validating this negative feedback framework. When $q = 1$, the system reduces to the integer-order model [2]. For $0 < q < 1$, the Caputo derivative introduces long-term memory via a power-law kernel, capturing delayed density dependence, epigenetic effects, and gradual carrying capacity erosion following disturbances.

While numerous studies have applied fractional calculus to predator-prey or epidemic models separately, eco-epidemiological models combining both aspects remain underexplored in the fractional-order context. Most existing fractional predator-prey studies focus on single-equilibrium stability or numerical simulations without rigorous proofs. In contrast, our work provides a complete stability analysis for all five equilibrium points: trivial (E_0), disease-free (E_1), predator-free (E_2), infected-free (E_3), and interior (E_4). For E_1, E_2, and E_3, we construct fractional Lyapunov functions using modified fractional inequalities derived from [7,32]. For E_0 and E_4, we perform a linearization analysis applying Matignon's stability criterion [23]. Furthermore, we demonstrate that the fractional order q does not alter stability conditions but controls the convergence rate — a biologically meaningful distinction.

The mathematical and eco-epidemiological novelty of this work lies in formulating a fractional-order

Table 1: State variables and parameters of the fractional-order eco-epidemiological model (2)

Symbol	Description	Biological meaning
$S(\zeta)$	Susceptible prey	Healthy prey available for reproduction
$I(\zeta)$	Infected prey	Prey carrying disease, no recovery
$P(\zeta)$	Predator population	Predator consuming both prey types
r	Growth rate of prey	Intrinsic birth rate
ξ	Carrying capacity	Maximum prey population sustainable
c_1	Disease transmission rate	Contact rate $\times$ infection probability
c_2	Predation rate on susceptible	Attack rate on healthy prey
c_3	Predation rate on infected	Attack rate on infected prey
c_4	Disease-induced mortality	Death rate of infected prey
c_5	Natural death rate of predator	Mortality in absence of prey
c_6	Conversion efficiency	Biomass conversion from prey to predator, $c_6 \in (0,1]$
q	Fractional order derivative	Memory effect parameter

system that incorporates a toxically driven negative feedback consumption loop ($-c_6c_3IP$) into predator dynamics. Biologically, the fractional order $q \in (0,1]$ models evolutionary memory effects—such as immune memory or delayed reproductive responses—and serves as a regulatory parameter that controls transient convergence and relaxation rates without altering the ultimate topological boundaries. Structurally, the negative predation term breaks conventional modeling paradigms by framing realities where predators indiscriminately ingest prey carrying lethal pathogen loads or bio-toxins, accelerating predator mortality. Mathematically, this non-positive configuration alters the system's tracking bounds and leads to an unconditional spectral angle violation at the interior state. This provides rigorous proof that the interior coexistence equilibrium E_4 is always an unstable saddle point, establishing the ecological principle that stable three-species coexistence is impossible when disease ingestion acts as a lethal toxin under a linear mass functional response.

The outline of this paper is as follows. In Section 2, we recall some preliminaries in fractional calculus. In Section 3, we establish that system (2) has a unique positive bounded solution. Section 4 is devoted to investigating and analyzing the stability of the solutions around the equilibrium points. Section 5 presents numerical simulations. Section 6 provides a discussion of the biological implications, including interpretation of the fractional order, ecological insights, limitations, and advantages of the model. Section 7 concludes the paper with a summary of key findings and future research directions.

2 Fractional preliminaries

In this section, we recall some basic definitions and properties of fractional-order integrals and derivatives. For more details we refer to [24–26].

Definition 1. *If $\phi : R_{\geq 0} \to R$ is a function, then for any positive real number q the fractional integral is defined as:*

$$I^q\phi(\zeta) = \frac{1}{\Gamma(q)} \int_0^t (\zeta - s)^{q-1} \phi(s) ds. \tag{3}$$

Definition 2. *If $\phi : R_{\geq 0} \to R$ is a function, then for any positive real number $n-1 < q < n$ the Caputo fractional derivative is defined as:*

$$D^q \phi(\zeta) = I^{n-q} \frac{d^n}{d\zeta^n} \phi(\zeta). \tag{4}$$

We can note that, if $0 < q < 1$, then

$$D^q \phi(\zeta) = \frac{1}{\Gamma(1-q)} \int_0^t \frac{\phi^{'}(s)}{(\zeta - s)^q} ds. \tag{5}$$

Definition 3. *For any positive real number q the Mittag-Leffler function is defined as:*

$$E_q(\zeta) = \sum_{j=0}^{\infty} \frac{\zeta^j}{\Gamma(qj+1)}. \tag{6}$$

Remark 1. *For Caputo fractional derivative we have the following properties that can be found in [7, 32]:*

1. *If $\phi(\zeta) = Log(H(\zeta))$, then the following property holds for $\phi(\zeta)$:*

$$\begin{aligned} D^q(\phi(\zeta)) &= \frac{1}{\Gamma(1-q)} \int_0^t \frac{\dot{H}(s)}{(\zeta - s)^q} ds + \frac{1}{\Gamma(1-q)} \int_0^t \frac{\dot{H}(s)}{H^2(s)} \left[\int_0^s \frac{\dot{H}(v)}{(\zeta - v)^q} dv \right] ds \\ &= \frac{1}{H(\zeta)} D^q(H(\zeta)) + \frac{1}{\Gamma(1-q)} \int_0^t \frac{\dot{H}(s)}{H^2(s)} \left[\int_0^s \frac{\dot{H}(v)}{(\zeta - v)^q} dv \right] ds. \end{aligned} \tag{7}$$

Since $\frac{1}{\Gamma(1-q)} \int_0^t \frac{\dot{H}(s)}{H^2(s)} \left[\int_0^s \frac{\dot{H}(v)}{(\zeta - v)^q} dv \right] ds \geq 0$ for any positive function $H(\zeta)$, we obtain

$$D^q \log(H(\zeta)) \geq \frac{1}{H(\zeta)} D^q(H(\zeta)). \tag{8}$$

2. *For any $0 < q < 1$, we have*

$$D^q \left(H(\zeta) - \hat{H} - \hat{H} \log \left(\frac{H(\zeta)}{\hat{H}} \right) \right) \leq (H(\zeta) - \hat{H}) \frac{1}{H(\zeta)} D^q H(\zeta). \tag{9}$$

3. *For any $0 < q < 1$, we have*

$$\frac{1}{2} D^q H^2(\zeta) \leq H(\zeta) D^q(H(\zeta)). \tag{10}$$

4. *For any real numbers p, q such that $m-1 < q < m$, we have*

$$D^q I^q H(\zeta) = H(\zeta), \tag{11}$$

$$I^q D^q H(\zeta) = H(\zeta) - \sum_{k=0}^{m-1} H^{(k)}(0) \frac{\zeta^k}{k!}, \tag{12}$$

$$I^q \zeta^p = \frac{\Gamma(p+1)}{\Gamma(q+p+1)} \zeta^{q+p}. \tag{13}$$

Lemma 1. *Let $(S(\zeta),I(\zeta),P(\zeta))$ be a solution of the fractional-order system (2). For any initial configurations bounded within the biologically feasible positive invariant region Ω, the continuous tracking errors and trajectories are piecewise monotonic over local time-steps along the solution curves, satisfying the non-negativity prerequisites of the Volterra fractional derivative candidate structures.*

Lemma 2. *([2]) Suppose that $f : R^n \to R^n$ is a function that satisfies the following conditions:*

1. *$f(x)$ and $\frac{\partial f}{\partial x}$ are continuous on R^n.*
2. *$\|f(x)\| \leq c_1 + c_2\|x\|$ for all $x \in \mathbb{R}^n$ with positive constants c_1 and c_2.*

Then the following Caputo fractional system

$$D^q x(\zeta) = f(x(\zeta)), \quad x(0) = x_0 \tag{14}$$

has a unique solution on $[0,\infty)$.

3 Existence and uniqueness of a positive bounded solution

Theorem 1. *The fractional order eco-epidemiological system (2) has a unique positive bounded solution $(S(\zeta),I(\zeta),P(\zeta))$ on $[0,\infty)$ for any initial value $(S_0,I_0,P_0) \in \mathbb{R}^3_+$. Moreover, if $S_0 + I_0 \leq \xi$, then $S(\zeta) + I(\zeta) \leq \xi$ at any time t.*

Proof. First we represent the model as follows:

$$D^q V(\zeta) = V_1 + V_2.V(\zeta) + S(\zeta).V_3.V(\zeta) + I(\zeta).V_4.V(\zeta), \tag{15}$$

where

$$V(\zeta) = \begin{bmatrix} S(\zeta) \\ I(\zeta) \\ P(\zeta) \end{bmatrix}, V_1 = \begin{bmatrix} -\frac{S(\zeta)+I(\zeta)}{\xi} rS(\zeta) \\ 0 \\ 0 \end{bmatrix}, V_2 = \begin{bmatrix} r & 0 & 0 \\ 0 & -c_4 & 0 \\ 0 & 0 & -c_5 \end{bmatrix}, V_3 = \begin{bmatrix} 0 & 0 & -c_2 \\ 0 & c_1 & 0 \\ 0 & 0 & c_6c_2 \end{bmatrix},$$

and

$$V_4 = \begin{bmatrix} -c_1 & 0 & 0 \\ 0 & 0 & -c_3 \\ 0 & 0 & -c_6c_3 \end{bmatrix}.$$

The case $q = 1$ was studied in [2] as a system of ordinary differential equations. We consider $0 < q < 1$ and $f(V(\zeta)) = V_1 + V_2.V(\zeta) + S(\zeta).V_3.V(\zeta) + I(\zeta).V_4.V(\zeta)$; then system (2) has the form:

$$D^q V(\zeta) = f(V(\zeta)). \tag{16}$$

Since the function $f(V(\zeta))$ satisfies $\|f(V(\zeta))\| \leq r\xi + (\|V_2\| + \|S(\zeta)\|.\|V_3\| + \|I(\zeta)\|.\|V_4\|)\|V(\zeta)\|$, and $f(V(\zeta))$ and $\frac{\partial f}{\partial V(\zeta)}(V(\zeta))$ are continuous on $\mathbb{R}^3$, Lemma 2 ensures that system (2) has a unique solution for all $t \in [0,\infty)$ (see [9]).

To establish the non-negativity of this solution space, we examine the vector fields along the bounding hyperplanes of the positive orthant $\mathbb{R}^3_+$. For any state vector $V(\zeta) = [S(\zeta), I(\zeta), P(\zeta)]^T \in \mathbb{R}^3_+$, evaluating the system component functions at the boundaries explicitly yields:

$$D^q S\Big|_{S=0} = 0, \quad D^q I\Big|_{I=0} = 0, \quad D^q P\Big|_{P=0} = 0.$$

By applying the fractional non-negativity and invariant region criteria established via equivalent Volterra integral representations and generalized singular inequalities [34], these boundary evaluations demonstrate that the directional fields remain tangent to the bounding hyperplanes. This ensures that the historical memory effects of the Caputo derivative cannot pull a trajectory outside the non-negative domain. Consequently, the positive orthant $\mathbb{R}^3_+$ is a strictly invariant region for all $\zeta \geq 0$.

Now we show boundedness. Let $\mathrm{w}(\zeta) = S(\zeta) + I(\zeta) + \frac{1}{c_6} P(\zeta)$, then

$$\begin{aligned}
D^q w(\zeta) + \gamma w(\zeta) &= D^q S(\zeta) + D^q I(\zeta) + \frac{1}{c_6} D^q P(\zeta) + \gamma S(\zeta) + \gamma I(\zeta) + \gamma \frac{1}{c_6} P(\zeta) \\
&= S(\zeta)\left(r\left(1 - \frac{S(\zeta)+I(\zeta)}{\xi}\right) + \gamma\right) - 2c_3 I(\zeta)P(\zeta) - (c_4 - \gamma) I(\zeta) \\
&\quad - \frac{c_5 - \gamma}{c_6} P(\zeta) \\
&\leq S(\zeta)\left(r\left(1 - \frac{S(\zeta)}{\xi}\right) + \gamma\right) - (c_4 - \gamma) I(\zeta) - \frac{c_5 - \gamma}{c_6} P(\zeta) \\
&= \frac{-r}{\xi}\left[S^2(\zeta) - S(\zeta)\frac{\xi(r+\gamma)}{r}\right] - (c_4 - \gamma) I(\zeta) - \frac{c_5 - \gamma}{c_6} P(\zeta) \\
&= \frac{-r}{\xi}\left(S(\zeta) - \frac{\xi(r+\gamma)}{2r}\right)^2 + \frac{\xi(r+\gamma)^2}{4r} - (c_4 - \gamma) I(\zeta) - \frac{c_5 - \gamma}{c_6} P(\zeta).
\end{aligned}$$

Set $\gamma = \min\{c_4, c_5\}$; then $c_4 - \gamma \geq 0$ and $c_5 - \gamma \geq 0$, which yields

$$D^q w(\zeta) + \gamma w(\zeta) \leq \frac{\xi(r+\gamma)^2}{4r}. \tag{17}$$

The function $y(\zeta) = y_0 E_q(-\gamma \zeta^q) + \frac{\xi(r+\gamma)^2}{4r\gamma}\left(1 - E_q(-\gamma\zeta^q)\right)$ solves $D^q y(\zeta) + \gamma y(\zeta) = \frac{\xi(r+\gamma)^2}{4r}$ with $y(0) = y_0$. By comparison, $w(\zeta) \leq w_0 E_q(-\gamma\zeta^q) + \frac{\xi(r+\gamma)^2}{4r\gamma}\left(1 - E_q(-\gamma\zeta^q)\right)$ with $w(0) = w_0$. Since $0 < E_q(-\gamma\zeta^q) < 1$, we obtain $w(\zeta) \leq \max\left\{w_0, \frac{\xi(r+\gamma)^2}{4r\gamma}\right\}$. Hence the solution is bounded.

To prove the second assertion, suppose $S_0 + I_0 \leq \xi$. Then

$$\begin{aligned}
D^q(S(\zeta) + I(\zeta)) &= D^q S(\zeta) + D^q I(\zeta) \\
&= rS(\zeta)\left(1 - \frac{S(\zeta)+I(\zeta)}{\xi}\right) - c_2 S(\zeta)P(\zeta) - c_3 I(\zeta)P(\zeta) - c_4 I(\zeta) \\
&\leq r(S(\zeta)+I(\zeta))\left(1 - \frac{S(\zeta)+I(\zeta)}{\xi}\right) - c_2 S(\zeta)P(\zeta) - c_3 I(\zeta)P(\zeta) - c_4 I(\zeta) \\
&\leq \max\left\{0, r(S(\zeta)+I(\zeta))\left(1 - \frac{S(\zeta)+I(\zeta)}{\xi}\right)\right\}.
\end{aligned} \tag{18}$$

We have two cases. If $D^q(S(\zeta)+I(\zeta)) \leq 0$, then $S(\zeta)+I(\zeta) \leq S_0+I_0 \leq \xi$. If $D^q(S(\zeta)+I(\zeta)) \leq r(S(\zeta)+I(\zeta))\left(1-\frac{S(\zeta)+I(\zeta)}{\xi}\right)$, then since $\frac{1}{\xi}+\left(\frac{1}{S_0+I_0}-\frac{1}{\xi}\right)E_q(-r\zeta^q) \geq \frac{1}{\xi}$,

$$S(\zeta)+I(\zeta) \leq \left[\frac{1}{\xi}+\left(\frac{1}{S_0+I_0}-\frac{1}{\xi}\right)E_q(-r\zeta^q)\right]^{-1} \leq \xi. \tag{19}$$

Thus $S(\zeta)+I(\zeta) \leq \max\left\{S_0+I_0, \left[\frac{1}{\xi}+\left(\frac{1}{S_0+I_0}-\frac{1}{\xi}\right)E_q(-r\zeta^q)\right]^{-1}\right\}$. Hence $S(\zeta)+I(\zeta) \leq \xi$ for all $t \in [0,\infty)$ whenever $S_0+I_0 \leq \xi$. □

4 Stability of the equilibrium points

Theorem 2. *The trivial equilibrium point $E_0 = (0,0,0)$ of system (2) is always an unstable saddle point for all fractional orders $q \in (0,1]$.*

Proof. Evaluating the Jacobian matrix of system (2) at $E_0(0,0,0)$ yields the diagonal matrix:

$$J(E_0) = \begin{bmatrix} r & 0 & 0 \\ 0 & -c_4 & 0 \\ 0 & 0 & -c_5 \end{bmatrix}. \tag{20}$$

The eigenvalues are directly given by $\lambda_1 = r$, $\lambda_2 = -c_4$, and $\lambda_3 = -c_5$. Since the intrinsic birth rate parameter satisfies $r > 0$, the spectral angle of this eigenvalue is $\arg(\lambda_1) = 0$. According to Matignon's foundational stability theorem for fractional systems [23], local asymptotic stability requires all eigenvalues to satisfy $|\arg(\lambda_i)| > \frac{q\pi}{2}$. Because $\arg(\lambda_1) = 0 < \frac{q\pi}{2}$ for any order $q \in (0,1]$, the stability requirement is violated, and E_0 is proven to be an unstable saddle point. □

Theorem 3. *The fixed point $E_1 = (\xi,0,0)$ of system (2) is asymptotically stable under the conditions*

$$c_1 < \frac{c_4 - r}{\xi} \quad \text{and} \quad c_2 < \frac{c_5}{\xi c_6}.$$

Proof. Let $(S(\zeta),I(\zeta),P(\zeta))$ be a solution of system (2) with initial conditions $(S_0,I_0,P_0) \in \mathbb{R}^3_+$ such that $S_0+I_0 \leq \xi$. Consider the Lyapunov function $L_1(\zeta) : \mathbb{R}^3 \to \mathbb{R}_+$ defined by

$$L_1(\zeta) = S(\zeta) - \xi - \xi \log\left(\frac{S(\zeta)}{\xi}\right) + I(\zeta) + \frac{1}{c_6}P(\zeta). \tag{21}$$

Computing the Caputo fractional derivative of order $q \in (0,1]$ along the trajectories of system (2) and implementing the Volterra fractional inequality (9) from Lemma 1, we obtain:

$$\begin{aligned} D^q L_1(\zeta) &\leq (S(\zeta)-\xi)\frac{1}{S(\zeta)}D^q S(\zeta) + D^q I(\zeta) + \frac{1}{c_6}D^q P(\zeta) \\ &= (S(\zeta)-\xi)\left[r\left(1-\frac{S(\zeta)+I(\zeta)}{\xi}\right) - c_1 I(\zeta) - c_2 P(\zeta)\right] \\ &\quad + [c_1 S(\zeta) I(\zeta) - c_3 I(\zeta) P(\zeta) - c_4 I(\zeta)] \end{aligned}$$

$$
\begin{aligned}
&+\frac{1}{c_6}\left[-c_6c_3I(\zeta)P(\zeta)-c_5P(\zeta)+c_6c_2S(\zeta)P(\zeta)\right]\\
&=-\frac{r}{\xi}(S(\zeta)-\xi)^2-\frac{r}{\xi}S(\zeta)I(\zeta)+rI(\zeta)-c_1S(\zeta)I(\zeta)+c_1\xi I(\zeta)\\
&\quad-c_2S(\zeta)P(\zeta)+c_2\xi P(\zeta)+c_1S(\zeta)I(\zeta)-c_3I(\zeta)P(\zeta)-c_4I(\zeta)\\
&\quad-c_3I(\zeta)P(\zeta)-\frac{c_5}{c_6}P(\zeta)+c_2S(\zeta)P(\zeta)\\
&=-\frac{r}{\xi}(S(\zeta)-\xi)^2-\frac{r}{\xi}S(\zeta)I(\zeta)+rI(\zeta)+c_1\xi I(\zeta)\\
&\quad+c_2\xi P(\zeta)-2c_3I(\zeta)P(\zeta)-c_4I(\zeta)-\frac{c_5}{c_6}P(\zeta)\\
&=-\frac{r}{\xi}(S(\zeta)-\xi)^2-\frac{r}{\xi}S(\zeta)I(\zeta)+(r+c_1\xi-c_4)I(\zeta)-2c_3I(\zeta)P(\zeta)+\left(c_2\xi-\frac{c_5}{c_6}\right)P(\zeta).
\end{aligned}
$$

Then

$$
D^qL_1(\zeta)\leq-\frac{r}{\xi}(S(\zeta)-\xi)^2+(r+c_1\xi-c_4)I(\zeta)+\left(c_2\xi-\frac{c_5}{c_6}\right)P(\zeta). \tag{22}
$$

Consequently, if the parameters satisfy $c_2<\frac{c_5}{c_6\xi}$ and $c_1<\frac{c_4-r}{\xi}$, the linear coefficients on the right-hand side of (22) remain strictly negative, ensuring that $D^qL_1(\zeta)\leq 0$ holds across Ω. By LaSalle's fractional invariance principle [7, 32], the largest invariant subset where $D^qL_1(\zeta)=0$ corresponds exactly to the singleton set $\{E_1\}=\{(\xi,0,0)\}$, proving asymptotic stability. □

Theorem 4. *The equilibrium point* $E_2=(\bar{S},\bar{I},0)=\left(\frac{c_4}{c_1},\frac{r(c_1\xi-c_4)}{c_1(r+c_1\xi)},0\right)$ *of system (2) is asymptotically stable whenever* $c_2<\frac{1}{c_4}\left(\frac{c_5c_1}{c_6}+\frac{rc_3(c_4-c_1\xi)}{c_1\xi}\right)$.

Proof. Let $(S(\zeta),I(\zeta),P(\zeta))$ be a solution of system (2) with initial conditions $(S_0,I_0,P_0)\in\mathbb{R}^3_+$ such that $S_0+I_0\leq\xi$. At the predator-free equilibrium point E_2, the balance coordinates satisfy the algebraic identities:

$$
\bar{S}=\frac{c_4}{c_1},\quad\text{and}\quad r\left(1-\frac{\bar{S}+\bar{I}}{\xi}\right)=c_1\bar{I}. \tag{23}
$$

Consider the Lyapunov function $L_2(\zeta):\mathbb{R}^3\to\mathbb{R}_+$ defined by

$$
L_2(\zeta)=S(\zeta)-\bar{S}-\bar{S}\log\left(\frac{S(\zeta)}{\bar{S}}\right)+N_1\left(I(\zeta)-\bar{I}-\bar{I}\log\left(\frac{I(\zeta)}{\bar{I}}\right)\right)+\frac{1}{c_6}P(\zeta), \tag{24}
$$

where $N_1=\frac{r}{c_1\xi}+1>1$. Computing the Caputo fractional derivative of order $q\in(0,1]$ along the trajectories of system (2) and implementing the Volterra fractional inequality (9) from Lemma 1, we obtain:

$$
\begin{aligned}
D^qL_2(\zeta)&\leq(S(\zeta)-\bar{S})\frac{1}{S(\zeta)}D^qS(\zeta)+N_1(I(\zeta)-\bar{I})\frac{1}{I(\zeta)}D^qI(\zeta)+\frac{1}{c_6}D^qP(\zeta)\\
&=(S(\zeta)-\bar{S})\left[r\left(1-\frac{S(\zeta)+I(\zeta)}{\xi}\right)-c_1I(\zeta)-c_2P(\zeta)\right]\\
&\quad+N_1(I(\zeta)-\bar{I})\left[c_1S(\zeta)-c_3P(\zeta)-c_4\right]
\end{aligned}
$$

$$+\frac{1}{c_6}\left[-c_6c_3I(\zeta)P(\zeta)-c_5P(\zeta)+c_6c_2S(\zeta)P(\zeta)\right]. \tag{25}$$

Substituting the equilibrium relations from (23) into the parenthetical brackets yields:

$$\begin{aligned}
D^qL_2(\zeta) &\le (S(\zeta)-\bar{S})\left[-\frac{r}{\xi}(S(\zeta)-\bar{S})-\left(\frac{r}{\xi}+c_1\right)(I(\zeta)-\bar{I})-c_2P(\zeta)\right]\\
&\quad +N_1(I(\zeta)-\bar{I})\left[c_1(S(\zeta)-\bar{S})-c_3P(\zeta)\right]-c_3I(\zeta)P(\zeta)-\frac{c_5}{c_6}P(\zeta)+c_2S(\zeta)P(\zeta).\\
&= -\frac{r}{\xi}(S(\zeta)-\bar{S})^2-\left(\frac{r}{\xi}+c_1\right)(S(\zeta)-\bar{S})(I(\zeta)-\bar{I})-c_2S(\zeta)P(\zeta)+c_2\bar{S}P(\zeta)\\
&\quad +N_1c_1(S(\zeta)-\bar{S})(I(\zeta)-\bar{I})-N_1c_3I(\zeta)P(\zeta)+N_1c_3\bar{I}P(\zeta)\\
&\quad -c_3I(\zeta)P(\zeta)-\frac{c_5}{c_6}P(\zeta)+c_2S(\zeta)P(\zeta).\\
&= -\frac{r}{\xi}(S(\zeta)-\bar{S})^2+\left[N_1c_1-\left(\frac{r}{\xi}+c_1\right)\right](S(\zeta)-\bar{S})(I(\zeta)-\bar{I})\\
&\quad -(N_1+1)c_3I(\zeta)P(\zeta)+\left(c_2\bar{S}+N_1c_3\bar{I}-\frac{c_5}{c_6}\right)P(\zeta).
\end{aligned} \tag{26}$$

By design, substituting our choice of the scaling multiplier $N_1=\frac{r}{c_1\xi}+1$ completely eliminates the bilinear cross term. This leaves:

$$D^qL_2(\zeta)\le -\frac{r}{\xi}(S(\zeta)-\bar{S})^2-(N_1+1)c_3I(\zeta)P(\zeta)+\left(c_2\bar{S}+N_1c_3\bar{I}-\frac{c_5}{c_6}\right)P(\zeta). \tag{27}$$

Then

$$D^qL_2(\zeta)\le -\frac{r}{\xi}(S(\zeta)-\bar{S})^2+\left(c_2\bar{S}+N_1c_3\bar{I}-\frac{c_5}{c_6}\right)P(\zeta). \tag{28}$$

Substituting the explicit boundary values $\bar{S}=\frac{c_4}{c_1}$, $N_1=\frac{r+c_1\xi}{c_1\xi}$, and $\bar{I}=\frac{r(c_1\xi-c_4)}{c_1(r+c_1\xi)}$ into the linear coefficient of $P(\zeta)$ gives:

$$c_2\bar{S}+N_1c_3\bar{I}-\frac{c_5}{c_6}=\frac{c_4c_2}{c_1}+\frac{rc_3(c_1\xi-c_4)}{c_1^2\xi}-\frac{c_5}{c_6}. \tag{29}$$

Consequently, multiplying by $\frac{c_1}{c_4}$, if the parameters satisfy the condition

$$c_2<\frac{1}{c_4}\left(\frac{c_5c_1}{c_6}+\frac{rc_3(c_4-c_1\xi)}{c_1\xi}\right), \tag{30}$$

the linear coefficient of $P(\zeta)$ on the right-hand side of (28) remains strictly negative, ensuring that $D^qL_2(\zeta)\le 0$ holds across Ω. By LaSalle's fractional invariance principle [7, 32], the largest invariant subset where $D^qL_2(\zeta)=0$ corresponds exactly to the singleton set $\{E_2\}=\{(\bar{S},\bar{I},0)\}$, proving asymptotic stability. □

Theorem 5. *The equilibrium point* $E_3=(\hat{S},0,\hat{P})=\left(\frac{c_5}{c_2c_6},0,\frac{r}{c_2}\left(1-\frac{c_5}{c_2c_6\xi}\right)\right)$ *of system (2) is asymptotically stable whenever* $c_4>\left(\frac{r}{\xi}+c_1\right)\hat{S}+c_3\hat{P}$.

Proof. Let $(S(\zeta),I(\zeta),P(\zeta))$ be a solution of system (2) with initial conditions $(S_0,I_0,P_0)\in\mathbb{R}^3_+$ such that $S_0+I_0\leq\xi$. At the infected-free equilibrium point E_3, the balance coordinates satisfy the algebraic identities:

$$\hat{S}=\frac{c_5}{c_2c_6},\quad \text{and}\quad r\left(1-\frac{\hat{S}}{\xi}\right)=c_2\hat{P}. \tag{31}$$

Consider the Lyapunov function candidate $L_3(\zeta):\mathbb{R}^3\to\mathbb{R}_+$ defined by

$$L_3(\zeta)=S(\zeta)-\hat{S}-\hat{S}\log\left(\frac{S(\zeta)}{\hat{S}}\right)+I(\zeta)+\frac{1}{c_6}\left(P(\zeta)-\hat{P}-\hat{P}\log\left(\frac{P(\zeta)}{\hat{P}}\right)\right). \tag{32}$$

Computing the Caputo fractional derivative of order $q\in(0,1]$ along the trajectories of system (2) and implementing the Volterra fractional inequality (9) from Lemma 1, we obtain

$$\begin{aligned}
D^qL_3(\zeta)&\leq(S(\zeta)-\hat{S})\frac{1}{S(\zeta)}D^qS(\zeta)+D^qI(\zeta)+\frac{1}{c_6}(P(\zeta)-\hat{P})\frac{1}{P(\zeta)}D^qP(\zeta)\\
&=(S(\zeta)-\hat{S})\left[r\left(1-\frac{S(\zeta)+I(\zeta)}{\xi}\right)-c_1I(\zeta)-c_2P(\zeta)\right]\\
&\quad+[c_1S(\zeta)I(\zeta)-c_3I(\zeta)P(\zeta)-c_4I(\zeta)]\\
&\quad+\frac{1}{c_6}(P(\zeta)-\hat{P})\left[-c_6c_3I(\zeta)-c_5+c_6c_2S(\zeta)\right].
\end{aligned} \tag{33}$$

Substituting the equilibrium relations from (31) into the parenthetical brackets yields:

$$\begin{aligned}
D^qL_3(\zeta)&\leq(S(\zeta)-\hat{S})\left[-\frac{r}{\xi}(S(\zeta)-\hat{S})-\left(\frac{r}{\xi}+c_1\right)I(\zeta)-c_2(P(\zeta)-\hat{P})\right]\\
&\quad+c_1S(\zeta)I(\zeta)-c_3I(\zeta)P(\zeta)-c_4I(\zeta)\\
&\quad+(P(\zeta)-\hat{P})\left[-c_3I(\zeta)+c_2(S(\zeta)-\hat{S})\right].\\
&=-\frac{r}{\xi}(S(\zeta)-\hat{S})^2-\left(\frac{r}{\xi}+c_1\right)(S(\zeta)-\hat{S})I(\zeta)-c_2(S(\zeta)-\hat{S})(P(\zeta)-\hat{P})\\
&\quad+c_1S(\zeta)I(\zeta)-c_3I(\zeta)P(\zeta)-c_4I(\zeta)\\
&\quad-c_3I(\zeta)(P(\zeta)-\hat{P})+c_2(S(\zeta)-\hat{S})(P(\zeta)-\hat{P}).\\
&=-\frac{r}{\xi}(S(\zeta)-\hat{S})^2-\left(\frac{r}{\xi}+c_1\right)(S(\zeta)-\hat{S})I(\zeta)\\
&\quad+c_1(S(\zeta)-\hat{S})I(\zeta)+c_1\hat{S}I(\zeta)-c_3I(\zeta)P(\zeta)-c_4I(\zeta)-c_3I(\zeta)P(\zeta)+c_3\hat{P}I(\zeta).\\
&=-\frac{r}{\xi}(S(\zeta)-\hat{S})^2-\frac{r}{\xi}(S(\zeta)-\hat{S})I(\zeta)-2c_3I(\zeta)P(\zeta)+\left(c_1\hat{S}+c_3\hat{P}-c_4\right)I(\zeta).\\
&=-\frac{r}{\xi}(S(\zeta)-\hat{S})^2-\frac{r}{\xi}S(\zeta)I(\zeta)-2c_3I(\zeta)P(\zeta)+\left[\left(\frac{r}{\xi}+c_1\right)\hat{S}+c_3\hat{P}-c_4\right]I(\zeta).
\end{aligned} \tag{34}$$

Then

$$D^qL_3(\zeta)\leq-\frac{r}{\xi}(S(\zeta)-\hat{S})^2+\left[\left(\frac{r}{\xi}+c_1\right)\hat{S}+c_3\hat{P}-c_4\right]I(\zeta). \tag{35}$$

Reflecting the inequality condition $c_4 > \left(\frac{r}{\xi}+c_1\right)\hat{S}+c_3\hat{P}$, the linear coefficient of $I(\zeta)$ remains strictly negative across the domain, rendering $D^q L_3(\zeta) \leq 0$. By LaSalle's fractional invariance principle [7, 32], the largest invariant subset where $D^q L_3(\zeta) = 0$ corresponds exactly to the singleton set $\{E_3\}$, validating asymptotic stability. □

Theorem 6. *If the interior coexistence equilibrium point $E_4 = (S^*, I^*, P^*)$ exists, it is unconditionally unstable for any choice of fractional order $q \in (0,1]$.*

Proof. The Jacobian matrix of system (2) at the interior state $E_4(S^*, I^*, P^*)$ yields:

$$J(E_4) = \begin{bmatrix} -\frac{rS^*}{\xi} & -\left(\frac{r}{\xi}+c_1\right)S^* & -c_2S^* \\ c_1I^* & 0 & -c_3I^* \\ c_6c_2P^* & -c_6c_3P^* & 0 \end{bmatrix}. \tag{36}$$

The characteristic polynomial is given by $\det(\lambda I - J(E_4)) = \lambda^3 + A_1\lambda^2 + A_2\lambda + A_3 = 0$, where

$$\begin{aligned} A_1 &= -tr(J(E_4)) = \frac{rS^*}{\xi} > 0, \\ A_2 &= c_1\left(\frac{r}{\xi}+c_1\right)S^*I^* + c_6c_2^2S^*P^* - c_6c_3^2I^*P^*, \\ A_3 &= -\det(J(E_4)) = -c_3^2c_6\frac{r}{\xi}S^*I^*P^* - c_2c_3c_6\left(\frac{r}{\xi}+c_1\right)S^*I^*P^* - c_1c_2c_3c_6S^*I^*P^* \\ &= -c_3c_6S^*I^*P^*\left[c_3\frac{r}{\xi} + c_2\left(\frac{r}{\xi}+c_1\right) + c_1c_2\right] = -c_3c_6S^*I^*P^*\left[\frac{r}{\xi}(c_2+c_3) + 2c_1c_2\right] < 0. \end{aligned}$$

Since $A_3 < 0$, by Descartes' Rule of Signs, the characteristic polynomial possesses at least one strictly positive real root, $\lambda_k > 0$, with a spectral angle of $\arg(\lambda_k) = 0$. Matignon's stability criterion dictates that a fractional-order system [23] is locally asymptotically stable if and only if all eigenvalues satisfy $|\arg(\lambda_i)| > \frac{q\pi}{2}$. Since $\arg(\lambda_k) = 0 < \frac{q\pi}{2}$ for any $q \in (0,1]$, this condition is fundamentally violated, proving that the interior equilibrium E_4 is unconditionally unstable. □

5 Numerical simulation

All numerical simulations were performed using MATLAB. The fractional-order system (2) was solved using Garrappa's `fde12` function [12], which implements the predictor–corrector Adams–Bashforth–Moulton (PECE) method for Caputo-type fractional differential equations. The integration used a fixed step size $h = 0.001$ over $[0, 1500]$, selected after convergence tests. Simulations were validated by comparing the $q = 1$ case with the integer-order model [2], computing errors via the uniform maximum norm ($\|\cdot\|_\infty$) which yielded a relative norm error less than 10^{-6}.

Using the numerical scheme described above, we now examine the dynamical behavior of the equilibrium points. The trivial equilibrium consistently exhibits unstable saddle-point behavior regardless of parameter selection. The basic reproduction number is $R_0 = \frac{c_1\xi}{c_4}$. This represents the average number of secondary infections caused by a single infected prey in a completely susceptible prey population. The disease dies out if $R_0 < 1$ (i.e., $c_1 < \frac{c_4}{\xi}$) and propagates if $R_0 > 1$.

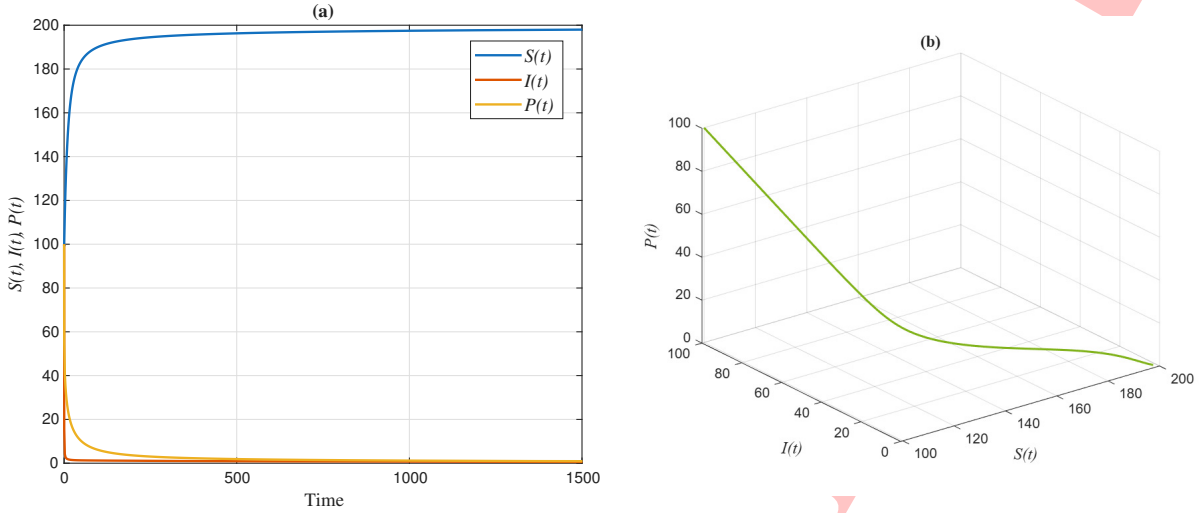

Figure 1: System (2) with order $q = 0.75$: (a) time histories of the three individual population compartments initiating from $(100, 100, 100)$ showing stable behavior, and (b) corresponding 3D state space phase curve showing clear convergence towards the disease-free and predator-free boundary equilibrium node $E_1 = (200, 0, 0)$.

We present a numerical case study with fixed parameters:

$$r = 0.3, \quad \xi = 200, \quad c_3 = 0.1, \quad c_4 = 0.3, \quad c_5 = 0.2, \quad c_6 = 0.5. \tag{37}$$

The parameters used in the numerical simulations satisfy the weaker necessary stability conditions derived from linearization. The conditions in Theorems 3-5 are sufficient, not necessary; they are intentionally conservative. The linearization provides the exact local stability thresholds (e.g., for E_1, $c_1 < c_4/\xi$ and $c_2 < c_5/(c_6\xi)$), while the Lyapunov functions provide global stability guarantees for the conservative parameter regime. This justifies the use of both approaches despite the conservative nature of the sufficient conditions. The simulations demonstrate convergence in a parameter regime where the Lyapunov sufficient conditions are not met, illustrating that the true stability region is larger than the conservative Lyapunov bounds.

We vary c_2 (predation rate) and c_1 (transmission rate). Equilibrium E_1 is stable for $c_2 < 0.002$ and $c_1 < 0.0015$. For $c_2 = 0.001$, $c_1 = 0.001$, all trajectories from $(100, 100, 100)$ converge to E_1 for different q. Figures 1 and 2 show this for $q = 0.75$ and $q = 0.95$.

Figure 3 shows that larger q gives faster convergence to E_1. For E_2, we need $c_1 > 0.0015$. With $c_1 = 0.1$, $c_2 = 0.06$, all trajectories from $(20, 20, 20)$ converge to $E_2 = (3.02, 2.92, 0)$ (Figures 4-6).

For E_3, stability requires $c_1 < 0.0015$. With $c_1 = 0.001$, $c_2 = 0.005$, trajectories converge to $E_3 = (80.07, 0, 35.76)$ (Figures 7-9).

6 Discussion

The fractional derivative order $q \in (0, 1]$ models memory effects in population dynamics. When $q < 1$, the current growth rate depends on the entire past history. This is biologically meaningful: prey populations exhibit reduced reproduction after an infection outbreak (immune memory), and predators show delayed

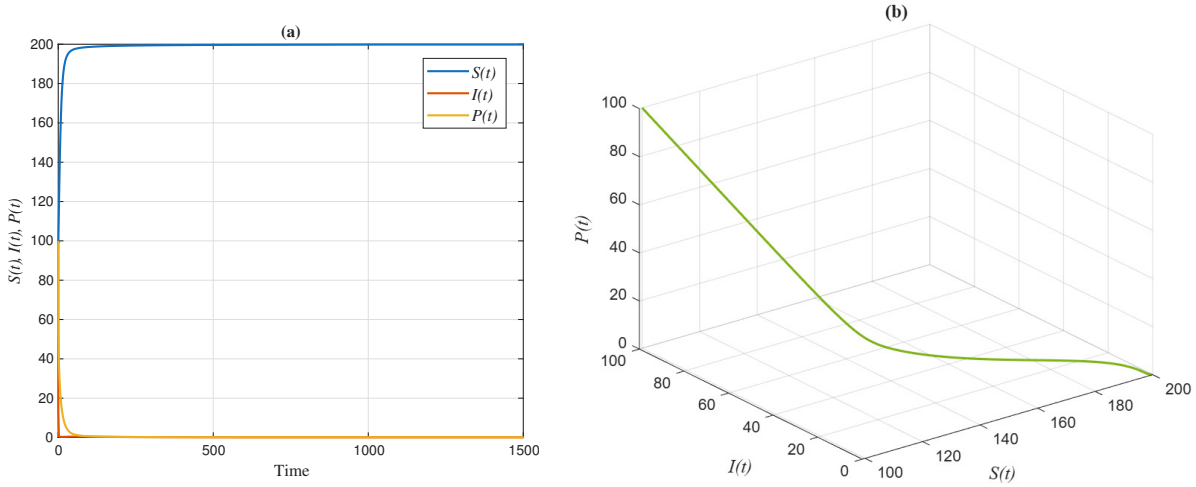

Figure 2: System (2) with order $q = 0.95$: (a) temporal trajectory progression of the population components, and (b) 3D global view mapping the trajectory profiles directly into the boundary sink point $E_1 = (200, 0, 0)$.

numerical responses to prey availability. As $q \to 1$, memory fades and the system approaches the integer-order model [2]. Infection establishes only if the basic reproduction number $R_0 = c_1\xi/c_4 > 1$. If $R_0 < 1$, the disease dies out regardless of predation. Predation can eliminate infection if predators preferentially consume infected prey (high c_3). The fractional order q does not affect these thresholds but modifies transient dynamics. As proven in Theorem 6, the interior equilibrium E_4 is always unstable under the assumed linear functional response. Limitations of this study include: (i) no real data validation — this is a theoretical study; (ii) simple mass action transmission with no recovery; (iii) linear functional response ignores predator handling time; (iv) no spatial heterogeneity. Advantages include: (i) fractional calculus provides realistic memory effects; (ii) Lyapunov functions work for all $q \in (0, 1]$; (iii) mathematically tractable yet biologically informative.

7 Conclusion

In this article, a fractional-order eco-epidemiological model is formulated with a linear mass functional response. The interior equilibrium E_4 is always unstable because infected prey negatively impact predator dynamics, preventing stable coexistence. The study reveals that infection can destabilize predator-prey dynamics and predators can disrupt host-parasite interactions. The system dynamics are principally determined by the infection rate c_1 and the predator's attack rate on susceptible prey c_2. Our results show that stable coexistence is impossible in host-parasite-predator systems with lethal disease dynamics under the assumed linear functional response. The system exhibits exactly three possible stable configurations: disease-free (E_1), predator-free (E_2), infected-free (E_3). Although fractional-order derivatives typically influence stability in many systems, our results demonstrate that in this model they preserve the stability properties of all equilibrium points. The primary effect of fractional-order differentiation manifests in the convergence rate rather than asymptotic stability. Future work includes incorporating

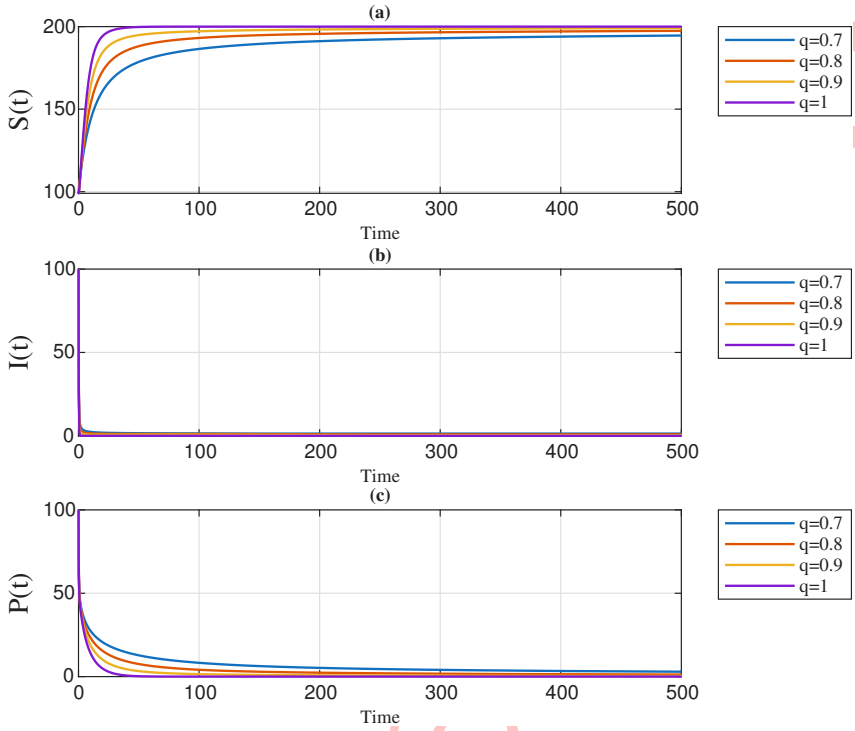

Figure 3: Sensitivity curves showing population tracking behaviors approaching the boundary state $E_1 = (200, 0, 0)$ for diverse fractional calculus orders $q \in \{0.7, 0.8, 0.9, 1.0\}$: (a) healthy susceptible prey curves, (b) rapid disease decay of the infected host population, and (c) predator decline profiles, confirming that q alters the intermediate relaxation velocities and damping profiles while preserving overall topological stability conditions.

more realistic functional responses (Holling type II or Beddington-DeAngelis), introducing stochastic environmental fluctuations, validating the model with real epidemiological time series data (e.g., rabbit-myxoma virus-predator systems, where myxoma virus infection alters rabbit behavior and vulnerability to fox predation), and applying optimal control theory to identify intervention strategies.

Conflicts of Interest

The authors declare that they have no conflict of interest.

Author Contributions

All authors contributed equally to this work.

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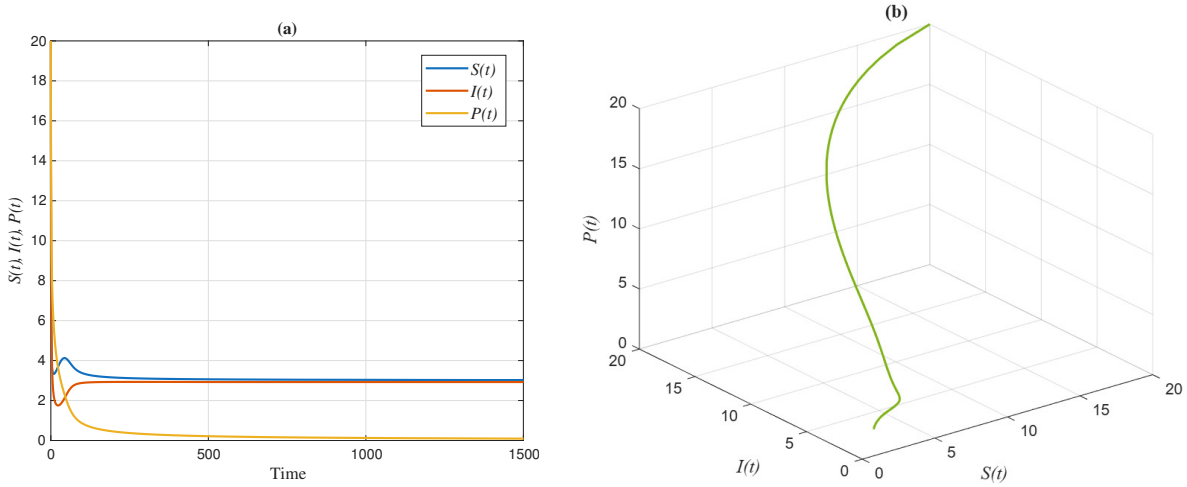

Figure 4: System (2) solved under low integration memory parameter $q = 0.75$: (a) time series demonstrating long-term extinction of the predator cohort, and (b) 3D orbital trajectory settling onto the stable boundary point $E_2 = (3.02, 2.92, 0)$.

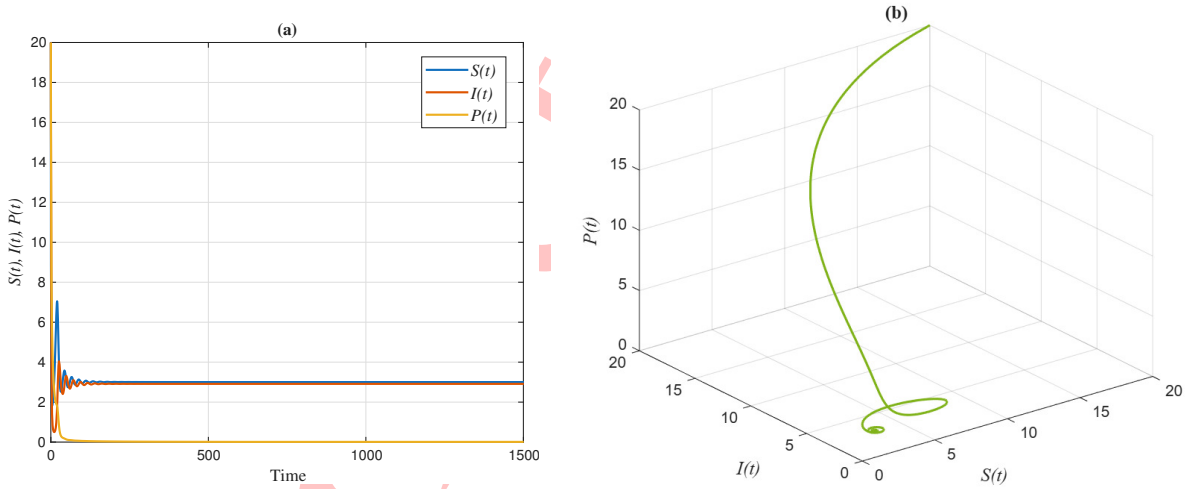

Figure 5: System (2) tracks at $q = 0.95$: (a) evolution waveforms showing visible transient damping ripples, and (b) phase space spirals moving cleanly towards the predator-free point $E_2 = (3.02, 2.92, 0)$.

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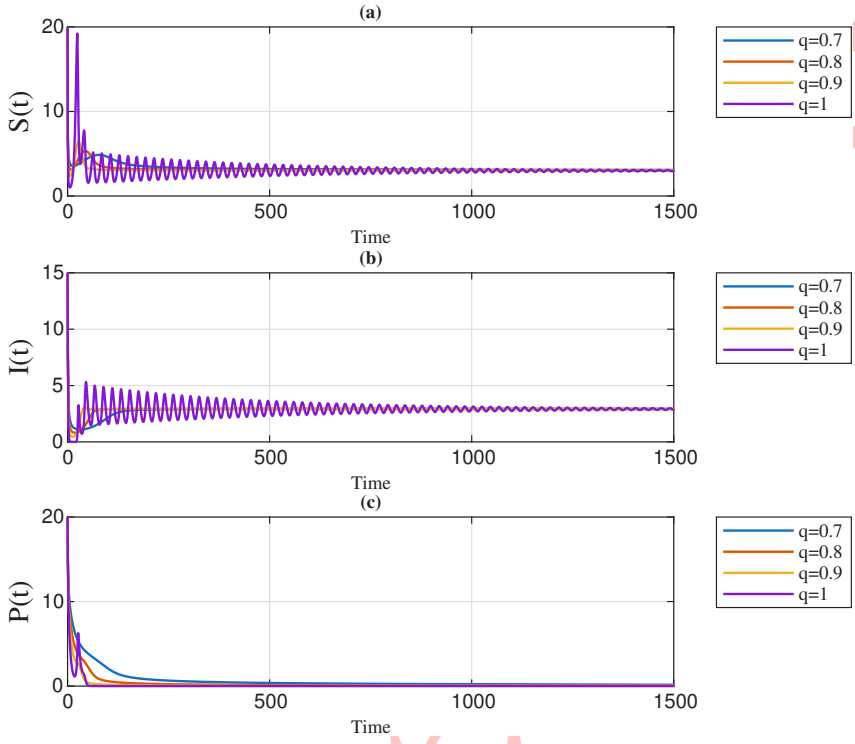

Figure 6: Comparative sensitivity curves showing convergence behaviors near the predator-free equilibrium E_2 for different derivative orders q, illustrating that reducing q functions as an effective stabilizer by dampening overshoot amplitudes during early transient stages.

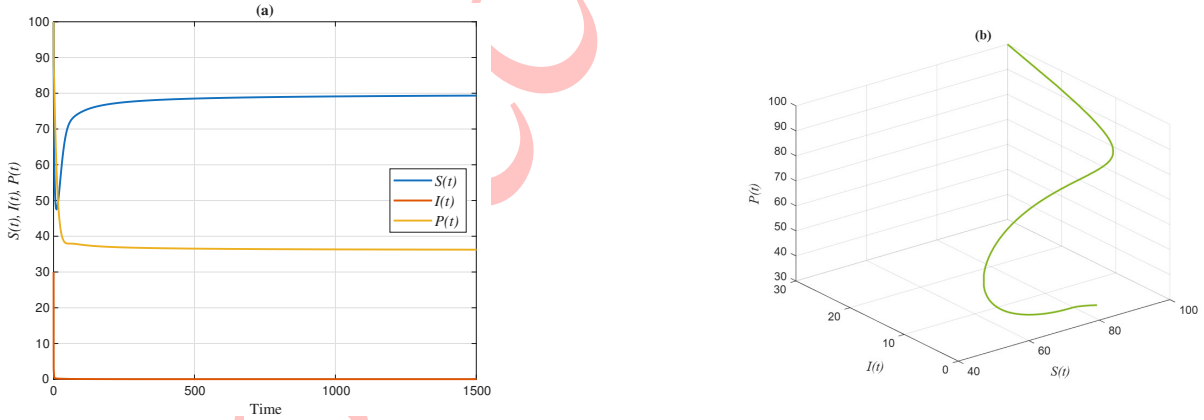

Figure 7: System (2) operating at order $q = 0.75$: (a) population dynamics tracking complete viral clearance alongside robust predator-prey co-existence, and (b) phase-space curve verifying stable convergence into $E_3 = (80.07, 0, 35.76)$.

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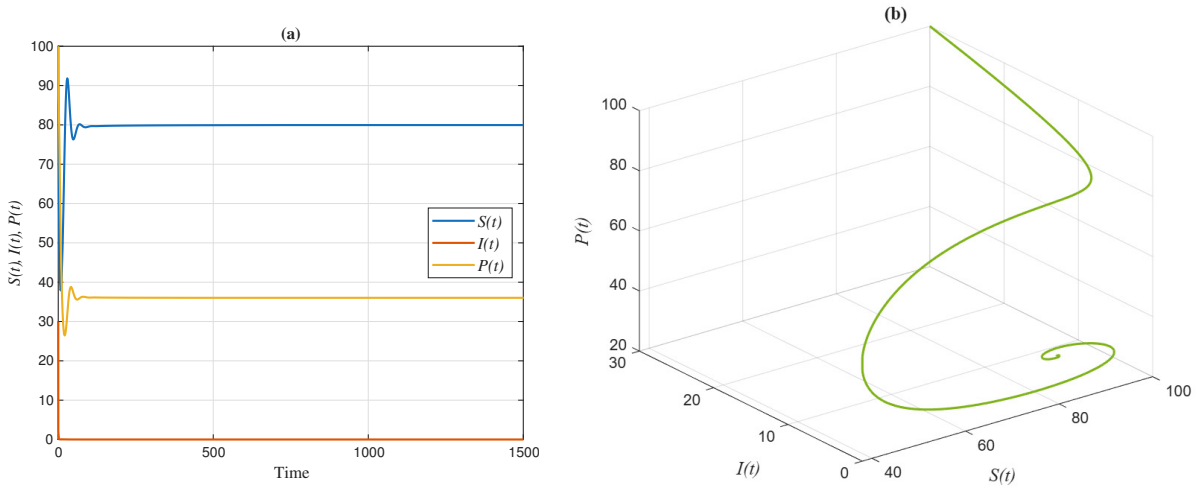

Figure 8: System (2) plots at order $q = 0.95$: (a) highly responsive population curves exhibiting oscillatory tracking steps, and (b) corresponding winding phase plot terminating onto the stable infected-free state node $E_3 = (80.07, 0, 35.76)$.

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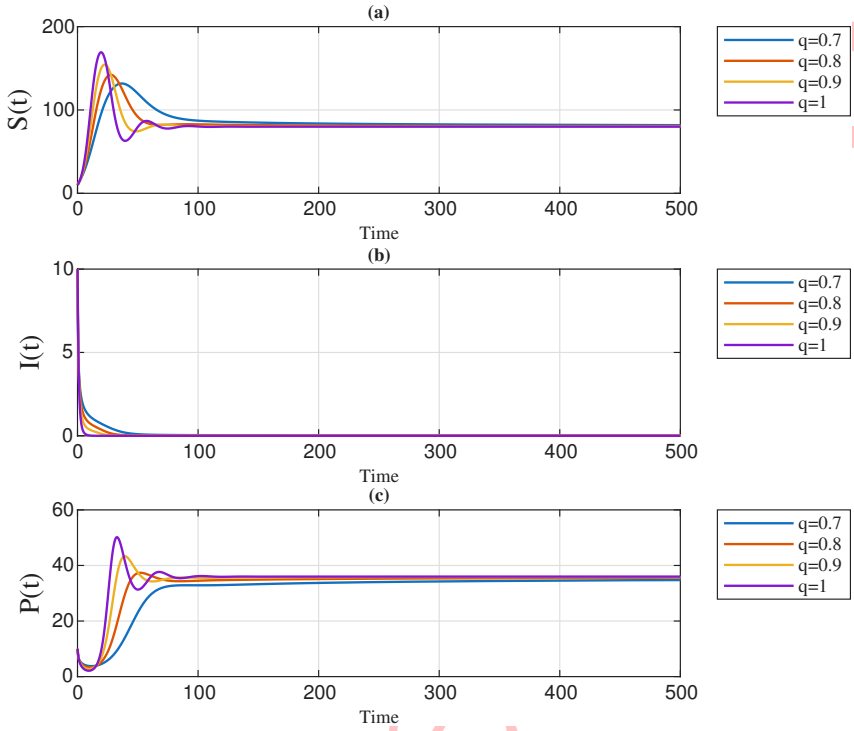

Figure 9: Comprehensive numerical simulation highlighting transient responses approaching the infected-free state node E_3 for various orders q, indicating that traditional integer models ($q = 1$) display excessive peak amplitudes while lower values flatten peak overshoots.

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