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A spectral collocation framework for fractional ODEs with nonlocal boundary conditions

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Abstract. Fractional differential equations with nonlocal boundary conditions naturally arise in models of heat conduction, viscoelasticity, anomalous diffusion, and population dynamics, where both memory effects and global constraints must be considered. Their numerical treatment is challenging due to the interplay between fractional operators and nonlocal conditions. In this work, we propose a collocation method based on orthogonal cardinal functions for solving fractional ordinary differential equations with nonlocal boundary conditions. To enable an accurate numerical approximation, the desired problem is first transformed into an equivalent Volterra integral equation. The orthogonal property of the cardinal functions guarantees stability and spectral accuracy, and their interpolation property makes it easier to enforce nonlocal constraints. Through numerical experiments, we verify the scheme's accuracy and efficiency and investigate a priori error estimates. In comparison to existing numerical methods, the creative application of orthogonal cardinal functions within a collocation framework offers an effective and reliable tool for fractional models with increased accuracy.

Keywords: Cardinal functions, error estimates, fractional differential equations, nonlocal boundary conditions, Volterra integral equation.
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1 Introduction

Ordinary differential equations (ODEs) with nonlocal boundary conditions (NBCs) are increasingly studied because they can capture global interactions across physical, biological, and engineering systems.

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Unlike classical boundary conditions, which constrain solutions at fixed endpoints, NBCs relate to integral constraints, averages, or values at points inside the domain [39]. This difference can be subtle but often leads to richer mathematical structures and new solution behaviors. For example, in heat conduction, it can make more sense to focus on total heat energy or average temperature rather than just endpoint values [11]. In the literature, population dynamics often involve conservation principles or feedback laws that naturally lead to integral conditions on the total population [4]. Similarly, in structure mechanics, multipoint conditions can represent beams or columns with interior supports or sensor placements [24]. The NBCs typically involve noncomact operators and nonlinear couplings, making them more complex and challenging than their classical counterparts. As known, addressing these problems often requires specialized tools, including Green's function reformulations and fixed-point theory [19, 39]. Moreover, NBCs act as a conceptual bridge to nonlocal partial differential equations and fractional-order models, where memory effects and long-range dependencies really matter [17]. It is well known that, learning about ODEs with NBCs is important not only for advancing theoretical ideas but also for building more realistic model in physics, biology, and engineering.

Fractional calculus, by extending differentiation and integration to non-integer orders, offers a powerful and flexible framework for modeling phenomena with intrinsic memory and hereditary effects [16, 32, 40]. In the last few decades, fractional differential equations (FDEs) have found remarkable success in fields such as viscoelasticity, anomalous diffusion, signal processing, electrochemistry, and control theory [6, 13, 36, 37]. Particularly, they have used where classical models fall short in representing long-range temporal or spatial dependencies [23, 40]. These equations are especially good at capturing the complexity of real-world dynamical systems because they are nonlocal, meaning that the fractional derivative at any point depends on the function's history across the whole domain. Various computational methods have been proposed to solve FDEs efficiently, including the implicit integration factor method [52], the Adomian decomposition method [15], and adaptive grid techniques to enhance numerical accuracy [38]. Recently, Izadi et al. [26] developed a quasilinearization-Lerch matrix collocation (QLM-Lerch) technique for fractional-order Duffing equations with integral boundary conditions, demonstrating superior accuracy compared to existing approaches. Wavelet-based methods [7, 46] and B-spline collocation approaches [30, 33] have also gained popularity for their flexibility in approximating solutions. The QLM-Lerch matrix collocation methodology introduced in [26] represents another recent advancement in this direction, offering high accuracy for nonlinear fractional systems with integral boundary conditions. Additionally, second-order accurate difference schemes [29], multistep schemes [22], collocation method [14, 18, 41], finite difference method [49, 50], classical finite element method [21], and Petrov-Galerkin finite element-meshfree formulation [35] have been developed to address FDEs. More recently, Izadi et al. [28] applied the local discontinuous Galerkin (LDG) finite-element method to multi-order fractional differential equations with applications to circuit modeling, demonstrating high accuracy and computational efficiency. In a related work, Izadi et al. [27] developed clique and QLM-clique matrix methods for solving nonlinear nonlocal two-point boundary value problems.

There has been a recent surge of interest in fractional ODEs featuring NBCs, as these issues seamlessly integrate the global memory attributes of fractional operators with the complex limitations dictated by nonlocal boundary relations [34, 39]. These challenges are not merely theoretical: they arise naturally in engineering, mechanics, and biology, where the evolution of a system is influenced both by its historical states and by integral or multipoint conditions at the boundaries. Consequently, these models are often very nonlinear and require a lot of computational load, which can be too much for current analytical and numerical methods [16, 34]. In [3, 48], the existence and uniqueness of the solution of this kind of ODE are investigated.

Cardinal functions are a class of basis functions that are employed in numerical analysis and approximation theory. They are especially useful when interpolating from discrete sets of points or generating function approximations, which are unavoidable in spectral or pseudospectral approaches. Cardinal functions are typically used as bases in high-accuracy spectral or pseudospectral methods. Cardinal functions actually help with spectral accuracy in numerical techniques, which yield extremely precise approximations of differential equation solutions [9]. Collocation method is a good technique to solve FDEs because they can get very accurate results by minimizing the residual of the equation at certain points [1, 2, 8, 34, 44]. There are clear benefits to using orthogonal cardinal functions (OCFs) as basis functions. Legendre and Chebyshev polynomials, for instance, achieve the high spectral accuracy [12]. The cardinal property lets you interpolate exactly at the chosen nodes. This makes it easier to deal with boundary conditions that are both local and nonlocal. As a result, OCFs are good for both approximating fractional operators and handling global constraints at the same time.

The innovation of this study resides in the formulation of a collocation method utilizing OCFs to address fractional ordinary differential equations with NBCs. To our knowledge, this combination has not been thoroughly studied in the literature. Current approaches to solve this problem frequently utilize finite differences [43], finite elements [20], or spectral approximations, neglecting the interpolation benefits offered by cardinal functions [51]. In contrast to existing collocation approaches such as those in [48], which employ standard Gauss collocation points and require numerical quadrature for coefficient computation, our method utilizes orthogonal cardinal functions (OCFs). The cardinality property enables exact interpolation at nodal points without the need for numerical integration to determine expansion coefficients. This significantly reduces computational overhead while preserving spectral accuracy. The presented method achieves high accuracy and efficiency by incorporating orthogonality and cardinality into the collocation framework, while directly addressing the NBCs present in fractional models. This method is a new addition to the field of numerical analysis of fractional boundary value problems, and it makes fractional calculus more useful for modeling complicated systems.

This article is organized as follows: Section 2 consists of some preliminaries. The mathematical model and the existence and uniqueness of the solution are involved in Section 3. Section 4 introduces OCFs and their key properties, along with matrix representations of fractional integral and derivative operators. Section 5 details the numerical scheme for solving fractional FDEs with NBCs using the collocation approach. Section 6 includes an a priori error estimate analysis. Section 7 presents numerical experiments and results. Finally, Section 8 provides concluding remarks and summarizes the study's key findings.

2 Preliminaries

We begin by introducing the necessary notation and defining the function spaces, complete with their associated norms, that will be utilized for the remainder of this work. Unless stated otherwise, C is a generic positive constant used consistently in this work. $C[a,b]$ is the Banach space of continuous functions endowed with norm $\|u\|_\infty = \sup\{|u(s)|, a \leq s \leq b\}$. The inner product and norm of $L^q[a,b]$ are stated by

$$(u,v)_q = \int_a^b u(s)v(s)ds, \qquad \|u\|^2_{L^q} = (u,u)^2_q, \qquad q \geq 1.$$

For a nonnegative integer $n \in \mathbb{N}$, the Sobolev space $\mathrm{H}^n([a,b])$ is defined as

$$\mathrm{H}^n([a,b]) = \left\{u \in L^2([a,b]) : \mathscr{D}^r u \in L^2([a,b]), \mathbb{N} \ni r \leq n\right\}.$$

This space is endowed with the norm and semi-norm

$$\|u\|^2_{H^n([a,b])} = \sum_{i=0}^{n} \|u^{(i)}(s)\|^2_{L^2([a,b])}, \quad |u|^2_{H^{n,N}([a,b])} = \sum_{i=\min\{n,N\}}^{N} \|u^{(i)}(s)\|^2_{L^2([a,b])}.$$

For $p \in \mathbb{N}$ and real number $\mu < 1$, let us denote by $C^{p,\mu}[0,b]$ the class of functions $u : [0,b] \to \mathbb{R}$ that are continuous on $[0,b]$, possess p continuous derivative on $(0,b]$ and satisfy the estimate (see, e.g. [10])

$$|u^{(j)}| \leq C/\omega_{j-1+\mu}, \qquad j = 1, \dots, p,$$

where $C = C(u)$ is a positive constant, and

$$\omega_{j-1+\mu} := \begin{cases} 1, & j < 1-\mu, \\ 1+|\log(s)|, & j = 1-\mu, \\ s^{1-\mu-j}, & j > 1-\mu. \end{cases}$$

We say a function belongs to $C^{p,\mu}[0,b]$ whenever $u \in C[0,b] \cap C^p(0,b]$ and

$$\|u\|_{p,\mu} := \sum_{j=1}^{p} \sup_{s\in(0,b]} \omega_{j-1+\mu}(s)|u^{(j)}(s)| < \infty,$$

endowed with the norm

$$\|u\|_{C^{p,\mu}(0,b]} = \|u\|_\infty + \|u\|_{p,\mu}.$$

For real number η, let υ be an integer, such that $\upsilon - 1 \leq \eta \leq \upsilon$. The Caputo fractional derivative (CFD) of order η are defined as [25]

$${}_0^C\mathscr{D}_s^\eta(u)(s) = \frac{1}{\Gamma(\upsilon-\eta)} \int_0^s (s-x)^{\upsilon-\eta-1} u^{(\upsilon)}(x)dx = \mathscr{I}_0^{\upsilon-\eta} \mathscr{D}_s^\upsilon(u)(s), \tag{1}$$

where $\mathscr{D}_s = \frac{d}{ds}$, and $\mathscr{I}_0^\eta$ denotes the fractional integral operator (FIO)

$$\mathscr{I}_0^\eta(u)(s) = \frac{1}{\Gamma(\eta)} \int_0^s (s-x)^{\eta-1} u(x)dx. \tag{2}$$

It is well known (see, e.g. [32]) that

$$\mathscr{I}_0^\eta {}_0^C\mathscr{D}_s^\eta(u)(s) = u(s) - \sum_{i=0}^{\upsilon-1} \frac{u^{(i)}(0)}{i!} x^i. \tag{3}$$

3 The mathematical model

The FDE we consider in this work reads

$$ {}_0^C\mathscr{D}_s^{\eta}(u)(s)+g(s)u(s)=f(s), \qquad 0\le s\le b,\ 0<\eta\le 1, \tag{4}$$

with non-local boundary values

$$ \nu_0 u(0)+\nu_1 u(b_1)+\nu_2\int_0^{b_2} u(x)dx+\nu_3\int_0^{b_3} {}_0^C\mathscr{D}_x^{\beta}(u)(x)dx=\nu_4. \tag{5}$$

Here $\nu_i\in\mathbb{R}$ for $i=1:4$, $b_1,b_2,b_3\in(0,b]$, $0<\beta\le\eta\le 1$ and ${}_0^C\mathscr{D}_s^{\eta}$ indicates the CFD (1) with respect to s. We assume that $g,f\in C[0,b]$.

Existence and regularity of the solution to (4)-(5) are established by the following theorem [48].

Theorem 1. *([48]) (i) Let $0<\beta\le\eta\le 1$ and $b_1,b_2,b_3\in(0,b]$. Further assume that $g,f\in C[0,b]$ and $\nu_0+\nu_1+\nu_2 b_2\le 0$. Suppose that the homogeneous form of the equation (1)-(2), obtained by $f=\nu_4=0$, has only trivial solution $u=0$. Then, there is a unique solution $u\in C[0,b]$ of the equation (4)-(5), and moreover ${}_0^C\mathscr{D}_s^{\eta}(u)\in C[0,b]$.*
(ii) By considering hypothesis (i), assume that $g,f\in C^{p,r}(0,b]$ for some $p\in\mathbb{N}$, $r\in\mathbb{R}$, $r<1$.
Then, the problem (4)-(5) has a unique solution $u\in C^{p,\mu}(0,b]$, and ${}_0^C\mathscr{D}_s^{\eta}(u)\in C^{p,\mu}(0,b]$, in which $\mu:=\max\{1-\eta,r\}$.

Converting a differential equation into an equivalent integral equation is a common technique. The corresponding integral equation of (4) can be obtained using (3) as follows:

$$ u(s)-u_0+\mathscr{I}_0^{\eta}(gu)(s)=\mathscr{I}_0^{\eta}(f)(s), \tag{6}$$

where $u_0=u(0)$ is unknown.

4 Orthogonal cardinal functions

Given $N\in\mathbb{N}$, let $\{\psi_m\}_{m=0}^{N}$ denote a family of orthogonal polynomials on $[a,b]$, with roots $\{s_{m,j}\}_{j=0}^{m-1}$. Note that these roots are guaranteed to be real, simple, and located within the interval $[a,b]$ [45]. There are two common structures for introducing the cardinal functions. The first is based on the Jacobi grid, namely the roots of ψ_{N+1}. The second uses the extrema of ψ_N, together with the endpoints a and b, these points are known as the Lobatto nodes.

- The cardinal functions associated with the Jacobi grid are denoted as,

$$ c_m(s)=\frac{\psi_{N+1}(s)}{\mathscr{D}_s(\psi_{N+1}(s_{N+1,m}))(s-s_{N+1,m})}, \qquad m=0,1,\dots,N, \quad s\in[a,b]. \tag{7}$$

- For Lobatto nodes, these functions are stated by:

$$ c_m(s)=\frac{(1-s^2)\mathscr{D}_s(\psi_N(s))}{\mathscr{D}_s\left((1-s_{N+1,m}^2)\mathscr{D}_s(\psi_N(s_{N+1,m}))\right)(s-s_{N+1,m})}, \qquad m=0,1,\dots,N, \tag{8}$$

in which

$$ \{s_{N+1,m}\}_{m=0}^{N}:=\{\hat{s}_{N,m}\}_{m=0}^{N-2}\cup\{a,b\}, \quad \text{where} \quad \mathscr{D}_s(\psi_N(\hat{s}_{N,m}))=0.$$

These constructions guarantees the cardinality property $c_m(s_{N+1,\bar{m}}) = \delta_{m,\bar{m}}$, $m, \bar{m} = 0, 1, \ldots, N$, which allows an exact interpolation of any continuous function at the nodal points:

$$u(s) \approx \mathscr{Q}_N(u(s)) = \sum_{m=0}^{N} u(s_{N+1,m}) c_m(s), \tag{9}$$

where $\mathscr{Q}_N : C[a,b] \to \Pi_N[a,b]$ denotes the projection onto the space of polynomials of degree at most N. Note that the cardinality omits the need for numerical quadrature to compute expansion coefficients.

The following lemma provides an error estimate for projection (9).

Lemma 1. *([12]) Let $M \geq 0$. Consider the approximation $\mathscr{Q}_N(u)$ obtained from the formula (9) at the nodes $\{s_{N+1,m}\}_{m=0}^{N}$. Then, the approximation error satisfies*

$$|u - \mathscr{Q}_N(u)|_{L^2([a,b])} \leq CN^{-n} |u|_{H^{n,N}([a,b])},$$

where the constant $C > 0$ is independent of N. Moreover, for any $m \geq 1$, and $1 \leq l \leq n$, the following bound holds:

$$\|u - \mathscr{Q}_N(u)\|_{\mathscr{H}^l([a,b])} \leq CN^{2l-1/2-n} |u|_{H^{n,N}([a,b])}. \tag{10}$$

Remark 1. *It is worth mentioning that, depending on the choice of orthogonal polynomials, different families of cardinal bases can be obtained. In this work, we use the Chebyshev and Legendre polynomials. Therefore, two types of cardinal functions, specifically Chebyshev cardinal functions (CCFs) and Legendre cardinal functions (LCFs), are constructed and utilized.*

Remark 2. *Because we will need Jacobi polynomials, we will briefly introduce these functions here. The Jacobi polynomials of degree m with indices $\alpha, \beta > -1$, specified by $J_m^{\alpha,\beta}$, are determined by*

$$J_m^{(\alpha,\beta)}(s) = \frac{1}{2^m} \sum_{j=1}^{m} \binom{m+\alpha}{j} \binom{m+\beta}{m-j} (s-1)^j (s+1)^{m-j}.$$

Note that the Jacobi polynomials with $\alpha = \beta = 0$ and $\alpha = \beta = -1/2$ correspond to the Legendre and Chebyshev polynomials, respectively.

4.1 Matrix form of the derivative operator

In this section, we develop a matrix representation for the derivative operator using CCFs and LCFs. This representation is motivated by the structure of orthogonal polynomials. As established in the literature (see, e.g., [45]), the leading coefficient of $\psi_{N+1}(s)$ is given by

$$\kappa_N = \begin{cases} \frac{\Gamma(2N+1)}{(b-a)^N \Gamma(N+1)^2}, & \text{for LCFs}, \\ \frac{\Gamma(2N)}{N!(b-a)^N \Gamma(N)}, & \text{for CCFs}. \end{cases} \tag{11}$$

Differentiating $J_{N+1}^{*}(s)$ (Shifted Jacobi polynomials) k-times yields

$$\mathscr{D}_s^k (J_{N+1}^{*(\alpha,\beta)})(s) = \rho_{N+1,k} J_{N+1-k}^{*(\alpha+k,\beta+k)}(s), \quad k \neq 1, \tag{12}$$

where

$$\rho_{N+1,k} = \begin{cases} \dfrac{\Gamma(N+k+1)}{2^{k-1}(b-a)^k\Gamma(N+1)}, & \text{for LCFs,} \\ \dfrac{\Gamma(N+k-1)}{2^{k-1}(b-a)^k\Gamma(N-1)}, & \text{for CCFs.} \end{cases} \tag{13}$$

Equations (11) and (13) provide an alternate formulation for the OCFs in (7) and (8):

$$c_m(s) = \sigma \prod_{j=0,\ j\neq m}^{N} (s - s_j), \qquad m = 0,1,\ldots,N, \quad s \in [a,b], \tag{14}$$

where the constant σ is defined differently depending on the definition of cardinal functions:

$$\sigma = \frac{\kappa_N^{\alpha,\beta}}{\mathscr{D}_s(J_{N+1,s}^{(\alpha,\beta)}(s))|_{s_{N+1,m}}}, \qquad \text{for equation (7)}, \tag{15}$$

and

$$\sigma = \frac{-4\rho_{N+1,1}\kappa_{N-2}}{(b-a)^2\mathscr{D}_s\left((1-s^2)\mathscr{D}_s(J_M^{(\alpha,\beta)}(s))\right)|_{s_{N+1,m}}}. \tag{16}$$

Differentiating both sides of equation (14) gives

$$\mathscr{D}_s(c_m)(s) = \sigma \sum_{\substack{l=0\\ l\neq m}}^{N} \prod_{\substack{j=0\\ j\neq m,l}}^{N} (s - s_{N+1,j}) = \sum_{\substack{l=0\\ l\neq m}}^{N} \frac{c_m(s)}{s - s_{N+1,l}}, \quad m = 0,1,\ldots,N. \tag{17}$$

The function $\mathscr{D}_s(c_m)(s)$ can be approximated as follows

$$\mathscr{D}_s(c_m)(s) \approx \sum_{m''=0}^{N} \mathscr{D}_s(c_m(s))|_{s=s_{N+1,n''}}\, c_{m''}(s), \quad m = 0,1,\ldots,N. \tag{18}$$

Upon substituting equation (17) into equation (18), the following explicit expression is obtained:

$$\mathscr{D}_s(c_m(s))|_{s=s_{N+1,m''}} = \begin{cases} \displaystyle\sum_{\substack{l=0\\ l\neq m}}^{N} \frac{1}{s_{N+1,m''} - s_{N+1,l}}, & m = m'', \\ \displaystyle\sigma \prod_{\substack{j=0\\ j\neq m,m''}}^{N} (s_{N+1,m''} - s_{N+1,j}), & m \neq m''. \end{cases} \tag{19}$$

We introduce the vector function $\Phi_N(s) \in \mathbb{R}^{N+1}$, whose components are defined as

$$[\Phi_N]_{m+1}(s) = c_m(s), \qquad m = 0,1,\ldots,N. \tag{20}$$

Motivated by this notation, the derivative operator can be approximated in matrix form

$$\mathscr{D}_s(\Phi_N)(s) \approx D\Phi_N(s), \tag{21}$$

where the entries of the operational matrix $D \in \mathbb{R}^{(N+1)\times(N+1)}$ are determined by equation (19).

4.2 Fractional integral operator in the matrix form

To determine the matrix representation of the FIO, an alternative definition of OCFs is required. Motivated by [47], $c_m(s)$ can be written

$$c_m(s) = \sigma \sum_{i=0}^{N} \rho_{m,i} s^{N-i}, \qquad m = 0,1,\ldots,N, \quad s \in [a,b], \tag{22}$$

where the constants $\rho_{m,i}$ are recursively defined by

$$\rho_{m,0} = 1, \quad \rho_{m,i} = \frac{1}{i}\sum_{k=0}^{i} v_{m,k}\rho_{m,i-k}, \quad v_{m,k} = \sum_{\substack{l=0\\ l\neq m}}^{N} (s_{N+1,l})^k, \quad m,i = 0,1,\ldots,N.$$

Thanks to the definition of the FIO $\mathscr{I}_0^{\eta}$ (see [32]), we have

$$\mathscr{I}_0^{\eta}(s^{\vartheta}) = \frac{\Gamma(\vartheta+1)}{\Gamma(\vartheta+\eta+1)} s^{\vartheta+\eta}, \qquad \eta \in \mathbb{R}_+, \quad s \in [a,b]. \tag{23}$$

Applying (23) to the expansion in equation (22) yields

$$\mathscr{I}_0^{\eta}(c_m)(s) = \sigma \mathscr{I}_0^{\eta}\left(\sum_{i=0}^{N} \rho_{m,i} s^{N-i}\right) = \sigma \sum_{i=0}^{N} \rho_{m,i} \frac{\Gamma(N-i+1)}{\Gamma(N-i+\eta+1)} s^{N-i+\eta}, \quad m = 0,1,\ldots,N. \tag{24}$$

To derive an explicit expression for $\mathscr{I}_0^{\eta}(c_m)(s)$, we approximate the FIO in terms of the OCFs as

$$\mathscr{I}_0^{\eta}(c_m)(s) \approx \sum_{m'=0}^{N} \mathscr{I}_0^{\eta}(c_m)(s_{N+1,m'})\, c_{m'}(s), \qquad m = 0,\ldots,N,$$

where the coefficients $\mathscr{I}_0^{\eta}(c_m)(s_{N+1,m'})$ are evaluated according to equation (24).

Now, we introduce the matrix $I_\eta \in \mathbb{R}^{(N+1)\times(N+1)}$, which represents the FIO in the OCFs, i.e.,

$$\mathscr{I}_0^{\eta}(\Phi_N)(s) \approx I_\eta \Phi_N(s), \tag{25}$$

where the entries of I_η are given explicitly by

$$[I_\eta]_{m+1,m'+1} = \mathscr{I}_0^{\eta}(c_m)(s_{N+1,m'}), \qquad m,m' = 0,\ldots,N. \tag{26}$$

This definition enables the proper computation of the action of the FIO on the OCFs, allowing for its use in spectral discretizations of fractional differential equations.

5 Outline of the method

As mentioned above, the goal is to solve equation (4). Instead of this equation, we consider the corresponding integral equation (6). Therefore, the collocation method is used to solve the weakly singular integral equation

$$u(s) - u_0 + \mathscr{I}_0^{\eta}(gu)(s) = \mathscr{I}_0^{\eta}(f)(s), \qquad s \in [0,b], 0 < \eta \leq 1, \tag{27}$$

with $u_0 = u(0)$, and non-local boundary values

$$\nu_0 u(0) + \nu_1 u(b_1) + \nu_2 \int_0^{b_2} u(x)dx + \nu_3 \int_0^{b_3} {}_0^C\mathscr{D}_x^\beta(u)(x)ds = \nu_4, \tag{28}$$

in which $\nu_i \in \mathbb{R}$ for $i = 1:4$, and $b_1, b_2, b_3 \in (0, b]$.

Let the unknown solution u can be approximated as

$$u(s) \approx \mathscr{Q}_N(u(s)) = \sum_{m=0}^{N} u(s_{N+1,m})c_m(s) = u_N(s). \tag{29}$$

In the matrix form, this approximation can be written by

$$u_N(s) = U^T \Phi_N(s), \tag{30}$$

where $U \in \mathbb{R}^{N+1}$. Substituting the approximation (29) into (27) leads to

$$u_N(s) - u_0 + \mathscr{I}_0^\eta(gu_N)(s) = \mathscr{I}_0^\eta(f)(s), \qquad s \in [0,b], 0 < \eta \leq 1. \tag{31}$$

Taking into account the fractional integration in the matrix form I_η, one can introduce the residual

$$r_N = \left(U^T - U_0^T + U^T G I_\eta - F^T I_\eta\right)\Phi_N(s). \tag{32}$$

Here

$$u_0 \approx U_0\Phi_N(s), \quad gu_N(s) \approx U^T G\Phi_N(s), \quad f \approx F^T\Phi_N(s).$$

The proposed technique employs a collocation method, whereby the approximate solution is forced to satisfy the governing equation at a finite number of collocation points. This requires minimizing the residual function r_N at those points. By choosing the nodes $\{s_{N+1,m}\}_{m=0}^N$ as collocation points, one obtain the following system of linear equations:

$$AU = F_0, \quad \text{with} \quad A := I + I_\eta^T G^T, \quad \text{and} \quad F_0 = U_0 + F. \tag{33}$$

This system consists of $N+1$ equations with $N+2$ unknowns. However, we have a non-local condition (28) that can provide us with one more equation. To this end, we substitute u_N instead of u in (28), viz,

$$\nu_0 u_N(0) + \nu_1 u_N(b_1) + \nu_2 \int_0^{b_2} u_N(x)dx + \nu_3 \int_0^{b_3} {}_0^C\mathscr{D}_x^\beta(u_N)(x)ds = \nu_4. \tag{34}$$

Taking into account the matrix form of the integral operator (namely, the matrix form of FIO with $\eta = 1$), and using (1), it results that

$$\nu_0 u_N(0) + \nu_1 u_N(b_1) + \nu_2 U^T I_1 \Phi_N(b_2) + \nu_3 U^T D I_{\upsilon_1-\beta} I_1 \Phi_N(b_3) = \nu_4, \qquad \upsilon_1 - 1 \leq \beta \leq \upsilon_1. \tag{35}$$

The implementation of the proposed method consists of the following steps:

Step 1: Choose the collocation nodes $\{s_{N+1,m}\}_{m=0}^N$ (Jacobi or Lobatto grid).
Step 2: Construct the cardinal basis functions $\{c_m(s)\}_{m=0}^N$ using (7) or (8).
Step 3: Compute the derivative matrix D via (19)-(21).
Step 4: Compute the fractional integral matrix I_η via (24)-(26).
Step 5: Construct the multiplication matrix G such that $gu_N(s) \approx U^T G\Phi_N(s)$.
Step 6: Form the collocation system $AU = F_0$, where $A = I + I_\eta^T G^T$ and $F_0 = U_0 + F$ (see (33)).
Step 7: Incorporate the nonlocal boundary condition (34) to obtain the final square linear system.
Step 8: Solve the resulting linear system for the unknown coefficients U.
Step 9: Return the approximate solution $u_N(s) = U^T\Phi_N(s)$.

6 A priori error estimate

In this section, we perform convergence analysis, providing a priori error estimate in L^2-norm.

Let $e_N = u - u_N$ represent the error. Considering Q_N as the projection operator onto the space $\Pi_N[0,b]$, one can write

$$e = u - u_N = \underbrace{(I - \mathscr{Q}_N)u}_{=:z_1} + \underbrace{\mathscr{Q}_N u - u_N}_{=:z_2}, \tag{36}$$

where z_1 is projection error and $z_2 \in \Pi_N$.

Now, we establish upper bounds for $\|e\|_{H^\gamma}$ and $\|e\|_{L^\infty}$ that will be used in the following Lemma.

Finding a bound for $\|e\|_{H^\gamma}$: It follows from (36) that

$$\|e\|_{H^\gamma} \le \|z_1\|_{H^\gamma} + \|z_2\|_{H^\gamma}, \qquad \gamma = n - \eta, n \in \mathbb{N}, \eta \in (0,1]. \tag{37}$$

Taking into account the spectral approximation estimate [45], since $\gamma = n - \eta < n$, then we have

$$\|z_1\|_{H^\gamma} = \|u - \mathscr{Q}_N u\|_{H^\gamma} \le CN^{\gamma-n}\|u\|_{H^n} \le CN^{-\eta}\|u\|_{H^n}. \tag{38}$$

Because $z_2 \in \Pi_N$, using the inverse estimate [45]

$$\|z_2\|_{H^\gamma} \le C_{inv}N^\gamma \|z_2\|_{L^2}. \tag{39}$$

On the other hand $\|z_2\|_{L^2} = \|\mathscr{Q}_N u - u_N\|_{L^2} \le \|e_N\|_{L^2} + \|z_1\|_{L^2}$. Using Lemma 1 for $\|z_1\|_{L^2}$:

$$\|z_1\|_{L^2} = \|u - \mathscr{Q}u\|_{L^2} \le CN^{-n}|u|_{H^{n,N}} \le CN^{-n}\|u\|_{H^n}. \tag{40}$$

Therefore

$$\|z_2\|_{H^\gamma} \le C_{inv}N^\gamma \left(\|e\|_{L^2} + CN^{-n}\|u\|_{H^n}\right). \tag{41}$$

Substituting (38) and (41) back into (37) leads to

$$\|e\|_{H^\gamma} \le CN^{-\eta}\|u\|_{H^n} + C_{inv}N^\gamma\|e\|_{L^2} + \underbrace{C_{inv}CN^{s-n}\|u\|_{H^n}}_{=CN^{-\eta}\|u\|_{H^n}}, \tag{42}$$

$$\le C\left(N^\gamma\|e\|_{L^2} + N^{-\eta}\|u\|_{H^n}\right). \tag{43}$$

Finding a bound for $\|e\|_{L^\infty}$: Considering (36), it is easy to show that

$$\|e\|_{L^\infty} \le \|z_1\|_{L^\infty} + \|z_2\|_{L^\infty}. \tag{44}$$

It follows from Lemma 1 that

$$\|z_1\|_{H^\gamma} \le CN^{\gamma-n}|u|_{H^{n,N}} \le CN^{-\eta}\|u\|_{H^n}, \qquad \gamma > 1/2,\ \gamma < n. \tag{45}$$

By embedding $H^\gamma \hookrightarrow C$,

$$\|z_1\|_{L^\infty} \le C'\|z_1\|_{H^\gamma} \le CN^{\gamma-n}\|u\|_{H^n} = CN^{-\eta}\|u\|_{H^n}. \tag{46}$$

Using the inverse estimate, one can obtain

$$\|z_2\|_{L^\infty} \le C'_{inv} N^{1/2} \|z_2\|_{L^2} \le C'_{inv} N^{1/2} \left(\|e\|_{L^2} + \|z_1\|_{L^2} \right). \tag{47}$$

Substituting (40) into (47) leads to

$$\|z_2\|_{L^\infty} \le C N^{1/2} \|e\|_{L^2} + C N^{1/2-n} \|u\|_{H^n}. \tag{48}$$

Since $N^{1/2-n} \le C N^{-\eta}$ for $n \gg \eta$, we simplify (48) as follows:

$$\|z_2\|_{L^\infty} \le C \left(N^{1/2} \|e\|_{L^2} + N^{-\eta} \|u\|_{H^n} \right). \tag{49}$$

Inserting (36) and (49) back into (44) yields

$$\|e\|_{L^\infty} \le C \left(N^{1/2} \|e\|_{L^2} + N^{-\eta} \|u\|_{H^n} \right). \tag{50}$$

Lemma 2. *Let $n \ge 1$ satisfies $n > \eta + \frac{1}{2}$, $0 < \eta \le 1$, and set $\gamma = n - \eta > 1/2$. Furthermore, assume that $e = u - u_N$ where u and u_N are the exact and approximate solutions of (27). If $g \in H^\gamma(0,b) \cap L^\infty(0,b)$ and $u \in H^n(0,b)$, then*

$$\|(I - \mathscr{Q}_N) \circ \mathscr{I}_0^\eta (ge)\|_{L^2} \le B N^{-\eta} \|g\|_\star \|e\|_{L^2} + C \|g\|_\dagger N^{-n} \|u\|_{H^n}, \tag{51}$$

where B, C are constants independent of N, and $\|g\|_\star$ and $\|g\|_\dagger$ are finite combination of $\|g\|_{L^\infty}$ and $\|g\|_{H^\gamma}$.

Proof. Thanks to Lemma 1, we can obtain

$$\|(I - \mathscr{Q}_N) \circ \mathscr{I}_0^\eta (ge)\|_{L^2} \le C N^{-n} |\mathscr{I}_0^\eta (ge)|_{H^n}. \tag{52}$$

According to the fractional integral mapping [32], one can write

$$|\mathscr{I}_0^\eta (ge)|_{H^n} \le C_m \|ge\|_{H^\gamma}. \tag{53}$$

Substituting (53) back into (52) yields

$$\|(I - \mathscr{Q}_N) \circ \mathscr{I}_0^\eta (ge)\|_{L^2} \le C_0 N^{-n} \|ge\|_{H^\gamma}, \qquad C_0 = C C_m. \tag{54}$$

Taking into account the Kato-Ponce (fractional Leibniz) inequality [31], we obtain

$$\|ge\|_{H^\gamma} \le C \left(\|g\|_{L^\infty} \|e\|_{H^\gamma} + \|g\|_{H^\gamma} \|e\|_{L^\infty} \right). \tag{55}$$

Inserting (42) and (50) into (55) results that

$$\|ge\|_{H^\gamma} \le C \Big(\|g\|_{L^\infty} \left(N^\gamma \|e\|_{L^2} + N^{-\eta} \|u\|_{H^n} \right) + \|g\|_{H^\gamma} \left(N^{1/2} \|e\|_{L^2} + N^{-\eta} \|u\|_{H^n} \right) \Big) \tag{56}$$

$$= \underbrace{C \left(\|g\|_{L^\infty} N^\gamma + \|g\|_{H^\gamma} N^{1/2} \right)}_{M_1} \|e\|_{L^2} + \underbrace{C \left(\|g\|_{L^\infty} + \|g\|_{H^\gamma} \right)}_{M_2} N^{-\eta} \|u\|_{H^n} \tag{57}$$

$$= M_1 \|e\|_{L^2} + M_2 N^{-\eta} \|u\|_{H^n}. \tag{58}$$

Replacing $\|ge\|_{H^\gamma}$ in equation (54) with (56) leads to

$$\|(I-\mathcal{Q}_N)\circ\mathcal{I}_0^\eta(ge)\|_{L^2} \le C_0N^{-n}\left(M_1\|e\|_{L^2}+M_2N^{-\eta}\|u\|_{H^n}\right) \tag{59}$$

$$= C\left(\left(\|g\|_{L^\infty}N^{-\eta}+\|g\|_{H^\gamma}N^{1/2-n}\right)\|e\|_{L^2}+\|g\|_{\dagger}N^{-n-\eta}\|u\|_{H^n}\right), \tag{60}$$

where $\|g\|_\dagger = \|g\|_{L^\infty}+\|g\|_{H^\gamma}$.

Since $n>\eta+\frac{1}{2}$, we have $1/2-n+\eta<0$, which implies that $N^{1/2-n}\le CN^{-\eta}$ for all sufficiently large N. Thus, we can absorb the $N^{1/2-n}$ into the $N^{-\eta}$ term. Therefore, it follows from (59) that

$$\|(I-\mathcal{Q}_N)\circ\mathcal{I}_0^\eta(ge)\|_{L^2} \le BN^{-\eta}\|g\|_\star\|e\|_{L^2}+C\|g\|_\dagger N^{-n}\|u\|_{H^n},$$

where B, C are constants independent of N. □

Theorem 2. *Let $0<\eta\le 1$, $n\ge 1$ with $n>\eta+\frac{1}{2}$, and set $\gamma=n-\eta>1/2$. Furthermore, assume that $e=u-u_N$ where u and u_N are the exact and approximate solutions of (27). Suppose that $g\in H^\gamma(0,b)\cap L^\infty(0,b)$, $f\in H^\gamma(0,b)$ and $u\in H^n(0,b)$. Define*

$$D := \|f\|_{H^\gamma}+\|g\|_\dagger\|u\|_{H^\gamma}, \quad \|g\|_\dagger := \|g\|_{L^\infty}+\|g\|_{H^\gamma}.$$

Then there exist constants B, C' (independent of N) so that, for all $s\in[0,b]$, if

$$1-BN^{-\eta}\|g\|_\star E_\eta\left(\frac{\|g\|_{L^\infty}}{\Gamma(\eta)}s^\eta\right)>0,$$

the error satisfies

$$\|e(s)\|_{L^2}\le C'N^{-n}(D+\|u\|_{H^n})E_\eta\left(\frac{\|g\|_{L^\infty}}{\Gamma(\eta)}s^\eta\right), \tag{61}$$

where $E_\eta(s)=\sum_{k=0}^\infty \frac{s^k}{\Gamma(\eta k+1)}$ indicates the Mittag-Leffler function.

Proof. Subtracting the collocation equation from (27) leads to

$$e(s)-\mathcal{I}_0^\eta(ge)(s)=(I-\mathcal{Q}_N)\circ\mathcal{I}_0^\eta(f-gu_N)(s). \tag{62}$$

Using $f-gu_N=(f-gu)+ge$, equation (62) can be rewritten as

$$e(s)-\mathcal{I}_0^\eta(ge)(s)=(I-\mathcal{Q}_N)\circ\mathcal{I}_0^\eta(f-gu)(s)+(I-\mathcal{Q}_N)\circ\mathcal{I}_0^\eta(ge)(s). \tag{63}$$

Thanks to the Lemma 1 and using the fractional integral mapping [42], we have

$$\|(I-\mathcal{Q}_N)\circ\mathcal{I}_0^\eta(f-gu)\|_{L^2}\le AN^{-n}\|f-gu\|_{H^\gamma}. \tag{64}$$

Now, considering the Kato-Ponce inequality and the Sobolev embedding $H^\gamma\hookrightarrow L^\infty$ (for $\gamma>1/2$), it results that

$$\|(I-\mathcal{Q}_N)\circ\mathcal{I}_0^\eta(f-gu)\|_{L^2}\le AN^{-n}\left(\|f\|_{H^\gamma}+\|g\|_\dagger\|u\|_{H^\gamma}\right). \tag{65}$$

From the exact error equality (63), one can obtain

$$|e(s)|\le|(I-\mathcal{Q}_N)\circ\mathcal{I}_0^\eta(f-gu)(s)|+|(I-\mathcal{Q}_N)\circ\mathcal{I}_0^\eta(ge)(s)|+\frac{\|g\|_{L^\infty}}{\Gamma(\eta)}\int_0^s(s-x)^{\eta-1}|e(x)|dx. \tag{66}$$

It follows from the fractional Grönwall inequality [5] that

$$|e(s)| \leq \Big(|(I-\mathscr{Q}_N)\circ \mathscr{I}_0^{\eta}(f-gu)(s)| + |(I-\mathscr{Q}_N)\circ \mathscr{I}_0^{\eta}(ge)(s)| \Big) E_\eta\left(\frac{\|g\|_{L^\infty}}{\Gamma(\eta)}s^\eta\right). \tag{67}$$

Taking L^2-norm and using equations (51) and (65), yields

$$\begin{aligned}\|e(s)\|_{L^2} &\leq \Big(\|(I-\mathscr{Q}_N)\circ \mathscr{I}_0^{\eta}(f-gu)(s)\|_{L^2} + \|(I-\mathscr{Q}_N)\circ \mathscr{I}_0^{\eta}(ge)(s)\|_{L^2} \Big) E_\eta\left(\frac{\|g\|_{L^\infty}}{\Gamma(\eta)}s^\eta\right)\\ &\leq \Big(BN^{-\eta}\|g\|_\star\|e\|_{L^2} + C\|g\|_\dagger N^{-n}\|u\|_{H^n} + AN^{-n}\left(\|f\|_{H^\gamma} + \|g\|_\dagger\|u\|_{H^\gamma}\right)\Big) E_\eta\left(\frac{\|g\|_{L^\infty}}{\Gamma(\eta)}s^\eta\right)\\ &\leq \Big(BN^{-\eta}\|g\|_\star\|e\|_{L^2} + C\|g\|_\dagger N^{-n}\|u\|_{H^n} + AN^{-n}D\Big) E_\eta\left(\frac{\|g\|_{L^\infty}}{\Gamma(\eta)}s^\eta\right).\end{aligned}$$

Consequently, we have

$$\left(1 - BN^{-\eta}\|g\|_\star E_\eta\left(\frac{\|g\|_{L^\infty}}{\Gamma(\eta)}s^\eta\right)\right)\|e(s)\|_{L^2} \leq \left(CN^{-n}\|u\|_{H^n} + AN^{-n}D\right) E_\eta\left(\frac{\|g\|_{L^\infty}}{\Gamma(\eta)}s^\eta\right).$$

If $1 - BN^{-\eta}\|g\|_\star E_\eta\left(\frac{\|g\|_{L^\infty}}{\Gamma(\eta)}s^\eta\right) > 0$, then

$$\|e(s)\|_{L^2} \leq \frac{CN^{-n}\|u\|_{H^n} + AN^{-n}D}{1 - BN^{-\eta}\|g\|_\star E_\eta\left(\frac{\|g\|_{L^\infty}}{\Gamma(\eta)}s^\eta\right)} E_\eta\left(\frac{\|g\|_{L^\infty}}{\Gamma(\eta)}s^\eta\right). \tag{68}$$

For large N the denominator is close to 1 and we obtain the asymptotic spectral rate with the Mittag-Leffler function

$$\|e(s)\|_{L^2} \leq C'N^{-n}(D + \|u\|_{H^n}) E_\eta\left(\frac{\|g\|_{L^\infty}}{\Gamma(\eta)}s^\eta\right). \tag{69}$$

□

7 Numerical results

To illustrate the method's effectiveness, two numerical experiments are involved. To this end, we often employ the L^2-error or the absolute error as measures of accuracy.

All computations were performed using Maple 2022 and MATLAB R2022a on a personal computer equipped with an Intel Core i7 processor and 16GB RAM.

Example 1. The first example is dedicated to FDE [48]

$${}_0^C\mathscr{D}_s^{\frac{1}{3}}(u)(s) + s^{\frac{1}{3}}u(s) = \frac{5\Gamma(\frac{29}{12})}{8\Gamma(\frac{29}{12}) + 5\Gamma(\frac{5}{3})}\left(1 + 2\frac{\Gamma(\frac{2}{3})}{\Gamma(\frac{1}{3})}s^{\frac{1}{3}}\right), \qquad 0 \leq s \leq 1,\ 0 < \eta \leq 1,$$

with non-local boundary values

$$u(0) + u(1) + \int_0^1 u(x)dx + \int_0^1 {}_0^C\mathscr{D}_x^{\frac{1}{4}}(u)(x)ds = 1.$$

The exact solution for this equation is $u(s) = \frac{5\Gamma(\frac{29}{12})}{8\Gamma(\frac{29}{12}) + 5\Gamma(\frac{5}{3})}s^{\frac{2}{3}}$.

To verify that the method is convergent, Tables 1 and 2 are tabulated. These tables are generated using the Chebyshev and Legendre polynomials, along with the Jacobi and Lobatto grids. The tables also provide the corresponding computation times. As expected, the approximation error decreases with an increasing number of basis functions N, indicating the method's improved accuracy. Table 3 compares the proposed scheme with the method presented in [48]. The experimental outcomes indicate that the introduced method yields a higher degree of accuracy than [48]. Figures 1 and 2 support our analysis in the preceding section and demonstrate the method's convergence.

Table 1: The L^∞ - errors evaluated using Chebyshev polynomials with different N, for Example 1

	Jacobi Grid			Lobatto Grid		
$s\backslash N$	10	20	30	10	20	30
0.1	7.54e-4	1.80e-4	9.23e-5	2.14e-3	4.45e-4	2.15e-4
0.2	2.53e-4	1.33e-4	7.99e-5	8.42e-4	4.32e-8	9.46e-5
0.3	2.90e-4	1.10e-4	7.10e-5	6.63e-4	2.90e-4	1.34e-4
0.4	4.29e-4	1.27e-4	6.69e-5	1.05e-3	2.88e-4	1.13e-4
0.5	2.81e-4	1.09e-4	6.42e-5	3.89e-4	1.48e-4	7.74e-5
0.6	2.43e-4	1.01e-4	6.16e-5	6.72e-4	2.92e-5	4.19e-5
0.7	3.29e-4	1.08e-4	5.88e-5	1.17e-4	1.39e-4	3.23e-5
0.8	2.66e-4	9.37e-5	5.51e-5	6.22e-4	1.41e-4	5.91e-5
0.9	2.39e-4	8.57e-5	4.96e-5	3.06e-4	8.16e-5	4.96e-6
1.0	1.81e-4	7.63e-5	4.59e-5	7.76e-5	1.12e-5	3.92e-6
CPU time	0.078	0.344	0.406	0.078	0.187	0.532

Table 2: The L^∞ - errors evaluated using Legendre polynomials with different N for Example 1

	Jacobi Grid			Lobatto Grid		
$s\backslash N$	10	20	30	10	20	30
0.1	6.82e-4	2.21e-4	1.24e-4	1.01e-3	1.20e-4	6.92e-5
0.2	4.33e-4	1.83e-4	1.09e-4	1.91e-4	4.30e-5	1.49e-5
0.3	3.79e-4	1.63e-4	9.87e-5	3.35e-4	8.11e-5	3.31e-5
0.4	4.22e-4	1.59e-4	9.17e-5	6.79e-4	1.37e-4	2.61e-5
0.5	3.35e-4	1.45e-4	8.63e-5	3.84e-4	2.74e-5	1.59e-5
0.6	3.12e-4	1.36e-4	8.17e-5	1.64e-4	3.24e-5	7.38e-6
0.7	3.34e-4	1.33e-4	7.77e-5	3.14e-4	9.43e-5	6.70e-6
0.8	2.85e-4	1.25e-4	7.38e-5	4.93e-4	1.98e-5	1.46e-5
0.9	2.84e-4	1.18e-4	7.01e-5	3.19e-5	2.14e-5	1.93e-5
1.0	2.02e-4	9.78e-5	6.18e-5	1.72e-4	2.83e-5	1.11e-5
CPU time	0.063	0.250	0.531	0.046	0.359	0.672

Example 2. The second example is dedicated to FDE

$$ {}_0^C\mathscr{D}_s^\eta(u)(s) + (s^{\frac{3}{2}} + s - 1)u(s) = \frac{\Gamma(\eta+3)\,s^2}{2} + s^{\eta+2}\left(s^{\frac{3}{2}} + s - 1\right), \qquad 0 \le s \le 1,\ 0 < \eta \le 1, $$

Table 3: Error comparison of Example 1

N	8	16	32
Chebyshev polynomials with Jacobi grid	2.40e-3	8.00e-4	2.15e-4
Chebyshev polynomials with Lobatto grid	9.43e-4	2.06e-4	6.57e-5
Chebyshev polynomials with Jacobi grid	1.77e-3	5.00e-4	6.92e-5
Chebyshev polynomials with Lobatto grid	8.54e-4	2.61e-4	1.16e-4
[48] Method 1 with Gauss collocation points	1.83e-2	1.16e-2	7.33e-3
[48] Method 2 with Gauss collocation points	5.60e-3	3.65e-3	2.35e-3

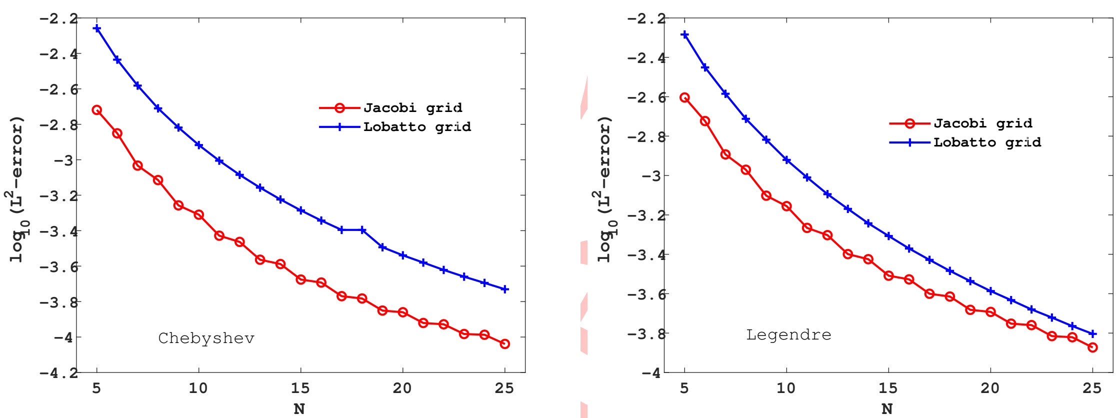

Figure 1: The L^2 - errors obtained from using Chebyshev (left) and Legendre (right) for Example 1

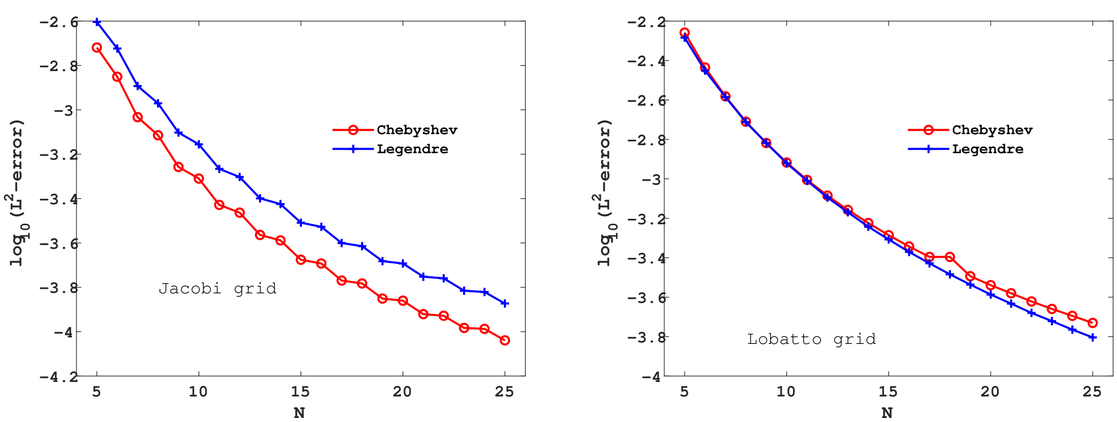

Figure 2: The L^2 - errors obtained from using Jacobi grid (left) and Lobatto grid (right) for Example 1

with non-local boundary values

$$u(0)+u(0.5)+\int_0^{0.3} u(s)ds+\int_0^{0.9} {}_0^C\mathscr{D}_s^{\frac{1}{2}}(u)(s)ds = 0.5^{\eta+2}+\frac{0.3^{\eta+3}}{\eta+3}+\frac{3^{2\eta+5}2^{-\eta-1.5}5^{-2.5-\eta}\Gamma(\eta+3)}{(2\eta+5)\Gamma(\eta+2.5)}.$$

The exact solution for this equation is $u(s)=s^{\eta+2}$.

Tables 4 and 5 are generated using the Chebyshev and Legendre polynomials, along with the Jacobi and Lobatto grids. The tables also provide the corresponding computation times. As expected, the approximation error decreases with an increasing number of basis functions N, indicating the method's improved accuracy. Figures 3 and 4 support our analysis in the preceding section and demonstrate the method's convergence. We know that if the function $u(s)$ has a first-order continuous derivative, then [16]

$$\lim_{\eta\to 1} {}_0^C\mathscr{D}_s^{\eta}(u(s)) = u'(s).$$

Figure 5 shows the effectiveness of the method in solving problems involving fractional derivatives. This figure demonstrates that as $\eta \to 1$, the solution also converges to the solution with $\eta = 1$.

Table 4: The L^∞ - errors evaluated using Chebyshev polynomials with different N, taking $\eta = 0.8$ for Example 2

	Jacobi Grid			Lobatto Grid		
$s\backslash N$	10	20	30	10	20	30
0.1	1.55e-6	9.75e-9	6.30e-10	1.52e-6	4.65e-9	5.71e-10
0.2	3.72e-7	6.30e-9	1.94e-11	9.93e-7	1.29e-8	4.07e-10
0.3	8.01e-7	2.48e-10	9.19e-11	2.00e-6	2.99e-9	1.42e-10
0.4	6.85e-7	1.29e-10	1.41e-10	1.01e-6	1.23e-8	1.24e-10
0.5	4.26e-8	3.86e-9	1.64e-10	5.00e-7	5.72e-9	1.25e-11
0.6	6.21e-7	3.93e-10	1.78e-10	9.09e-7	9.52e-9	1.04e-10
0.7	6.25e-7	9.17e-10	1.77e-10	1.34e-6	3.17e-9	7.92e-11
0.8	1.19e-7	2.79e-9	1.74e-10	2.59e-8	1.36e-9	9.30e-11
0.9	6.36e-7	2.51e-9	2.64e-10	7.47e-7	1.79e-9	5.95e-11
1.0	5.75e-7	2.30e-9	2.40e-10	4.80e-7	2.54e-9	8.75e-12
CPU time	0.250	0.500	0.907	0.516	0.578	1.016

8 Conclusion

In this paper, we proposed a collocation method based on orthogonal cardinal functions (OCFs) for solving fractional ordinary differential equations with nonlocal boundary conditions. To do this, the desired equation is rendered into a weakly singular integral equation of the Volterra type. Then, the spectral collocation method is used to reduce the obtained Volterra equation into a system of algebraic equations. There are two challenging issues: the first is the existence of the nonlocal boundary condition, and the second is related to the fractional derivative. By combining the interpolation properties of cardinal functions with the spectral accuracy of orthogonal polynomials, the method effectively handles the global constraints inherent in nonlocal problems while maintaining high computational efficiency.

Table 5: The L^∞ - errors evaluated using Legendre polynomials with different N, taking $\eta = 0.8$ for Example 2

	Jacobi Grid			Lobatto Grid		
$s\backslash N$	10	20	30	10	20	30
0.1	1.19e-6	8.98e-9	6.55e-10	1.71e-6	9.07e-9	4.85e-11
0.2	7.62e-8	6.14e-9	3.64e-10	8.67e-7	8.19e-9	3.54e-10
0.3	5.64e-7	3.15e-9	4.01e-10	1.24e-6	3.08e-9	4.34e-11
0.4	5.92e-7	3.55e-9	4.38e-10	7.63e-7	6.16e-9	1.53e-11
0.5	1.82e-7	5.43e-9	4.52e-10	2.99e-7	2.92e-9	5.21e-11
0.6	5.45e-7	3.72e-9	4.49e-10	6.77e-7	4.80e-9	7.56e-11
0.7	4.86e-7	3.58e-9	4.31e-10	8.51e-7	2.38e-9	5.16e-11
0.8	2.06e-7	4.41e-9	4.10e-10	5.22e-8	1.10e-9	3.56e-11
0.9	5.41e-7	4.23e-9	4.25e-10	6.63e-7	2.44e-11	4.04e-11
1.0	9.03e-7	6.31e-9	5.04e-10	3.30e-7	1.37e-9	3.66e-11
CPU time	0.484	0.703	1.203	0.547	0.562	0.875

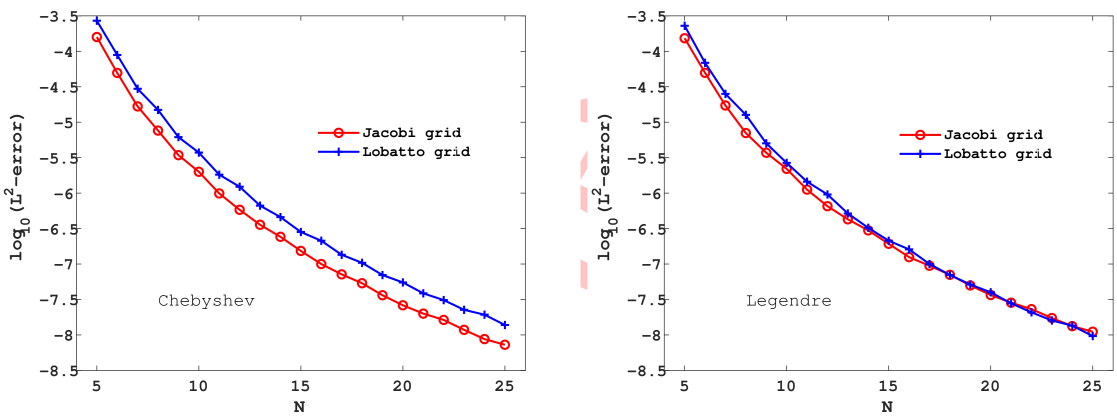

Figure 3: The L^2 - errors obtained from using Chebyshev (left) and Legendre (right), taking $\eta = 0.5$ for Example 2

Theoretical analysis investigates an a priori error estimate, and numerical experiments confirmed the method's spectral convergence and superiority over existing collocation approaches.

The main advantages of the presented approach are its ease of use, accuracy at the spectral level, and natural fit for nonlocal conditions, which often pose challenges for traditional finite difference or finite element methods. However, further research could explore adaptive strategies to enhance efficiency in large-scale problems and extend this framework to nonlinear and variable-order fractional models. Overall, the proposed approach advances the field of numerical methods for fractional calculus and offers a practical way to model and simulate complex systems with memory and nonlocal effects.

In conclusion, the developed scheme exhibits the following capabilities:

- Flexibility: easily applicable to various types of nonlocal boundary conditions (integral, multi-point, fractional constraints);

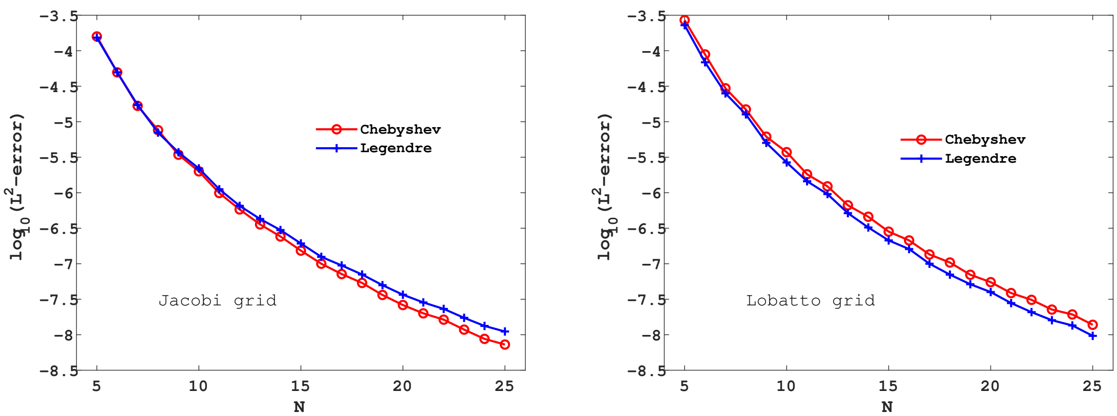

Figure 4: The L^2 - errors obtained from using Jacobi grid (left) and Lobatto grid (right), taking $\eta = 0.5$ for Example 2

- Spectral accuracy: achieves spectral-type accuracy (algebraic order $O(N^{-n})$ convergence rates with increasing basis functions;
- Computational efficiency: reduced computational load due to the cardinality of the bases and by avoiding integration for computing the Fourier coefficients.

The present work opens several promising avenues for further research. We will try to apply the method to real-world models in anomalous diffusion, heat transfer, epidemiology, or population dynamics, where both memory and nonlocality are essential. In subsequent work, it may be possible to develop modifications of the method for variable-order fractional differential equations.

Ethical Statement

The authors declare that this work is original and has not been published or submitted elsewhere. The authors also declare that there are no conflicts of interest and that the manuscript complies with the ethical standards of publication.

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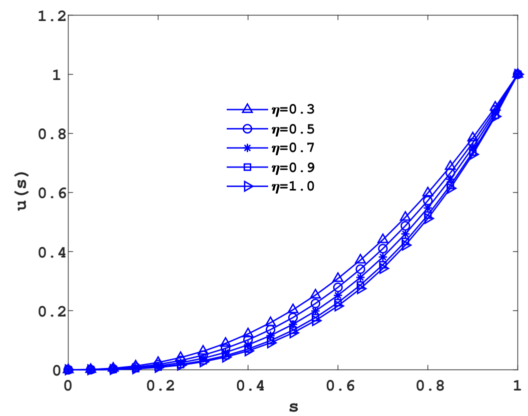

Figure 5: The numerical solution with various choices of η, for Example 2

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