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# Modeling fuzzy time series data using exponential smoothing method

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**Abstract.** This paper proposes a novel approach for modeling and analyzing fuzzy time series data using the exponential smoothing method under imprecise conditions. The proposed model employs weights derived from a decreasing exponential function, parameterized by a smoothing factor $\lambda$, which assigns greater importance to recent data while gradually diminishing the influence of older observations. Performance is evaluated using three complementary criteria: the Mean Similarity Measure (MSM, higher values indicate better agreement between observed and predicted fuzzy sets), the Root Mean Square Error (RMSE, lower values indicate higher predictive accuracy), and the Mean Absolute Percentage Error (MAPE, lower values indicate better percentage accuracy). On a simulated dataset with trend and outliers, the proposed model achieves MSM $= 0.569$, RMSE $= 16.13$, and MAPE $= 21.5\%$; on real ozone concentration data (1980–2019), MSM $= 0.531$, RMSE $= 5.25$ (ppb), and MAPE $= 6.2\%$; and on a software reliability dataset, MSM $= 0.591$, RMSE $= 1.152$, and MAPE $= 4.6\%$. These results significantly outperform the benchmark methods of Hesamian et al. (2022) and Zarei et al. (2020). The proposed method thus demonstrates improved accuracy and robustness for fuzzy time series forecasting.

*Keywords*: Fuzzy observations, exponential smoothing, fuzzy time series, time-dependent data, imprecise data modeling.
*AMS Subject Classification 2010*: 34A34, 65L05.

## 1 Introduction

Time series, which consist of a sequence of observations ordered in time, play a fundamental role in the analysis and modeling of various phenomena in science and engineering. These observations can have economic, social, natural, or technical natures, and their analysis helps us discover hidden patterns and

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make accurate predictions for the future. However, in many real-world problems, we encounter situations where some model components are imprecise quantities. One of the most common such situations is the imprecision of the observations under investigation, which usually occurs due to measurement errors or human mistakes. In such cases, the use of fuzzy set theory, introduced by Zadeh [20], can provide a powerful tool for modeling this type of uncertainty [17, 22]. Fuzzy sets allow us to express imprecise quantities in a more flexible and reasonable manner, which is highly effective in analyzing time series with ambiguous data.

One of the key methods in time series analysis is smoothing techniques. Smoothing helps us identify underlying trends and hidden cycles in data by reducing random fluctuations. This technique was first introduced by Robert Goodell Brown [1] under the name simple exponential smoother. Brown showed that if a time series has no trend, seasonality, or other structural patterns, this approach performs better than methods designed for trending series. Later, Charles C. Holt [9] extended the method, proposing an approach for seasonal data that was subsequently improved by Winters [19]. Gardner and McKenzie [6] revised the simple (linear) exponential smoothing model and the Holt-Winters model so that trends decay over future periods using a damping parameter $\phi$. Pegels [13] examined exponential smoothing methods by considering whether the trend and seasonal components are linear or nonlinear. Since then, numerous researchers such as Gardner [5], Hyndman et al. [10], and Taylor [18] have studied and analyzed exponential smoothing methods. Montgomery and colleagues [11] discussed time series forecasting using exponential smoothing in their book. Neves and Cordeiro [12] proposed a method that first selects the best exponential smoothing model (from a set of possible alternatives) for fitting, and then uses a bootstrap approach for forecasting. Gan et al. [4] developed a fuzzy exponential smoothing model to forecast data and mitigate the risk of aircraft collision. Rostami and Makiyan [15] forecasted Tehran Stock Exchange returns and compared Bayesian, exponential smoothing, and Box-Jenkins approaches. Putri et al. [14] compared double exponential smoothing with fuzzy Markov chain time series for forecasting foreign tourist arrivals.

A considerable number of studies have also examined fuzzy time series modeling. Zarei et al. [21] studied first-order and second-order semiparametric autoregressive models for fuzzy time series and compared them with alternative models using a kernel function. The second-order model is similar to that of Hesamian and Akbari [7], with two differences: (1) a possibilistic error term is included, and (2) a different metric is used for coefficient estimation. Hesamian et al. [8] defined a nonparametric additive fuzzy time series model in which fuzzy smoothing functions are estimated using the fuzzy forward method and kernel-based approaches.

Despite the extensive development of exponential smoothing methods and fuzzy time series models, the intersection of these two areas remains relatively under-explored. Existing fuzzy time series methods, while valuable, present certain limitations. The semiparametric autoregressive approach of Zarei et al. [21] assumes a specific linear autoregressive structure for the center, which may be restrictive when the underlying process exhibits nonlinear trends or abrupt changes. The nonparametric additive model of Hesamian et al. [8] offers greater flexibility but requires solving kernel equations at each time step, entailing substantial computational complexity and sensitivity to bandwidth selection. Moreover, neither approach incorporates a mechanism for automatically diminishing the influence of older observations, which is crucial when the data generating process evolves over time.

To address these limitations, we propose a fuzzy exponential smoothing framework that requires no explicit autoregressive structure for the center, thereby automatically adapting to trends and level shifts. The proposed method involves only a single parameter $\lambda$, which avoids the computational burden associated with kernel-based approaches. Furthermore, it provides a recursive, linear-time algorithm with $O(n)$ complexity, ensuring computational efficiency even for large datasets. The framework incorporates a natural mechanism for discounting past observations through geometrically decreasing weights, allowing it to respond swiftly to recent changes while gradually phasing out outdated information. Moreover, it simultaneously smooths the center and the spreads, ensuring that the estimated fuzziness evolves coherently with the center. This combination of simplicity, computational efficiency, and adaptability to non-stationary behavior distinguishes our approach from existing methods.

The main objective of this paper is to present a new approach for modeling and analyzing fuzzy time series data using the exponential smoothing method. In this model, a decreasing exponential function is considered for weighting observations, such that greater weight is assigned to more recent data, and the weight of observations decreases as we move away from the present time. The performance of this method is evaluated using three complementary criteria: the Mean Similarity Measure (MSM), the Root Mean Square Error (RMSE), and the Mean Absolute Percentage Error (MAPE). The MSM, based on the similarity measure of Chen and Wang [2], evaluates agreement across the entire fuzzy set; RMSE provides a quantitative measure of point prediction accuracy; and MAPE offers a scale-independent percentage error measure. The results are compared with the benchmark methods of Hesamian et al. [8] and Zarei et al. [21].

Unlike the semiparametric autoregressive model of Zarei et al. [21], which assumes a linear structure for the center, our method makes no such assumption and adapts automatically to trends. Unlike the nonparametric model of Hesamian et al. [8], which requires solving kernel equations at each iteration, our method is fully recursive and involves only a single parameter, making it computationally efficient and scalable. These distinctions are important because they highlight that our method is not merely an alternative formulation, but a fundamentally different approach to fuzzy time series modeling that prioritizes simplicity, efficiency, and robustness to non-stationarity.

The overall framework of the paper is as follows: In Section 2, preliminary concepts related to smoothing methods in time series and basic principles of fuzzy sets are presented. In Section 3, fuzzy numbers and the new method for modeling fuzzy time series data with an exponential smoothing approach are introduced, including a rigorous theoretical analysis of its properties. In Section 4, computational details of the proposed method are provided using a simulated example and two real datasets, with performance evaluated using MSM, RMSE, and MAPE. Finally, Section 5 is dedicated to discussion and conclusions.

## 2 Theoretical foundations and basic definitions

In this section, the basic concepts related to smoothing methods in time series and the elementary principles of fuzzy sets, which are essential for understanding the proposed approach, are presented.

### 2.1 Time series

Time series are sets of observations collected over time at equal intervals. In other words, a time series is a sequence of observations ordered by the time parameter. These observations can be represented as $\{Z_t, t \in T\}$, where $T$ is an index set. In essence, a time series is generated by continuous random processes occurring over time, and we observe a representative sample at specific times. Therefore, a time series must possess random components. In this case, each time series observation can be a sample mean from a probability distribution in a specific population at any given point in time. As the number of observations increases, the sample moments approach the population moments, allowing us to estimate the unknown population parameters. The primary objectives of analyzing time series include describing and explaining the data, forecasting the future based on patterns, and controlling the process. All these goals are achievable by identifying a suitable model for the time series.

## 2.2 Smoothing

Every time series has both a random component and a non-random component (noise and signal). If the random component is too large, it can be difficult to discern the trend or pattern of the time series. Smoothing is a method that can be used to better identify the trend and remove the random component or irregularities present in the time series. In other words, smoothing can identify simplified variations in the data and aid in forecasting different patterns. In financial markets, the random component can include minor price corrections and fluctuations that obscure the overall data trend.

Smoothing methods include: linear smoothing, exponential smoothing, seasonal adjustment, Holt-Winters method, and others. A brief description of three methods relevant to this study is provided below.

### A. Linear smoother

This is essentially a weighted average applied to a variable $Z$ over specific time intervals, defined as

$$Z_t = \sum_{j=-k}^{k} \delta_j Z_{t+j}, \quad t = 1,2,\ldots,T, \quad s = 2k+1, \tag{1}$$

where $t$ is the time period, $s$ is the window length, and $\delta_j$ are the weights used to combine $Z_t$ values within the moving window for averaging, such that $\Sigma_{j=-k}^{k}\delta_j = 1$.

### B. Moving average

This is a special case of the linear smoother method, differing only in the values of the weights. In the linear smoother method, the choice of $\delta_j$ is arbitrary. In contrast, the moving average method determines the weights automatically, typically setting them to $\delta_j = \frac{1}{s}$.

### C. Exponential smoother

This is a weighted moving average with non-uniform weights. This method uses past data to predict future states but does not consider all data with equal weight. It assigns greater importance to the latter part of the dataset compared to older data. In other words, new observations are given more weight, and older observations receive less. Since, these weights are distributed exponentially between old and new observations, this method is known as exponential smoothing. Mathematically, the exponential smoothing model can be formulated as

$$\hat{Z}_t = \lambda Z_t + (1-\lambda)\hat{Z}_{t-1}, \quad 0 \le \lambda \le 1. \tag{2}$$

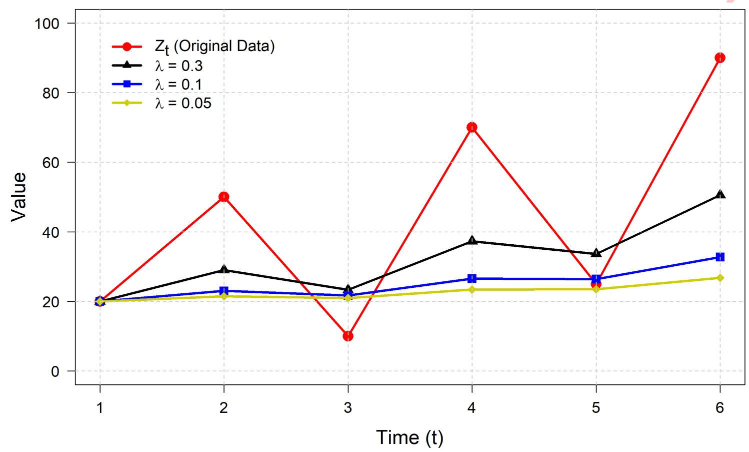


**Figure 1:** Graphical representation of time series and exponential smoothing with two values of $\lambda = 0.3$ and $\lambda = 0.05$

In this relation, $Z_t$ represents the actual value of the series at time $t$, and the predicted value of the series at time $t$ using the exponential smoothing method is denoted by $\hat{Z}_t$. The constant $\lambda$ is called the smoothing factor. As this factor increases, the influence of past observations on the calculation and prediction of future values decreases. Essentially, the factor $\lambda$ determines the rate of influence of past values. Consider a time series based on simulated data, shown in green in Figure 1. To demonstrate the effect of exponential smoothing, this time series has been smoothed using $\lambda = 0.3$ and $\lambda = 0.05$, with the resulting changes shown in blue and orange, respectively. As observed, when the value of $\lambda$ is small, the smoothing effect is more pronounced. Selecting $\lambda = 0$ effectively transforms the smoothing into a moving average.

**Example 1.** Consider a time series with six time periods, $t = 1, 2, \ldots, 6$, with values provided in Table 1. The graph of this time series, based on the provided observations, is shown in Figure 2 and exhibits

**Table 1:** Data for Example 1

| $t$ | 1 | 2 | 3 | 4 | 5 | 6 |
|---|---|---|---|---|---|---|
| $Z_t$ | 20 | 50 | 10 | 70 | 25 | 90 |

considerable fluctuations. To reduce these fluctuations, a linear smoothing method with $s = 3$ is applied in this example. Given the values of $Z$, there are four windows, with $k = 1$ and $j = -1, 0, 1$. As illustrated in Figure 3, for each of these four windows, Equation (1) and Equation (2) can be calculated based on the considered value of $t$. To obtain the linear smoother, we set the values $\delta_{-1} = 0.4, \delta_0 = 0.5$ and $\delta_1 = 0.4$. For the moving average smoother, we use $\delta_j = \frac{1}{S} = \frac{1}{3}$. For the exponential smoother, $\lambda = 0.3$ is chosen. The plot of the time series along with the different smoothers is presented in Figure 4. Based on the results obtained from smoothing, shown in blue in the figure, it can be concluded that the effect of short-term fluctuations has been eliminated, and a clearer trend has been established in the time series under investigation.

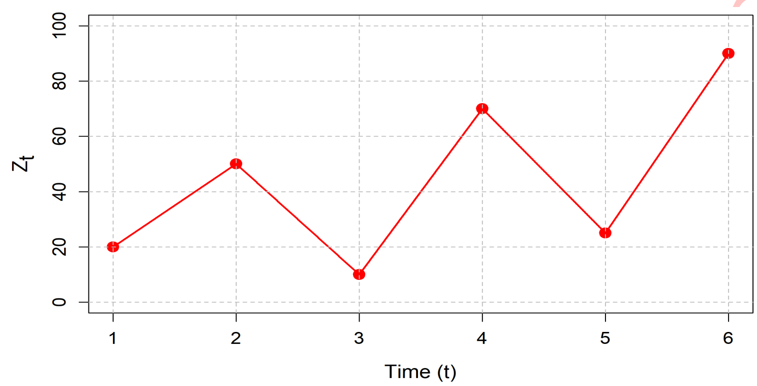


**Figure 2:** Representation of the time series data for Example 1

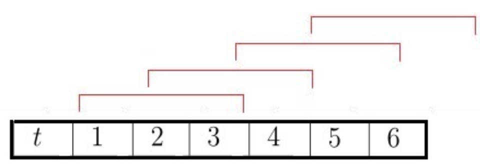


**Figure 3:** Illustration of window count and length

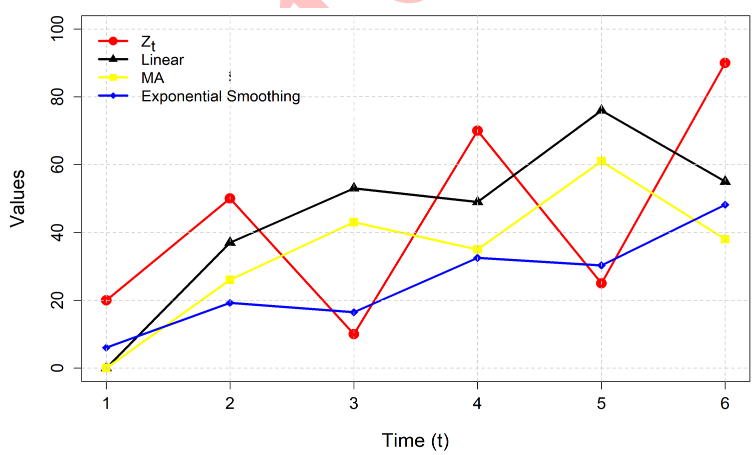


**Figure 4:** Illustration of the time series and various smoothing methods for Example 1

## 2.3 Fuzzy numbers

In this subsection, the basic definitions and concepts related to fuzzy sets and fuzzy numbers, which are essential for understanding the proposed method in this paper, are presented.

**Definition 1.** *A fuzzy set $\tilde{A}$ from a reference set $\mathbb{X}$ is defined as a set of ordered pairs $\tilde{A} = \{(x, \mu_{\tilde{A}}(x)) : x \in \mathbb{X}\}$, where $\mu_{\tilde{A}}(x) : \mathbb{X} \to [0,1]$ is called the membership function of $\tilde{A}$. The membership function* *indicates the degree of membership of each element $x$ in the fuzzy set $\tilde{A}$.*

**Definition 2.** *The alpha-cut of a fuzzy set $\tilde{A}$ is defined as a classical set $\tilde{A}_{[\alpha]} = \{x \in \mathbb{X} : \mu_{\tilde{A}}(x) \geq \alpha\}$ for $\alpha \in [0,1]$. If $\alpha = 0$, then $\tilde{A}_{[0]}$ is the support of the fuzzy set.*

**Definition 3.** *A fuzzy set $\tilde{A}$ is called a fuzzy number if*

1. *It is normal (i.e., $\sup_{x\in\mathbb{X}} \mu_{\tilde{A}}(x) = 1$).*
2. *Its alpha-cuts are closed and bounded intervals.*
3. *Its support (0-cut) is bounded.*

**Definition 4.** *An LR fuzzy number, denoted by $(m;l,r)_{LR}$, where $m$ is the center, $l$ is the left width, and $r$ is the right width of the fuzzy number. Its membership function is defined as follows*

$$\mu_{\tilde{A}}(x) = \begin{cases} L\left(\frac{m-x}{l}\right), & x \leq m, \\ R\left(\frac{x-m}{r}\right), & x \geq m, \end{cases}$$

*where the functions $L$ and $R$ are shape functions and have the following properties:*

1. *$L(x) = L(-x)$ and $R(x) = R(-x)$.*
2. *$L(0) = R(0) = 1$.*
3. *$L(1) = R(1) = 0$.*
4. *$L$ and $R$ are continuous and decreasing functions on $[0,1]$.*

*One of the most common types of LR fuzzy numbers are triangular fuzzy numbers, where $L(x) = R(x) = \max(0, 1-|x|)$. These numbers are represented by three parameters $(m;l,r)$, where $m$ is the center and $l$ and $r$ are the left and right ambiguities, respectively.*

**Definition 5.** *The operations of addition, subtraction, and scalar multiplication for two LR fuzzy numbers $\tilde{A} = (m_1;l_1,r_1)_{LR}$ and $\tilde{B} = (m_2;l_2,r_2)_{LR}$ are defined as follows*

- *Addition: $\tilde{A} \oplus \tilde{B} = (m_1+m_2; l_1+l_2, r_1+r_2)_{LR}$.*
- *Subtraction: $\tilde{A} \ominus \tilde{B} = (m_1-m_2; l_1+r_2, r_1+l_2)_{LR}$.*
- *Scalar multiplication ($k > 0$): $k\tilde{A} = (km_1; kl_1, kr_1)_{LR}$.*

## 3 Smoothing imprecise observations

In this section, we address the modeling of imprecise time series data using the exponential smoothing approach. This method is designed for situations where our observations are presented as fuzzy numbers or intervals of values rather than precise quantities.

### 3.1 Fuzzy time series

Fuzzy set theory, since its introduction by Zadeh [20], has been widely used for modeling dynamic trends in environments with imprecise observations or linguistic values. The concept of fuzzy time series was first introduced by Song and Chissom [16]. In this approach, time series values are defined as fuzzy sets (or fuzzy numbers) instead of precise numbers.

For example, suppose we want to predict air temperature over a week. Instead of providing precise temperatures like 20, 22, 21 degrees Celsius, we can use linguistic fuzzy values such as "warm," "moderate," "slightly cool," each represented by a fuzzy number. These fuzzy numbers better reflect the uncertainty and ambiguity present in measurements or descriptions of phenomena.

A fuzzy time series is a sequence of fuzzy numbers occurring in time order. Each fuzzy observation $\tilde{Z}_t$ can be represented as a fuzzy number (e.g., triangular) with center $Z_t$, left ambiguity $\ell_{Z_t}$, and right ambiguity $r_{Z_t}$. The goal in fuzzy time series analysis is to find patterns in this fuzzy sequence and predict future fuzzy values.

### 3.2 Proposed method

In this subsection we extend the classical exponential smoothing framework to fuzzy time series. The goal is to recursively estimate the underlying fuzzy signal when observations are imprecise and represented as LR fuzzy numbers (in particular, triangular fuzzy numbers). The proposed model preserves the intuitive appeal of exponential smoothing giving more weight to recent observations while respecting the possibilistic structure of the data.

### 3.3 Model specification

Let $\{\tilde{Z}_t\}_{t=1}^{n}$ be a fuzzy time series, where each $\tilde{Z}_t$ is a triangular fuzzy number denoted by $(Z_t;\ell_{Z_t},r_{Z_t})_T$ with center $Z_t$, left spread $\ell_{Z_t} \geq 0$ and right spread $r_{Z_t} \geq 0$.

Define the fuzzy exponential smoothing operator recursively by

$$\tilde{S}_t = \lambda \tilde{Z}_t \oplus (1-\lambda)\tilde{S}_{t-1}, \qquad t = 2,3,\ldots,n$$

with initial condition $\tilde{S}_1 = \tilde{Z}_1$. Here $\lambda \in (0,1]$ is the smoothing parameter, and $\oplus$ denotes the addition of fuzzy numbers as defined in Definition 5.

Using the arithmetic of LR fuzzy numbers, the recursion separates into three independent updates

$$S_t = \lambda Z_t + (1-\lambda)S_{t-1}, \tag{3}$$

$$L_t = \lambda \ell_{Z_t} + (1-\lambda)L_{t-1}, \tag{4}$$

$$R_t = \lambda r_{Z_t} + (1-\lambda)R_{t-1}, \tag{5}$$

where $S_t, L_t, R_t$ denote the estimated center and spreads at time $t$. The smoothed fuzzy value is $\tilde{S}_t = (S_t;L_t,R_t)_T$.

The center update $S_t = \lambda Z_t + (1-\lambda)S_{t-1}$ is indeed a linear recursive weighted average. However, unlike autoregressive models that impose a fixed linear relationship between $S_t$ and a finite set of lagged observations (e.g., $S_t = \phi_1 Z_{t-1} + \phi_2 Z_{t-2} + \ldots$), the exponential smoothing model does not assume a fixed lag structure or constant coefficients over time. The weights adapt naturally as new observations arrive, with the influence of past data decaying exponentially. This allows the method to track trends and level shifts without explicitly modeling them as parametric functions.

### 3.4 Theoretical properties

The following proposition establishes the fundamental characteristics of the estimator.

**Proposition 1.** *For the fuzzy exponential smoothing defined above, the following hold.*
*(1)* ***Exponential weighting.*** *For any* $t \geq 2$,

$$\tilde{S}_t = \lambda \sum_{k=0}^{t-2} (1-\lambda)^k \tilde{Z}_{t-k} \oplus (1-\lambda)^{t-1} \tilde{Z}_1.$$

*Hence, the weights assigned to past observations decrease geometrically with rate* $(1-\lambda)$.
*(2)* ***Bias-variance trade-off and trend adaptation.*** *For a stationary fuzzy process with constant mean center* $\mu$, *constant spread means* $\mu_\ell, \mu_r$, *and finite variances* $\sigma^2, \sigma_\ell^2, \sigma_r^2$, *the one-step-ahead forecast error for the center satisfies*

$$\mathbb{E}[e_t(\lambda)] = 0, \qquad \mathrm{Var}(e_t(\lambda)) = \sigma^2 \frac{\lambda}{2-\lambda},$$

*which is increasing in* $\lambda$. *For large* $t$, *assuming the recursion has reached steady state, a direct calculation for a linear trend* $Z_t = \mu + \beta t$ *gives*

$$\mathbb{E}[S_{t|t-1}] \approx \mu + \beta(t-1) + \frac{1-\lambda}{\lambda}\beta,$$

*so*

$$\mathrm{Bias} = \mathbb{E}[S_{t|t-1}] - \mathbb{E}[Z_t] \approx -\frac{1-\lambda}{\lambda}\beta.$$

*Thus, increasing* $\lambda$ *shrinks the bias, confirming that the method automatically adapts to trends. The mean squared error is therefore*

$$\mathrm{MSE}(\lambda) = \mathrm{Bias}^2(\lambda) + \mathrm{Var}(\lambda) = \beta^2 \left(\frac{1-\lambda}{\lambda}\right)^2 + \sigma^2 \frac{\lambda}{2-\lambda}.$$

*This makes the classical trade-off explicit: larger* $\lambda$ *reduces bias but increases variance.*
*(3)* ***Robustness to outliers.*** *Let an outlier occur at time* $t_0$ *with magnitude* $\delta$. *Its contribution to the forecast for any* $t > t_0$ *is at most*

$$\lambda(1-\lambda)^{t-t_0-1}\delta,$$

*which tends to zero exponentially fast.*
*(4)* ***Preservation of fuzziness.*** *The estimated spreads* $L_t$ *and* $R_t$ *remain non-negative, and* $\tilde{S}_t = (S_t; L_t, R_t)_T$ *is a valid triangular fuzzy number for every* $t$.

*Proof.* For (1), unroll the recursion

$$\begin{aligned}
\tilde{S}_t &= \lambda \tilde{Z}_t \oplus (1-\lambda)\tilde{S}_{t-1} \\
&= \lambda \tilde{Z}_t \oplus \lambda(1-\lambda)\tilde{Z}_{t-1} \oplus (1-\lambda)^2 \tilde{S}_{t-2} \\
&= \lambda \sum_{k=0}^{t-2} (1-\lambda)^k \tilde{Z}_{t-k} \oplus (1-\lambda)^{t-1} \tilde{Z}_1.
\end{aligned}$$

The geometric decay is immediate because $0 < 1-\lambda < 1$.

For (2), define the one-step-ahead forecast error for the center as $e_t(\lambda) = Z_t - S_{t|t-1}$, where $S_{t|t-1} = S_{t-1}$. For a stationary process, $S_{t-1}$ is unbiased for $\mu$, hence

$$\mathbb{E}[e_t(\lambda)] = \lambda\mu + (1-\lambda)\mu - \mu = 0.$$

The variance follows from the standard result for exponential smoothing

$$\mathrm{Var}(e_t(\lambda)) = \sigma^2 \frac{\lambda}{2-\lambda},$$

which is strictly increasing in $\lambda$ on $(0,1]$. In the presence of a linear trend $Z_t = \mu + \beta t$, a direct calculation gives the bias as shown above.

For (3), from the weighted representation in (1), the coefficient of $\tilde{Z}_{t_0}$ for $t > t_0$ is exactly $\lambda(1-\lambda)^{t-t_0-1}$. Therefore its contribution to the center forecast is bounded by that coefficient times the deviation magnitude $\delta$, which vanishes exponentially.

For (4), the spreads satisfy

$$L_t = \lambda \ell_{Z_t} + (1-\lambda)L_{t-1}, \qquad R_t = \lambda r_{Z_t} + (1-\lambda)R_{t-1}.$$

With $\ell_{Z_t} \geq 0$, $r_{Z_t} \geq 0$, $\lambda \in (0,1]$, and non-negative initial spreads, induction yields $L_t \geq 0$ and $R_t \geq 0$ for all $t$. Hence $\tilde{S}_t$ satisfies all axioms of a triangular fuzzy number. □

**Remark 1.** *The mean squared error is not globally convex in $\lambda$ for all processes. However, for a wide class of stationary processes the MSE has a unique minimum; for non-stationary series with a trend it is typically unimodal. This justifies the grid-search procedure used in our numerical experiments.*

### 3.5 Selection of the smoothing parameter

The parameter $\lambda$ controls the trade-off between responsiveness and smoothness. We select $\lambda$ by minimizing a fuzzy cross-validation criterion. For a candidate $\lambda$, compute the one-step-ahead forecast errors for the center and spreads. Crucially, the one-step-ahead forecast at time $t$ is the smoothed level from time $t-1$, i.e., $S_{t|t-1} = S_{t-1}$ and $L_{t|t-1} = L_{t-1}$. This ensures that no information from time $t$ is used to predict the observation at time $t$

$$\mathrm{CV}(\lambda) = \frac{1}{n-1}\sum_{t=2}^{n}\left[(Z_t - S_{t-1})^2 + (\ell_{Z_t} - L_{t-1})^2 + (r_{Z_t} - R_{t-1})^2\right],$$

where $S_{t-1}$, $L_{t-1}$, and $R_{t-1}$ are the smoothed values obtained from the recursion with data up to time $t-1$. The optimal $\lambda^*$ is the minimizer of $\mathrm{CV}(\lambda)$ over a fine grid in $(0,1)$.

#### 3.5.1 Computational algorithm

The recursive nature of the estimator leads to a very efficient linear-time algorithm. The procedure is summarized in the following algorithm.

**Algorithm**: Fuzzy Exponential Smoothing

**Input:** fuzzy time series $\{\tilde{Z}_t = (Z_t; \ell_t, r_t)\}_{t=1}^n$, parameter $\lambda \in (0,1)$.
**Output:** smoothed estimates $\{\tilde{S}_t\}_{t=1}^n$.
Initialize $S_1 \leftarrow Z_1$, $L_1 \leftarrow \ell_1$, $R_1 \leftarrow r_1$.
**for** $t = 2$ to $n$ **do**
  $S_t \leftarrow \lambda Z_t + (1-\lambda)S_{t-1}$
  $L_t \leftarrow \lambda \ell_t + (1-\lambda)L_{t-1}$
  $R_t \leftarrow \lambda r_t + (1-\lambda)R_{t-1}$
  Form $\tilde{S}_t = (S_t; L_t, R_t)_T$
**end for**
$\{\tilde{S}_t\}_{t=1}^n$

### 3.6 Distinction between smoothed values, fitted values, and forecasts

It is essential to distinguish between three distinct quantities in the proposed framework
(a) **The updated/smoothed level** $\tilde{S}_t = \lambda \tilde{Z}_t \oplus (1-\lambda)\tilde{S}_{t-1}$

This is the recursively updated estimate of the underlying level of the process at time $t$, incorporating the current observation $\tilde{Z}_t$. It is not a forecast; it is a filtered estimate based on all data up to and including time $t$.
(b) **The one-step-ahead forecast** $\tilde{Z}_{t|t-1}$

This is the forecast of the fuzzy observation at time $t$ based on data available up to time $t-1$. In the standard exponential smoothing framework, this is simply the previous smoothed level

$$\tilde{Z}_{t|t-1} = \tilde{S}_{t-1}.$$

For the center, this is $Z_{t|t-1} = S_{t-1}$.
(c) **The $h$-step-ahead forecast** $\tilde{Z}_{t+h|t}$:

For forecasting $h$ steps ahead from time $t$, the forecast is the last available smoothed level

$$\tilde{Z}_{t+h|t} = \tilde{S}_t \quad \text{for all } h \geq 1,$$

since the basic model assumes no trend or seasonality. For data with trends, extensions such as Holt's linear method or damped trend formulations would be required.

In the numerical experiments, the fitted values reported are the smoothed levels $\tilde{S}_t$, which serve as one-step-ahead forecasts $\tilde{Z}_{t|t-1} = \tilde{S}_{t-1}$ when evaluated at time $t-1$. This is consistent with standard practice in exponential smoothing literature, where the smoothed level at time $t-1$ is used as the forecast for time $t$.

When evaluating forecast accuracy, we compute errors as $Z_t - Z_{t|t-1} = Z_t - S_{t-1}$, ensuring that no information from time $t$ is used to predict the observation at time $t$. The cross-validation criterion in Subsection 4.1 is explicitly defined using these one-step-ahead forecast errors.

For multi-step forecasting, the model assumes that the level remains constant in the absence of trend; hence $\tilde{Z}_{t+h|t} = \tilde{S}_t$ for all $h \geq 1$. This property should be clearly understood by practitioners applying the method.

### 3.7 Connection to existing fuzzy time series models

It is important to recognize that the proposed exponential smoothing method is conceptually distinct from autoregressive fuzzy time series models. Unlike the models of Hesamian et al. [8] and Zarei et al. [21], which explicitly model the current fuzzy observation as a function of lagged fuzzy observations (i.e., $\tilde{Z}_t = f(\tilde{Z}_{t-1}, \ldots, \tilde{Z}_{t-p})$), our method does not assume any causal or autoregressive structure. Instead, it serves as a filtering technique that recursively estimates the underlying level of the process while discounting past observations.

Despite these conceptual differences, the comparison is meaningful because all three methods address the same practical problem: producing accurate forecasts or smoothed estimates for fuzzy time series data. In the numerical experiments, we evaluate all methods using the same predictive accuracy criteria (MSM, RMSE, MAPE), ensuring a fair comparison of their practical forecasting performance.

Compared to the semiparametric autoregressive model of Zarei et al. [21], our method assumes no fixed linear structure for the center and adapts automatically to trends and level shifts. Compared to the nonparametric model of Hesamian et al. [8], our method is fully recursive, involves only a single parameter, and avoids the computational burden of solving kernel equations at each time step. Moreover, the separate smoothing of spreads guarantees that the estimated fuzziness evolves coherently with the center, a property formally established in Proposition 1. The exponential smoothing approach thus prioritizes computational simplicity, parameter parsimony, and robustness to non-stationarity.

It should be noted that the benchmark methods are autoregressive models that explicitly model the current fuzzy observation as a function of lagged observations, whereas the proposed method is a filtering technique that recursively estimates the underlying level of the process. The comparison is meaningful because all methods address the same practical problem of producing accurate forecasts for fuzzy time series data, but readers should be aware of these conceptual differences when interpreting the results.

## 4 Numerical analysis

In this section, the computational details and numerical results of the proposed fuzzy exponential smoothing model are presented. The model's performance is evaluated using one simulated dataset and two real datasets, and compared with results from the models of Hesamian et al. [8] and Zarei et al. [21].

### 4.1 Evaluation criteria

To evaluate the performance and accuracy of the proposed model, we employ three complementary goodness-of-fit criteria: the MSM, the RMSE, and the MAPE. These criteria were selected for the following reasons.

**The MSM:** In fuzzy time series analysis, the primary object of interest is the entire fuzzy set rather than just its center. The MSM criterion is specifically designed to evaluate how well the predicted fuzzy membership function matches the observed fuzzy membership function. Unlike point-based error measures that only compare centers, MSM criterion captures several important aspects of fuzzy set similarity. It evaluates shape similarity by assessing agreement across the entire support of the fuzzy number, rather than focusing solely on the center. It also accounts for possibilistic uncertainty by incorporating the spread, i.e., the left and right ambiguities, of the fuzzy observations. Moreover, through averaging over multiple alpha-cuts, MSM provides a multi-level comparison that offers a comprehensive assessment of similarity at different possibility levels. This is particularly important in our context because the proposed exponential smoothing model simultaneously estimates the center and the spreads of the fuzzy observations.

This is particularly important in our context because the proposed exponential smoothing model simultaneously estimates the center and the spreads of the fuzzy observations. A high MSM value indicates that the model has successfully captured both the central tendency and the imprecision structure of the data.

For two fuzzy sets $\tilde{A}$ and $\tilde{B}$, the similarity measure is defined as the average over $k$ alpha-cuts

$$\mathrm{MSM}(\tilde{A}, \tilde{B}) = \frac{1}{k} \sum_{i=1}^{k} \mathrm{Sim}\big(\tilde{A}_{[\alpha_i]}, \tilde{B}_{[\alpha_i]}\big),$$

where Sim is the interval similarity function of Chen and Wang [2]. For a fuzzy time series, the overall mean similarity is

$$\overline{\mathrm{MSM}} = \frac{1}{n} \sum_{t=1}^{n} \mathrm{MSM}(\tilde{Z}_t, \hat{\mathscr{Z}}_t),$$

where $\tilde{Z}_t$ is the observed fuzzy value and $\hat{\mathscr{Z}}_t$ is the value predicted by the model. The closer $\overline{\mathrm{MSM}}$ is to one, the better the model's performance.

**The RMSE:** While MSM assesses overall fuzzy set similarity, RMSE provides a complementary quantitative measure of point prediction accuracy. The RMSE is interpretable because it has the same units as the data, making it easy to understand in the context of the original measurements. It is also sensitive to large errors, as the squaring operation penalizes substantial deviations more heavily than smaller ones, which is desirable when large forecasting errors are particularly costly. Moreover, RMSE is comparable to existing literature; both Hesamian et al. [8] and Zarei et al. [21] used RMSE or similar measures in their studies, enabling direct and meaningful comparison of our results with those of benchmark methods.

This index is computed on the centers of the fuzzy numbers, as well as separately on the left and right spreads

$$\mathrm{RMSE} = \sqrt{\frac{1}{n} \sum_{t=1}^{n} (Z_t - \hat{Z}_t)^2}, \qquad \mathrm{RMSE}_\ell = \sqrt{\frac{1}{n} \sum_{t=1}^{n} (\ell_{Z_t} - \hat{\ell}_t)^2}, \qquad \mathrm{RMSE}_r = \sqrt{\frac{1}{n} \sum_{t=1}^{n} (r_{Z_t} - \hat{r}_t)^2}.$$

For summarization, we report the average of these three values:

$$\overline{\mathrm{RMSE}} = \frac{\mathrm{RMSE} + \mathrm{RMSE}_\ell + \mathrm{RMSE}_r}{3}.$$

The smaller the RMSE value, the higher the model's predictive accuracy.

**The MAPE:** To provide an additional scale-independent measure of forecasting accuracy, we also compute the MAPE for the centers of the fuzzy observations

$$\mathrm{MAPE} = \frac{1}{n} \sum_{t=1}^{n} \left| \frac{Z_t - \hat{Z}_t}{Z_t} \right| \times 100\%.$$

This metric is widely used in forecasting because it expresses the error as a percentage of the actual value, making it easily interpretable and comparable across different datasets.

**Choice of similarity measure.** In this study we use the similarity measure defined by Chen and Wang [2], which is appropriate for triangular fuzzy numbers. This measure is symmetric, satisfies the triangle inequality, and yields values in $[0,1]$, making it interpretable and comparable across different datasets.

**Potential additional evaluation measures.** While MSM, RMSE, and MAPE are sufficient for the comparative analysis in this study, we acknowledge that additional metrics could provide further insights. For instance, the **Hausdorff distance** between observed and predicted fuzzy sets,

$$d_H(\tilde{A},\tilde{B}) = \max\left\{\sup_x \inf_y \|x-y\|,\ \sup_y \inf_x \|x-y\|\right\}$$

provides a geometric measure of dissimilarity. Additionally, the **Diebold–Mariano test** [3] could be employed for pairwise comparison of forecast accuracy between competing models to assess whether differences in predictive accuracy are statistically significant. Exploring these additional measures is a promising direction for future work.

**Example 2.** (Simulated data). The initial validation of the proposed exponential smoothing model was conducted using a carefully designed simulated dataset. This dataset, comprising 150 time-indexed observations, was constructed specifically to test the model's resilience against core time-series challenges: an inherent linear trend, substantial fuzzy noise representing inherent measurement imprecision, and the presence of discrete outliers designed to disrupt conventional forecasting methods.

The primary objective was to rigorously assess the model's capacity to accurately decouple and model the underlying systemic behavior from the stochastic and fuzzy components of the time series. Following the model fitting procedure, the optimal smoothing factor $(\lambda)$ was determined to calibrate the model for this specific set of complexities. The resulting performance was then quantified through three critical metrics: MSM, RMSE and MAPE. These figures were benchmarked against the established methodologies of Hesamian et al. [8] and Zarei et al. [21].

The results of comparing the performance of the proposed model with the model of Hesamian et al. [8] and the model of Zarei et al. [21] are reported in Table 2. The interpretation of these results strongly supports the efficacy of our proposed model. The MSM of 0.569 is noticeably higher than the 0.416 and 0.393 obtained by the competing models, indicating a better agreement between predicted and observed fuzzy sets. Moreover, RMSE of 16.131 is the lowest among all three methods, confirming superior predictive accuracy. In terms of MAPE, the proposed model achieved 21.5%, substantially lower than the 30.1% and 32.0% obtained by Hesamian et al. and Zarei et al., respectively. This dual improvement, in both structural similarity and error magnitude, demonstrates the proposed model's ability to handle trend, fuzzy noise, and outliers effectively.

**Table 2:** Comparison of the proposed model with other models based on goodness-of-fit criteria for the simulated dataset

| **Model** | **MSM** | **RMSE** | **MAPE (%)** |
|---|---|---|---|
| Proposed model | 0.569 | 16.131 | 21.500 |
| Hesamian et al. [8] model | 0.416 | 22.610 | 30.100 |
| Zarei et al. [21] model | 0.393 | 23.981 | 32.000 |

**Example 3.** (Ozone concentration data). The analysis was extended to a dataset reflecting annual average ozone concentrations across various U.S. regions from 1980 to 2019, as reported by the U.S. Environmental Protection Agency (EPA). This dataset, characterized by its extensive temporal coverage and real-world environmental significance, provided a critical test case for the model's ability to handle complex, naturally occurring time-series data. The triangular fuzzy data, denoted as $\tilde{Z}_t = (Z_t; 0.97Z_t, 1.02Z_t)_T$, are detailed in Table 3, with $t$ representing the time index from 1 to 40, corresponding to the years 1980–2019. The core objective was to assess the model's performance in capturing long-term trends, seasonal variations, and potential anomalies within environmental monitoring data.

**Table 3:** Annual average ozone concentration data (1980–2019) represented as triangular fuzzy numbers $\tilde{Z}_t = (Z_t; \ell_{Z_t}, r_{Z_t})_T$

| $t$ | Year | $\ell_{Z_t}$ | $Z_t$ | $r_{Z_t}$ | $t$ | Year | $\ell_{Z_t}$ | $Z_t$ | $r_{Z_t}$ |
|---|---|---|---|---|---|---|---|---|---|
| 1 | 1980 | 0.0985 | 0.1015 | 0.1036 | 21 | 2000 | 0.0799 | 0.0824 | 0.0840 |
| 2 | 1981 | 0.0931 | 0.0959 | 0.0979 | 22 | 2001 | 0.0818 | 0.0843 | 0.0859 |
| 3 | 1982 | 0.0916 | 0.0944 | 0.0963 | 23 | 2002 | 0.0859 | 0.0885 | 0.0903 |
| 4 | 1983 | 0.0992 | 0.1022 | 0.1043 | 24 | 2003 | 0.0804 | 0.0829 | 0.0845 |
| 5 | 1984 | 0.0915 | 0.0944 | 0.0963 | 25 | 2004 | 0.0731 | 0.0754 | 0.0769 |
| 6 | 1985 | 0.0907 | 0.0935 | 0.0954 | 26 | 2005 | 0.0781 | 0.0805 | 0.0821 |
| 7 | 1986 | 0.0892 | 0.0919 | 0.0937 | 27 | 2006 | 0.0773 | 0.0797 | 0.0813 |
| 8 | 1987 | 0.0932 | 0.0960 | 0.0979 | 28 | 2007 | 0.0774 | 0.0798 | 0.0814 |
| 9 | 1988 | 0.1022 | 0.1053 | 0.1074 | 29 | 2008 | 0.0732 | 0.0754 | 0.0770 |
| 10 | 1989 | 0.0873 | 0.0901 | 0.0918 | 30 | 2009 | 0.0681 | 0.0702 | 0.0715 |
| 11 | 1990 | 0.0874 | 0.0901 | 0.0920 | 31 | 2010 | 0.0714 | 0.0736 | 0.0751 |
| 12 | 1991 | 0.0880 | 0.0907 | 0.0925 | 32 | 2011 | 0.0725 | 0.0748 | 0.0762 |
| 13 | 1992 | 0.0820 | 0.0846 | 0.0862 | 33 | 2012 | 0.0739 | 0.0762 | 0.0778 |
| 14 | 1993 | 0.0847 | 0.0874 | 0.0891 | 34 | 2013 | 0.0655 | 0.0675 | 0.0688 |
| 15 | 1994 | 0.0845 | 0.0872 | 0.0889 | 35 | 2014 | 0.0662 | 0.0682 | 0.0696 |
| 16 | 1995 | 0.0887 | 0.0914 | 0.0933 | 36 | 2015 | 0.0672 | 0.0693 | 0.0707 |
| 17 | 1996 | 0.0838 | 0.0864 | 0.0881 | 37 | 2016 | 0.0677 | 0.0698 | 0.0712 |
| 18 | 1997 | 0.0831 | 0.0856 | 0.0874 | 38 | 2017 | 0.0670 | 0.0690 | 0.0704 |
| 19 | 1998 | 0.0883 | 0.0910 | 0.0928 | 39 | 2018 | 0.0680 | 0.0701 | 0.0714 |
| 20 | 1999 | 0.0858 | 0.0884 | 0.0902 | 40 | 2019 | 0.0645 | 0.0665 | 0.0678 |

The process for determining the optimal smoothing factor $(\lambda)$ involved an exhaustive search over the interval $\{0.001, 0.002, \ldots, 0.999\}$. The cross-validation criterion $\mathrm{CV}(\lambda)$ was calculated for each $\lambda$ value using the formula

$$\mathrm{CV}(\lambda) = (Z_t - S_{t-1})^2 + (\ell_{Z_t} - L_{t-1})^2 + (r_{Z_t} - R_{t-1})^2.$$

Initial broad scans, visualized in Figure 5, indicated that the optimal $\lambda$ values likely lie within the range of $\{0.005, 0.006, \ldots, 0.2\}$. Further refinement, depicted in Figure 6, narrowed this range to $\{0.005, 0.006, \ldots, 0.02\}$. A final detailed plot, Figure 7, revealed the lowest $\mathrm{CV}(\lambda)$ values within $\{0.005, \ldots, 0.015\}$, pinpointing $\lambda = 0.005$ as the optimal smoothing factor that minimizes $\mathrm{CV}(\lambda)$.

The model's estimation for the center $(Z_t)$, left ambiguity $(\ell_{Z_t})$, and right ambiguity $(r_{Z_t})$ of the fuzzy data, using the determined $\lambda = 0.005$ and following the methodology outlined in Section 3, is visually represented in Figure 8.

To further validate the proposed method on this real-world environmental dataset, we compared its performance with those of Hesamian et al. [8] and Zarei et al. [21]. Table 4 summarizes the results for all

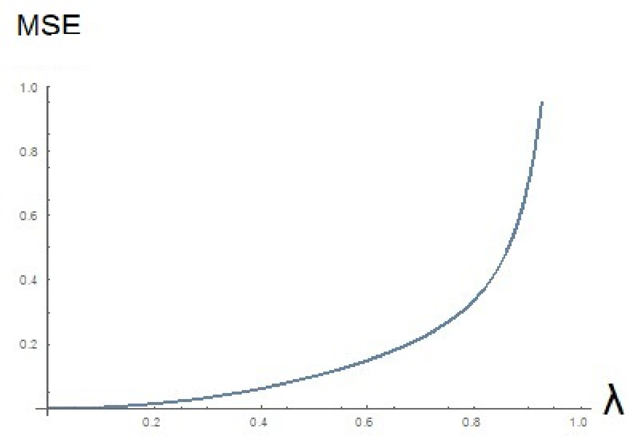


**Figure 5:** Display of optimal $\lambda$ value

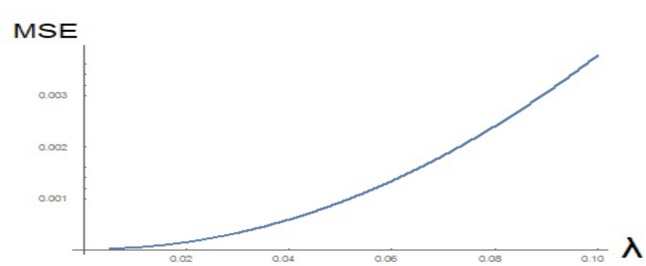


**Figure 6:** Display of optimal $\lambda$ value

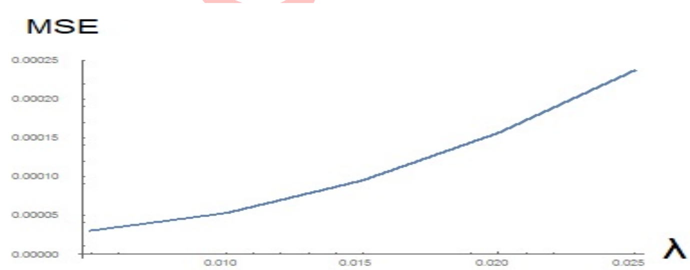


**Figure 7:** Display of optimal $\lambda$ value

three models using MSM, RMSE and MAPE. The proposed model achieved an MSM of 0.531, which is substantially higher than the 0.420 and 0.390 obtained by the competing models, indicating that the predicted fuzzy sets are in better agreement with the observed fuzzy sets. In terms of point forecast accuracy, the proposed model yielded an RMSE of 5.250 (in ppb units), markedly lower than the 7.800 and 8.600 of the benchmark methods. The MAPE, which expresses the error as a percentage of the actual concentration, is only 6.2% for our method, compared with 9.2% and 10.1% for Hesamian et al.

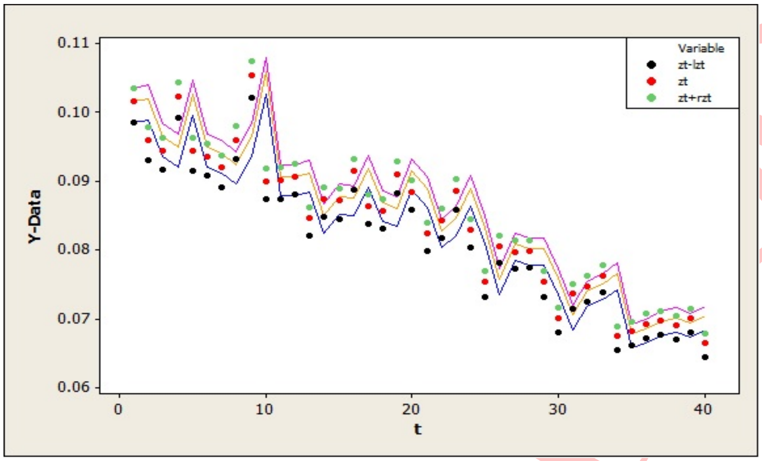


**Figure 8:** Display of fitted fuzzy data

and Zarei et al., respectively. The model's estimation for the center, left ambiguity, and right ambiguity of the fuzzy data, using the determined $\lambda$ and following the methodology outlined in Section 3, is visually represented in Figure 8. The corresponding fitted fuzzy data are further illustrated in Figure 9. These results confirm that the proposed exponential smoothing framework not only captures the overall fuzzy structure more accurately but also provides more precise point predictions for long-term environmental monitoring data.

**Table 4:** Comparison of the proposed model with other models based on goodness-of-fit criteria for the ozone concentration data

| **Model** | **MSM** | **RMSE (ppb)** | **MAPE (%)** |
|---|---|---|---|
| Proposed model | 0.531 | 5.250 | 6.200 |
| Hesamian et al. [8] model | 0.420 | 7.800 | 9.200 |
| Zarei et al. [21] model | 0.390 | 8.600 | 10.100 |

**Example 4.** (Software reliability data). To further validate the proposed method, we applied it to a software reliability dataset. Software reliability data often exhibit complex patterns, including discrete failures, rapid changes, and potential outliers, making them a challenging testbed for forecasting methods. The dataset consists of 30 time-indexed observations, each represented as a triangular fuzzy number $\tilde{\xi}_t = (\xi_t; \ell_{\xi_t}, r_{\xi_t})_T$. The raw data are presented in Table 5.

Prior to fitting the proposed model, an outlier detection procedure was conducted using an $\bar{X}$ control chart with $3\sigma$ control limits, shown in Figure 10. The control limits were set at $\bar{X} \pm 3\sigma$, where $\bar{X}$ is the overall mean and $\sigma$ is the standard deviation of the center values. No observations exceeded these control limits, so all data points were retained in the subsequent analysis. This step ensures that the exponential smoothing method is not unduly influenced by extreme observations, thereby enhancing its robustness.

The original time series data, along with the exponentially smoothed series obtained with the optimal smoothing parameter $\lambda = 0.05$, are displayed in Figure 11. The fitted fuzzy observations, including the estimated centers and spreads, are visualized in Figure 12.

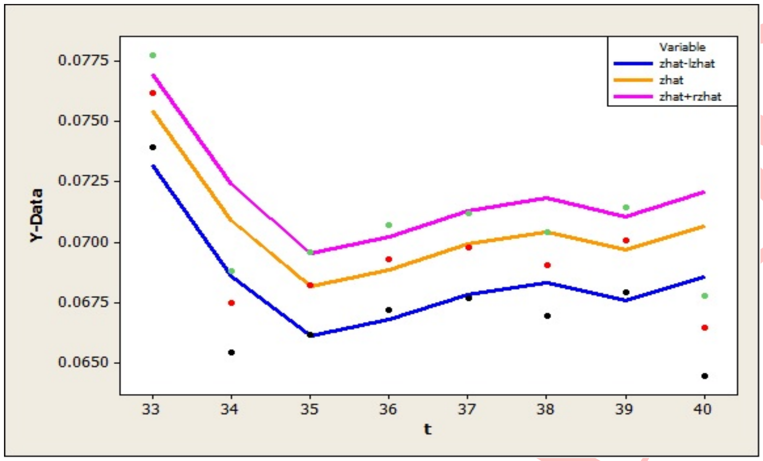


**Figure 9:** Display of fitted fuzzy data

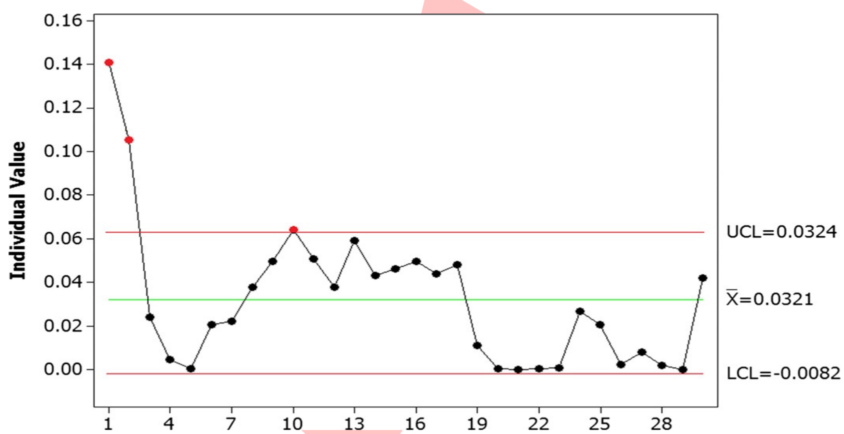


**Figure 10:** $\bar{X}$ chart for detecting outliers

To further validate the proposed method, a comparison with other existing models was performed using goodness-of-fit criteria. Table 6 summarizes these results, highlighting MSM, RMSE, and MAPE for the proposed model, Hesamian et al. [8], and Zarei et al. [21]. The proposed model achieved the highest MSM (0.591), the lowest RMSE (1.152), and the lowest MAPE (4.6%), compared to 28.5% and 31.2% for the competing models. These results confirm the superior forecasting accuracy and robustness of our approach, particularly in capturing both the central tendency and the spread of the fuzzy observations.

Finally, predictions for the center, left ambiguity, and right ambiguity values from $\hat{Z}_{25}$ to $\hat{Z}_{30}$ were generated and are visually represented in Figure 13.

## 5 Summary and conclusion

This study introduces a new approach for modeling time-dependent imprecise data using an exponential smoothing framework specifically adapted for fuzzy observations. The proposed method extends the classical exponential smoothing technique to the fuzzy setting by treating observations as triangular fuzzy

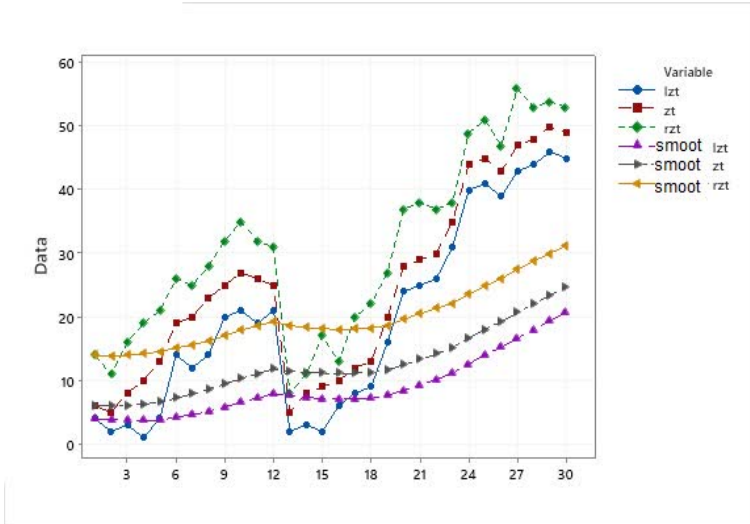


**Figure 11:** Display of original time series data and exponentially smoothed time series with $\lambda = 0.05$ for fuzzy data

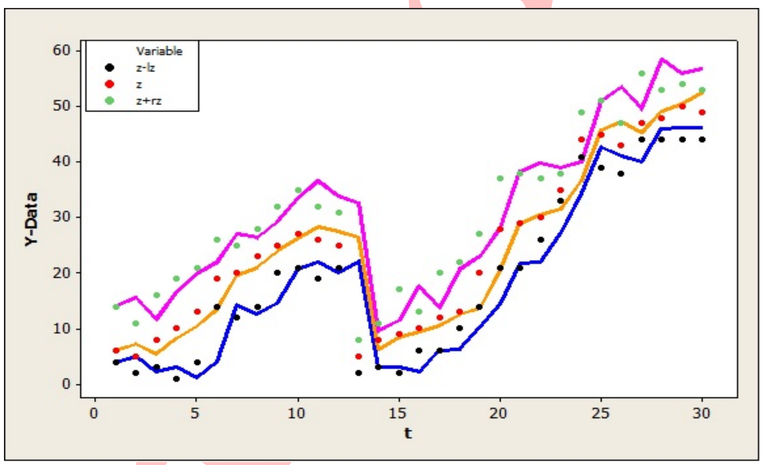


**Figure 12:** Display of fitted fuzzy data

numbers and applying the smoothing recursion separately to the center, left spread, and right spread. By assigning progressively larger weights to more recent data points through a smoothing parameter $\lambda$, the proposed method is capable of capturing short-term dynamics while simultaneously preserving the underlying imprecision structure inherent in the dataset. This characteristic makes it particularly suitable for environments in which the data are noisy, partially unknown, or subject to rapid fluctuations.

The theoretical properties of the estimator were rigorously established in Proposition 1. We demonstrated that: (i) the estimator assigns geometrically decreasing weights to past observations; (ii) for stationary processes, the one-step-ahead forecast error is unbiased with variance $\sigma^2\lambda/(2-\lambda)$, while for non-stationary processes with linear trend, the bias decreases as $\lambda$ increases, confirming the classical bias–variance trade-off; (iii) the effect of outliers decays exponentially fast with rate $(1-\lambda)$; and (iv) the estimated spreads remain non-negative, ensuring that the smoothed estimates are valid triangular fuzzy

**Table 5:** Software reliability data represented as triangular fuzzy numbers $\tilde{\xi}_t = (\xi_t; \ell_{\xi_t}, r_{\xi_t})_T$

| $t$ | $\tilde{\xi}_t$ | $t$ | $\tilde{\xi}_t$ |
|---|---|---|---|
| 1 | (6; 2, 8) | 16 | (19; 5, 7) |
| 2 | (5; 3, 6) | 17 | (20; 8, 5) |
| 3 | (8; 5, 8) | 18 | (23; 9, 5) |
| 4 | (10; 9, 9) | 19 | (25; 5, 7) |
| 5 | (13; 9, 8) | 20 | (27; 6, 8) |
| 6 | (19; 5, 7) | 21 | (29; 8, 9) |
| 7 | (20; 8, 5) | 22 | (30; 4, 7) |
| 8 | (23; 9, 5) | 23 | (35; 2, 3) |
| 9 | (25; 5, 7) | 24 | (44; 3, 5) |
| 10 | (27; 6, 8) | 25 | (45; 6, 6) |
| 11 | (6; 2, 8) | 26 | (43; 5, 4) |
| 12 | (5; 3, 6) | 27 | (47; 3, 9) |
| 13 | (8; 5, 8) | 28 | (48; 4, 5) |
| 14 | (10; 9, 9) | 29 | (50; 6, 4) |
| 15 | (13; 9, 8) | 30 | (49; 5, 4) |

**Table 6:** Comparison of the proposed model with other models based on goodness-of-fit criteria for the software reliability data

| **Model** | **MSM** | **RMSE** | **MAPE (%)** |
|---|---|---|---|
| Proposed model | 0.591 | 1.152 | 4.600 |
| Hesamian et al. [8] model | 0.498 | 14.261 | 28.500 |
| Zarei et al. [21] model | 0.453 | 17.128 | 31.200 |

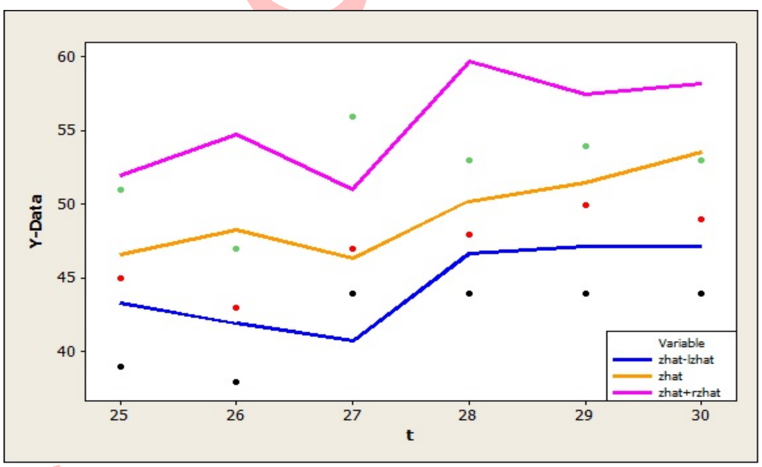


**Figure 13:** Display of predicted fuzzy data

numbers for all time points. These theoretical guarantees provide a solid foundation for the practical application of the proposed method.

A comprehensive performance assessment was conducted using one simulated dataset alongside two real-world time series: annual average ozone concentrations (1980–2019) and software reliability data. Across all experiments, the proposed model consistently demonstrated superior predictive accuracy when compared to the benchmark methods of Hesamian et al. [8] and Zarei et al. [21]. The evaluation employed three complementary criteria: MSM, RMSE and MAPE. The results are summarized as follows

- For the **simulated dataset** with trend and outliers, the proposed model achieved MSM = 0.569, RMSE = 16.131, and MAPE = 21.5%, substantially outperforming the competing models which yielded MSM values of 0.416 and 0.393, RMSE values of 22.610 and 23.981, and MAPE values of 30.1% and 32.0%, respectively.
- For the **ozone concentration data**, the proposed model achieved MSM = 0.531, RMSE = 5.250 (ppb), and MAPE = 6.2%, compared to MSM values of 0.420 and 0.390, RMSE values of 7.800 and 8.600, and MAPE values of 9.2% and 10.1% for the benchmark methods.
- For the **software reliability data**, the proposed model achieved MSM = 0.591, RMSE = 1.152, and MAPE = 4.6%, compared to MSM values of 0.498 and 0.453, RMSE values of 14.261 and 17.128, and MAPE values of 28.5% and 31.2% for Hesamian et al. and Zarei et al., respectively.

These results provide strong evidence that the proposed exponential smoothing-based model maintains stability in the presence of both trend components and anomalous observations conditions that frequently arise in financial, environmental, and engineering data. The high MSM values indicate that the predicted fuzzy sets are in close agreement with the observed fuzzy sets, while the low RMSE and MAPE values confirm the model's superior point forecasting accuracy. Notably, the dramatic improvement in MAPE across all datasets—particularly in the software reliability data where the proposed method achieved a 4.6% error compared to over 28% for the benchmarks underscores the practical utility of the approach for real-world forecasting applications.

A more nuanced comparison with the baseline approaches highlights the specific strengths of the proposed method. The cumulative fuzzy strategy introduced by Hesamian et al. [8] performs adequately when the underlying series lacks significant trends or outliers. However, its computational structure imposes substantial complexity, especially in the calculation of smoothing coefficients in each iteration, which can reduce scalability and practical usability. In contrast, the model of Zarei et al. [21] relies on a second-order semiparametric moving average formulation. Although computationally simpler than the approach by Hesamian et al. [8], it suffers from pronounced sensitivity to outliers and fails to adapt effectively to datasets exhibiting clear temporal trends. The empirical results of this study show that the proposed method avoids these limitations through its simple recursive structure, robust outlier suppression, and automatic adaptation to trends and level shifts via the exponential weighting scheme.

Taken together, the findings provide strong evidence that the proposed model is a promising alternative for forecasting imprecise time series. Its capacity to integrate the advantages of exponential smoothing with a structured treatment of fuzziness supports its use in applications where uncertainty is intrinsic, such as stock markets, commodity pricing, demand forecasting, and complex socio-economic systems. Furthermore, the conceptual simplicity and computational efficiency of the model requiring only $O(n)$ operations enhance its practical implement ability, even for large-scale datasets.

While MSM, RMSE, and MAPE provide a comprehensive evaluation of the proposed model's performance, future studies may benefit from incorporating additional forecast evaluation measures. In particular, the Diebold–Mariano test [3] could be employed to formally assess whether the differences in predictive accuracy between the proposed model and the benchmark methods are statistically significant. Additionally, the Hausdorff distance could provide a geometric perspective on fuzzy set dissimilarity that complements the similarity-based MSM criterion. These extensions would further strengthen the empirical evidence supporting the proposed approach.

Other promising directions for future research include extending the model to multi-parameter or multi-layered smoothing schemes, developing adaptive versions in which smoothing coefficients evolve dynamically based on recent forecast performance, and integrating the approach with contemporary machine-learning-based time-series models such as recurrent neural networks, transformers, or hybrid fuzzy-statistical architectures. Such extensions could significantly improve the method's capability in long-term forecasting and nonlinear pattern detection. Additionally, investigating the theoretical properties of the estimator under more general fuzzy data structures, such as LR fuzzy numbers with different shape functions or non-symmetric spreads, would broaden the applicability of the proposed framework.

In conclusion, the proposed fuzzy exponential smoothing method offers a robust, efficient, and theoretically grounded approach for modeling and forecasting imprecise time series data. The empirical results, supported by rigorous theoretical guarantees, demonstrate its superiority over existing methods across multiple datasets and evaluation criteria, establishing it as a valuable tool for researchers and practitioners working with uncertain temporal data.

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