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The application of copula functions in financial risk management

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Abstract. Modeling dependence structures is essential in modern financial risk management, especially when linear correlation fails to capture nonlinear co-movements. This study presents a case study of the dependence structure between Bitcoin (BTC) and Tether (USDT) over the period May 2024 to May 2025 using a copula-based framework, and the results are specific to the selected asset pair and time period. We consider several copula families, including the Farlie-–Gumbel-–Morgenstern (FGM) copula as a baseline benchmark model, alongside more flexible alternatives. Daily log-returns of BTC and USDT from May 2024 to May 2025 are analyzed. Empirical results confirm that both upper and lower tail dependence coefficients converge to zero. Copula parameters are estimated using maximum likelihood and rank-based methods. Model comparison based on goodness-of-fit criteria indicates that the Frank copula provides the best representation of the dependence structure, while the FGM copula exhibits limitations due to its restricted dependence range. Portfolio risk is then assessed through Value-at-Risk and Conditional Value-at-Risk. The results show that risk estimates are sensitive to the choice of copula model. At the same time, the framework enables generation of synthetic joint distributions for stress testing. Stress scenarios indicate that portfolio losses can significantly increase under adverse shocks.

Keywords: Copula function, FGM copula, weak dependence, tail dependence, financial risk, value-at-risk (VaR), conditional value-at-risk (CVaR), cryptocurrency markets.
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1 Introduction and preliminaries

Modeling dependence structures is a cornerstone of modern risk management. Classical measures such as linear correlation or covariance are inadequate to capture nonlinear and tail-oriented co-movements

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that frequently arise in financial markets [11]. This limitation has motivated the development of copula theory, which separates marginal behavior from dependence. Sklar's theorem established the mathematical foundation [32], and subsequent comprehensive treatments were given in [21, 30].

Several researches have been applied copula methodology [13]. Early constructions such as the Farlie–Gumbel–Morgenstern (FGM) copula [12, 29] provided tractable but limited models of weak dependence, while nonparametric approaches such as the empirical copula were pioneered by [7–9] and later surveyed in [16]. Tail dependence has been recognized as especially important in risk contexts, with formal dependence measures [5, 23, 24] and widely applied in extreme value and systemic risk analysis [3, 19, 31].

Within financial risk management, the need to move beyond historical simulation and correlation-based methods is well documented. Standard approaches such as VaR [15, 22, 25, 33, 34] and expected shortfall [1, 4, 6] remain benchmarks, but they require accurate modeling of joint extremes to be effective. Copula-based approaches provide precisely this flexibility, enabling stress testing, scenario design, and simulation frameworks unavailable to purely historical methods [11, 17].

In recent years, copula methods have been increasingly applied to cryptocurrency markets, which exhibit nonlinear dependence, regime shifts, and episodes of contagion. For instance, systemic risk and co-movements in digital assets have been quantified using copula-based models [14, 28]. However, despite this growing literature, there remains a gap in systematically studying cases where empirical tail dependence is absent, and in evaluating how different copula families perform under such conditions.

The present study addresses this gap by investigating the dependence structure between Bitcoin (BTC) and Tether (USDT) returns over the period May 2024–May 2025. Building on prior copula theory and tail dependence literature, we aim to assess the presence (or absence) of tail dependence, compare alternative copula models in weak-dependence settings, and analyze the implications for portfolio risk measurement and stress testing.

Tail dependence plays a central role in financial risk management because it characterizes the likelihood of joint extreme events and strongly influences dependence modeling and portfolio risk assessment [5, 21]. Since this concept is fundamental to the copula framework adopted in this study, a formal treatment is provided in Subsection 1.2.

The main contribution of this study is to provide a comprehensive copula-based analysis of the dependence structure between BTC and USDT returns under a weak-dependence and zero-tail-dependence setting. Unlike many studies that primarily focus on strongly tail-dependent financial assets, this work investigates the suitability and limitations of different copula families in modeling moderate dependence structures. In particular, the study demonstrates that although the FGM copula provides analytical simplicity and zero tail dependence, it fails to adequately capture the observed dependence level, while the Frank copula offers a more flexible and statistically appropriate representation for risk estimation and stress testing.

1.1 Copula

Let R denote the real line, i.e. $(-\infty, \infty)$. The extended real line is $\overline{R} = [-\infty, +\infty]$, and the extended real plane is $\overline{R}^2 = \overline{R} \times \overline{R}$. The Cartesian product of two closed intervals is a rectangle $B = [x_1, x_2] \times [y_1, y_2]$, whose four vertices are $(x_1, y_1), (x_1, y_2), (x_2, y_1)$, and (x_2, y_2). The product $I \times I$, with $I = [0, 1]$, is the unit square I^2.

Definition 1. *A function $\mathscr{C}$ mapping I^2 to $[0, 1]$ is called a copula if it satisfies the following properties:*

(i) For all x,y in I

$$\mathscr{C}(x,0)=0=\mathscr{C}(0,y) \qquad \textit{and} \qquad \mathscr{C}(x,1)=x,\ \mathscr{C}(1,y)=y. \tag{1}$$

(ii) For all x_1,x_2,y_1,y_2 in I such that $x_1 \le x_2$ and $y_1 \le y_2$,

$$\mathscr{C}(x_2,y_2)-\mathscr{C}(x_2,y_1)-\mathscr{C}(x_1,y_2)+\mathscr{C}(x_1,y_1)\ge 0. \tag{2}$$

If $\mathscr{C}$ is a copula, then $\max(x+y-1,0)\le\mathscr{C}(x,y)\le\min(x,y)$ for any $(x,y)\in \mathrm{Dom}\mathscr{C}$. These are bounds, and they are usually expressed as $M(x,y)=\min(x,y)$ and $w(x,y)=\max(x+y-1,0)$. We now revisit it in the bivariate context, as per the subsequent theorem, with regard to A. Sklar [32], who created copulas.

Theorem 1 (Sklar's Theorem)**.** *Let H be a bivariate distribution function with marginal distribution functions F and G. Then there exists a copula $\mathscr{C}$ such that, for all $x,y\in\overline{R}$,*

$$H(x,y)=\mathscr{C}(F(x),G(y)). \tag{3}$$

If both F and G are continuous, then $\mathscr{C}$ is uniquely determined. Otherwise, it is uniquely determined on $\mathrm{Rang}(F)\times\mathrm{Rang}(G)$.

Conversely, if $\mathscr{C}$ is a copula and F and G are distribution functions, then

$$\mathscr{C}(x,y)=H(F^{(-1)}(x),G^{(-1)}(y)), \tag{4}$$

defines a bivariate distribution function with margins F and G

Proof. See [32]. □

We define the product copula by $\mathscr{C}(x,y)=xy$. To generate observations (x,y) from random variables with a specified joint distribution function H, copulas provide a useful framework. From Theorem 1, one can proceed in a way analogous to the transformation method, starting from independent random variables uniformly distributed on $(0,1)$, we generate a pair whose joint distribution is given by the copula $\mathscr{C}$.

This construction requires the conditional distribution of Y given $X=x$, which we denote by $\mathscr{C}_x(y)$. It is defined as

$$\mathscr{C}_x(y)=P(Y\le y\mid X=x)=\lim_{\Delta x\to 0}\frac{\mathscr{C}(x+\Delta x,y)-\mathscr{C}(x,y)}{\Delta x}=\frac{\partial}{\partial x}\mathscr{C}(x,y).$$

Theorem 2. *The function $\mathscr{C}^{(w)}(u_1,\dots,u_d;\theta)$ is a valid copula if*

1. *$\mathscr{C}^{(w)}$ is grounded and has uniform marginals.*
2. *$\mathscr{C}^{(w)}(u_1,\dots,u_d;\theta)\ge 0$ almost everywhere.*
3. *θ is chosen such that $\mathscr{C}^{(w)}$ is d-increasing over $[0,1]^d$.*

Proof. See [30]. □

To generate random observations from a bivariate copula, the conditional inversion method can be implemented through the following algorithm.

Algorithm 1: Sampling a pair (X,Y) from a bivariate copula $\mathscr{C}$, see [30]

Draw $x \sim \mathrm{Unif}(0,1)$
Draw $t \sim \mathrm{Unif}(0,1)$
Compute $y = C_x^{-1}(t)$
Return $(x,\, y)$

1.2 Tail dependence

Tail dependence measures the likelihood of extreme co-movements between variables, crucial for financial risk where market crashes often amplify joint losses. The upper tail coefficient λ_U captures extreme gains, while the lower tail λ_L focuses on losses; both range in [0,1]. In cryptocurrency markets like BTC—USDT, zero tail dependence ($\lambda_U = \lambda_L = 0$) implies no significant joint extremes, as seen in our empirical analysis. This aligns with the FGM copula, which inherently exhibits tail independence, making it suitable as a baseline model for weak-dependence regimes. Empirically, tail dependence can be estimated via the empirical copula,

$$\hat{\lambda}_L \approx \frac{1}{k}\sum_{i=1}^{n} \mathbf{1}_{\{U_i \le k/n,\, V_i \le k/n\}}$$

for threshold $k \approx \sqrt{n}$.

The threshold choice $k \approx \sqrt{n}$ is a commonly used heuristic in empirical tail dependence estimation, providing a practical balance between tail sparsity and estimation stability in finite samples [5].

Using the empirical threshold $k \approx \sqrt{n}$ with $n = 362$ (yielding $k \approx 19$), the BTC–USDT data exhibit empirical lower and upper tail dependence estimates that remain close to zero, supporting the conclusion of negligible tail dependence.

Definition 2. *Let (X,Y) be a random vector with continuous marginal distribution functions F and G. The upper tail dependence coefficient of (X,Y) is defined as*

$$\lambda_U = \lim_{u \to 1} P\big(Y > G^{-1}(u) \mid X > F^{-1}(u)\big)\,, \tag{5}$$

provided that the limit exists and takes a value in $[0,1]$.

Likewise, assuming the limit exists and belongs to $[0,1]$, the lower tail dependence coefficient of (X,Y) is given by

$$\lambda_L = \lim_{u \to 0} P\big(Y \le G^{-1}(u) \mid X \le F^{-1}(u)\big)\,. \tag{6}$$

Theorem 3 ([21]). *Let $\lambda_L.\lambda_U, F, G, X, Y$ is the copula of X,Y with diagonal section $\delta_{\mathscr{C}}$, then*

$$\lambda_U = 2 - \lim_{t \to 1} \frac{1 - \mathscr{C}(t,t)}{1-t} = 2 - \delta'_{\mathscr{C}}(1^-) \tag{7}$$

exist if the limits (5) and (6) exist.

Proof. We have

$$
\begin{aligned}
\lambda_U &= \lim_{t\to 1^-} P[Y > G^{-1}(t) | X > F^{-1}(t)] \\
&= \lim_{t\to 1^-} \frac{\overline{\mathscr{C}}(t,t)}{1-t} = \lim_{t\to 1^-} \frac{1-2t+\mathscr{C}(t,t)}{1-t} \\
&= 2 - \lim_{t\to 1^-} \frac{1-\mathscr{C}(t,t)}{1-t} = 2 - \delta'_{\mathscr{C}}(1^-).
\end{aligned}
$$

For λ_L, the proof is the same. □

If $\lambda_U \in (0,1)$, then the copula $\mathscr{C}$ exhibits upper tail dependence; otherwise, it is said to be upper tail independent.

1.3 The bivariate FGM copula

The FGM copula represents one of the earliest and simplest parametric families for modeling bivariate dependence, particularly suited for scenarios with weak overall association and no tail dependence.

The FGM copula is adopted in this study as a baseline benchmark because it represents one of the simplest copula families capable of modeling both positive and negative dependence while preserving zero asymptotic tail dependence. Introduced independently by Farlie [12] and Morgenstern [29], it extends the independence copula $\Pi(u,v) = uv$ by incorporating a perturbation term $\theta uv(1-u)(1-v)$, where $\theta \in [-1,1]$ controls the strength and sign of dependence. This structure allows for positive ($\theta > 0$) or negative ($\theta < 0$) associations but limits the range of dependence measures; for instance, Spearman's rho is bounded by $|\rho_S| \leq 1/3$ and Kendall's tau by $|\tau| \leq 2/9$.

In financial applications, such as the BTC—USDT pair analyzed here, the FGM copula is advantageous for its analytical tractability. The copula density $\mathscr{C}(u,v) = 1+\theta(1-2u)(1-2v)$ facilitates straightforward parameter estimation via methods like maximum likelihood or inference functions for margins (IFM), and simulations can be efficiently generated using conditional inversion. However, its inability to capture strong or tail-dependent relationships restricts its use to weakly correlated assets, aligning with empirical findings of moderate monotonic dependence ($\rho_S \approx 0.41$) but zero tail coefficients in cryptocurrency returns. This parsimony makes FGM a useful baseline model in risk assessment, although more flexible copula families (e.g., Clayton, Gumbel, or Frank) may provide a better fit when the dependence structure is more complex [12,30].

The FGM copula is a well-known copula family introduced for modeling weak dependence [12,29]. It is defined for two variables as follows:

Definition 3. *Let $U,V \sim \mathscr{U}(0,1)$ be uniform random variables. The FGM copula is defined by:*

$$\mathscr{C}(u,v) = uv + \theta uv(1-u)(1-v), \quad \theta \in [-1,1].$$

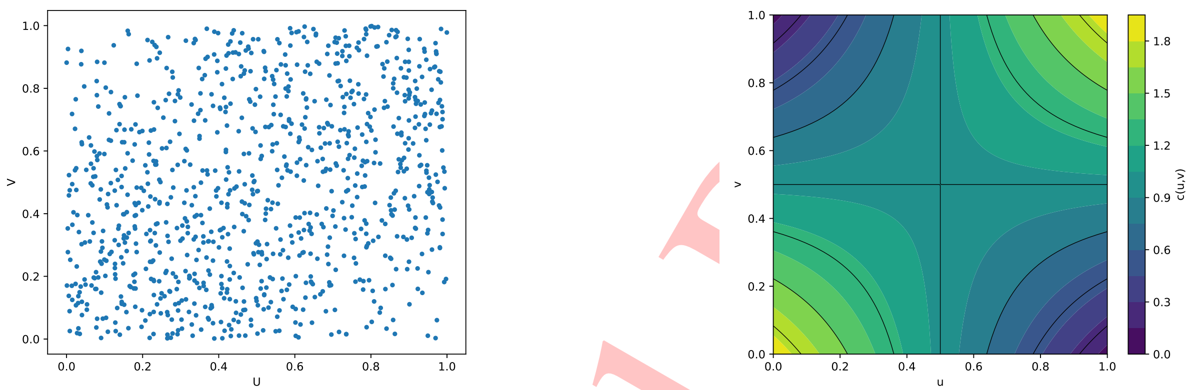
Theorem 4 ([30])**.** *The function $\mathscr{C}(u,v) = uv + \theta uv(1-u)(1-v)$ is a copula if and only if $\theta \in [-1,1]$.*

Proof. 1. $\mathscr{C}(u,v)$ is grounded and satisfies boundary conditions.

2. $\mathscr{C}(u,1) = u$ and $\mathscr{C}(1,v) = v$, satisfying uniform marginals.

3. For all $(u_1, v_1), (u_2, v_2)$ such that $u_1 \leq u_2$, $v_1 \leq v_2$, the 2-increasing condition holds if and only if $\theta \in [-1, 1]$.

. □



(a) Scatter plot of FGM samples

(b) Theoretical contour of FGM copula density

Figure 1: Scatter plots and contour plots of 1000 random samples from the FGM copula

1.4 Lower and upper tail dependence in the FGM copula

A fundamental property of the FGM copula is its absence of asymptotic tail dependence. In particular, both the lower and upper tail dependence coefficients are equal to zero for all admissible values of the dependence parameter $\theta \in [-1, 1]$. This property reflects the fact that the FGM copula concentrates dependence primarily in the central region of the distribution while remaining asymptotically independent in the tails.

Consequently, although the FGM copula can represent weak positive or negative dependence, it is not suitable for modeling joint extreme events. Nevertheless, its zero-tail-dependence structure makes it a useful baseline benchmark for weak-dependence settings such as the BTC—USDT case considered in this study.

2 Financial risk modeling

This section introduces core concepts in financial risk modeling, emphasizing portfolio-level metrics that are essential for quantifying potential losses in volatile markets such as cryptocurrencies. Building on the dependence structures discussed in Section 1, this section bridges theoretical copula frameworks with practical risk measures, highlighting how weak dependence, modeled through simple copula structures (e.g., FGM as a baseline benchmark), influences portfolio outcomes. In financial practice, risk modeling extends beyond individual assets to aggregated portfolios, where diversification benefits or lack thereof—can significantly impact overall exposure. Traditional metrics like VaR provide a threshold for potential losses at a given confidence level, while Expected Shortfall (ES/CVaR) offers a more comprehensive view by averaging losses beyond that threshold, addressing VaR's shortcomings in tail events [1,22]. For cryptocurrency pairs like BTC—USDT with zero tail dependence, risk measures under simplified copula specifications may still provide useful benchmarks for stress testing, although model selection remains essential for reliable risk estimation, given that extreme co-movements are empirically negligible. The presented framework illustrates how dependence structures influence multivariate risk probabilities and portfolio-level risk measures, underscoring the importance of coherent methodologies in financial risk analysis. This foundation sets the stage for integrating copulas into risk assessment, enabling more robust portfolio optimization and capital allocation in uncertain environments [2,11]. Below are some of the essentials we need.

Definition 4. *Let* $\mathbf{R} = (R_1, R_2, \ldots, R_d)$ *be the vector of asset returns, and* $\mathbf{w} = (w_1, w_2, \ldots, w_d)$ *be the portfolio weights such that* $\sum_{i=1}^{d} w_i = 1$*. Then the portfolio return is*

$$R_p = \sum_{i=1}^{d} w_i R_i = \mathbf{w}^\top \mathbf{R}. \tag{8}$$

Definition 5. *Value-at-Risk at level* $\alpha \in (0,1)$ *is defined as:*

$$VaR_\alpha(R_p) = \inf\{x \in R \mid P(R_p \le x) \ge \alpha\}. \tag{9}$$

Further details are provided in [22, 27].

2.1 Expected Shortfall (ES / CVaR)

Expected Shortfall or Conditional Value-at-Risk (CVaR) is:

$$ES_\alpha(R_p) = E[R_p \mid R_p \le VaR_\alpha(R_p)]. \tag{10}$$

It is a coherent risk measure and preferred over VaR. Further details are provided in [1, 10, 18, 20, 35].

Theorem 5. *Let* $\mathscr{C}(u,v,w)$ *be a copula function and* F_1, F_2, F_3 *be marginal distributions. Then the joint lower-tail probability is approximated as:*

$$P(X_1 < q_1, X_2 < q_2, X_3 < q_3) \approx \mathscr{C}(q_1, q_2, q_3), \quad \text{as } q_i \to 0^+. \tag{11}$$

If the lower-tail dependence coefficient λ_L *is high, then*

$$P(R_p \le x) \text{ increases} \Rightarrow ES_\alpha \text{ increases}.$$

Proof. See [11]. □

Theorem 5 is included to provide an explicit conceptual bridge between copula-based dependence modeling and portfolio risk measures. Although the result follows directly from Sklar's theorem and standard tail dependence arguments, its inclusion highlights the mechanism through which stronger lower-tail dependence may increase joint downside probabilities and, consequently, portfolio Expected Shortfall.

3 Financial risk modeling with copulas: from classical measures to dependence structures

Financial risks are typically classified into market risk, credit risk, and operational risk, with market and credit risk receiving particular attention due to their direct relationship with the dependence structure among assets and financial events. Traditional approaches such as covariance, linear correlation, and historical simulation, while widely used, often fail to capture two critical aspects of financial data: (i) *tail dependence*, reflecting strong co-movements during extreme market events, and (ii) *asymmetric dependence*, where correlations tend to intensify during downturns compared to periods of market growth. These limitations underscore the need for more advanced tools to model dependence structures.

Copula functions provide a powerful framework to address these shortcomings by separating marginal distributions from the joint dependence structure. They enable accurate estimation of risk measures such as VaR and CVaR, particularly in portfolios exposed to nonlinear and asymmetric dependence. For instance, copulas such as the *t-Copula* or *Gumbel Copula* allow explicit modeling of tail dependence, leading to more realistic joint risk scenarios. Beyond improving precision in risk estimation, copulas offer flexibility in stress-testing, compatibility with multi-asset portfolios, and the ability to simulate extreme events.

3.1 The role of tail dependence in risk measurement

1. Enhanced Risk Estimation: Ignoring tail dependence may lead to underestimation of extreme losses, particularly during market downturns.

2. Stress Testing: Copulas with tail dependence allow realistic simulation of joint extreme events.

3. Credit Risk: Joint default probabilities are strongly influenced by tail dependence.

4. Systemic Risk: Tail dependence provides insights into contagion and systemic failures.

3.2 Modeling weak dependence in zero-tail regimes

In such settings, copulas such as the Gaussian and FGM copulas may be considered due to their structural simplicity and absence of tail dependence. In weak-dependence settings with negligible tail dependence, simple copula families such as the Gaussian and FGM copulas may serve as useful benchmark models. However, empirical applications often require more flexible copula structures when the observed dependence exceeds the theoretical limitations of simple families.

Therefore, in empirical applications, it is essential to consider more flexible copula families capable of capturing a wider range of dependence patterns. The choice of copula model plays a critical role in accurately representing dependence structures and ensuring reliable risk estimation.

Overall, copula-based models provide a flexible and robust framework for financial risk analysis, allowing for improved modeling of dependence structures beyond traditional correlation-based approaches.

4 Empirical analysis

4.1 Data description

This study examines the dependence structure between BTC and USDT using daily closing price data over the period from May 2024 to May 2025. The dataset consists of synchronized daily observations for both assets.

Logarithmic returns are computed as

$$r_t = \log\left(\frac{P_t}{P_{t-1}}\right), \tag{12}$$

where P_t denotes the closing price at time t.

After removing missing and non-finite observations, the final sample includes $n = 362$ paired return observations. The empirical dependence between BTC and USDT returns is measured using Spearman's rank correlation coefficient, yielding

$$\rho_s \approx 0.4099, \tag{13}$$

which indicates a moderate level of dependence.

To provide an initial visual assessment of the dependence structure, the scatter plot of the pseudo-observations (U,V) is presented in Figure 2. The plot reveals a moderate level of dependence between BTC and USDT returns. Moreover, the absence of strong clustering in the lower-left and upper-right corners suggests weak tail dependence between the two assets.

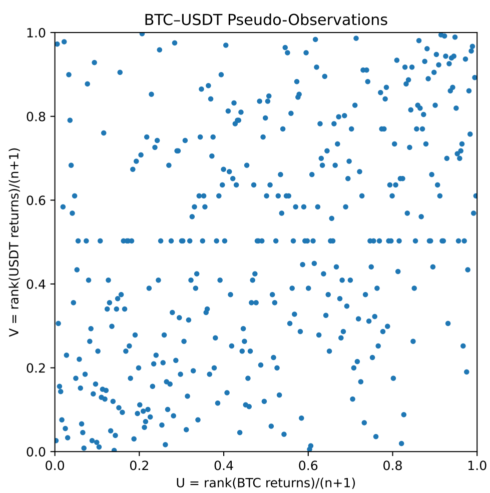


Figure 2: Scatter plot of pseudo-observations (U,V) constructed from BTC and USDT daily log-returns. The absence of pronounced clustering in the lower-left and upper-right corners suggests weak tail dependence.

The observed dependence pattern motivates the use of flexible copula models capable of capturing moderate dependence structures.

To complement the numerical goodness-of-fit results, contour plots of the fitted copula densities are presented. While information criteria such as AIC and BIC provide quantitative measures of model performance, contour plots offer a valuable visual tool for assessing how well each copula captures the underlying dependence structure in the data.

In particular, contour plots illustrate the shape of the joint density implied by each copula and allow for a direct comparison with the empirical distribution of the pseudo-observations. This visual inspection is especially useful in identifying differences in dependence patterns, including symmetry, strength of association, and tail behavior.

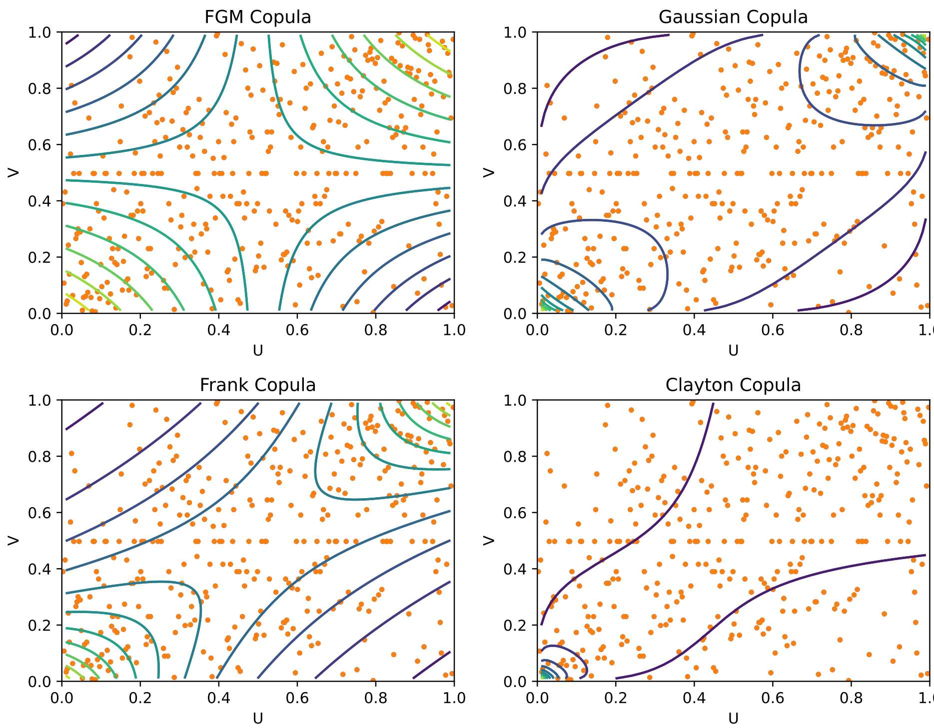


Figure 3: Contour plots of fitted copula densities for BTC–USDT pseudo-observations: (a) FGM, (b) Gaussian, (c) Frank, and (d) Clayton. The plots provide a visual comparison of the dependence structures implied by each copula model

The contour plots provide important insights into the dependence structure between BTC and USDT returns. The FGM copula exhibits a relatively simple and restricted contour pattern, reflecting its limited flexibility and inability to adequately capture moderate dependence. This observation is consistent with its inferior performance in the goodness-of-fit analysis.

The Gaussian copula produces smooth and symmetric contour lines around the diagonal, indicating a linear dependence structure. While it provides a reasonable approximation, it lacks flexibility in capturing more complex dependence patterns, particularly in the tails.

In contrast, the Frank copula displays more flexible and curved contour shapes that align more closely with the empirical distribution of the pseudo-observations. This suggests that the Frank copula is better suited to capturing the moderate dependence structure observed in the data. The visual evidence provided by the contour plots is consistent with the information criteria results, which identified the Frank copula as the best-fitting model.

The Clayton copula, on the other hand, shows a strong concentration of contour lines in the lower-left region, indicating pronounced lower-tail dependence. However, this feature is not strongly supported by the empirical data, which do not exhibit significant clustering in the joint lower tail. As a result, the Clayton copula tends to overemphasize extreme co-movements and provides a less accurate representation of the observed dependence structure.

Overall, the contour plots reinforce the conclusion that the Frank copula provides the most appropriate and flexible representation of the dependence structure between BTC and USDT returns.

4.2 Copula model selection and goodness-of-fit analysis

These preliminary observations motivate the need for a formal copula-based modeling framework to accurately capture the dependence structure. To model the dependence structure, the return series are transformed into pseudo-observations using the empirical rank transformation:

$$U_i = \frac{\operatorname{rank}(r_{BTC,i})}{n+1}, \qquad V_i = \frac{\operatorname{rank}(r_{USDT,i})}{n+1}. \tag{14}$$

We first consider the FGM copula. The relationship between Spearman's rho and the FGM parameter is given by

$$\rho_s = \frac{\theta}{3}. \tag{15}$$

Based on the empirical estimate $\rho_s \approx 0.4099$, the implied parameter is

$$\hat{\theta} = 3\rho_s \approx 1.2297 \tag{16}$$

which lies outside the admissible parameter range $\theta \in [-1, 1]$. This indicates that the FGM copula is not capable of capturing the observed dependence level.

To address this limitation, we compare the FGM copula with alternative copula families, including Gaussian, Frank, and Clayton copulas. All models are estimated using maximum likelihood, and their performance is evaluated using the log-likelihood, Akaike Information Criterion (AIC), and Bayesian Information Criterion (BIC).

Table 1: Copula model comparison for BTC–USDT returns

Copula	Parameter	LogLik	AIC	BIC
Frank	2.8009	34.1834	-66.3667	-62.4751
Gaussian	0.4011	30.3380	-58.6759	-54.7843
FGM	0.9477	27.0074	-52.0148	-48.1231
Clayton	0.5464	25.1337	-48.2673	-44.3757

The results indicate that the Frank copula provides the best fit, as it achieves the highest log-likelihood and the lowest AIC and BIC values. The Gaussian copula also performs reasonably well, while the FGM copula shows inferior performance due to its structural limitations. Table 1 [1] summarizes the estimated

[1] The reported parameter has a different interpretation across copula families. For the Frank, FGM, and Clayton copulas, the parameter corresponds to the dependence parameter θ, whereas for the Gaussian copula it represents the correlation parameter ρ.

copula parameters together with the corresponding log-likelihood, AIC, and BIC values for the fitted models. The differences in AIC values suggest that the dependence structure is neither purely linear (as in the Gaussian copula) nor strongly concentrated in the tails (as in the Clayton copula), but rather exhibits a moderate and symmetric dependence pattern. Although both the Gaussian and Frank copulas exhibit zero tail dependence, the Frank copula provides greater flexibility in capturing nonlinear dependence patterns in the central region of the distribution. These findings are consistent with the empirical observation of weak tail dependence, indicating that extreme co-movements between BTC and USDT are not pronounced. It is important to note that the FGM copula exhibits zero tail dependence, meaning it cannot capture joint extreme movements. This limitation reduces its suitability for financial risk modeling, where tail dependence is often critical. Therefore, in the analysis, the FGM copula is retained only as a benchmark model, while the Frank copula is selected as the primary model for subsequent risk analysis.

4.3 Risk analysis: VaR, CVaR, and stress testing

Based on the goodness-of-fit results, the Frank copula is selected as the primary model for the BTC–USDT dependence structure. Nevertheless, to examine the sensitivity of risk estimates to copula choice, we compute VaR and CVaR under four fitted copula models: FGM, Gaussian, Frank, and Clayton.

Let R_p denote the return of an equally weighted BTC–USDT portfolio:

$$R_p = 0.5R_{BTC} + 0.5R_{USDT}. \tag{17}$$

The equally weighted portfolio specification is adopted as a standard benchmark and simplifying assumption commonly used in illustrative financial risk analyses, allowing the effect of copula choice on portfolio risk estimates to be examined without introducing optimization-driven allocation effects. Results may differ under alternative portfolio weighting schemes, particularly when asset allocations are optimized to reflect different risk-return objectives.

The CVaR, also known as Expected Shortfall, is defined as

$$\mathrm{CVaR}_\alpha = E\left[R_p \mid R_p \leq \mathrm{VaR}_\alpha\right]. \tag{18}$$

The joint return distribution is simulated using each fitted copula combined with the empirical marginal distributions of BTC and USDT returns. Table 2[2] reports the estimated VaR and CVaR at the 95% and 99% confidence levels.

Table 2: VaR and CVaR comparison under different copula models

Copula	VaR 95%	CVaR 95%	VaR 99%	CVaR 99%
Frank	-1.8959%	-2.6492%	-3.1145%	-3.5534%
Gaussian	-1.9173%	-2.6776%	-3.1391%	-3.5755%
FGM	-1.9112%	-2.6720%	-3.1326%	-3.5586%
Clayton	-1.9126%	-2.6871%	-3.1646%	-3.5971%

The results show that the estimated risk measures are sensitive to the choice of copula model. The Frank copula, which provides the best goodness-of-fit according to AIC and BIC, yields a 95% VaR of

[2] VaR and CVaR values are reported as portfolio returns in percentage form. Negative values indicate portfolio losses, consistent with standard financial risk reporting conventions.

-1.8959% and a 99% VaR of -3.1145%. This means that, under the Frank copula, the one-day loss of the equally weighted BTC–USDT portfolio is expected to exceed approximately 1.90% with probability 5%, and approximately 3.11% with probability 1%.

From a practical perspective, these results suggest that portfolio risk is primarily driven by moderate co-movements rather than extreme joint crashes.

The Clayton copula produces the most conservative tail-risk estimates, particularly at the 99% level, with a VaR of -3.1646% and a CVaR of -3.5971%. This is consistent with the lower-tail dependence property of the Clayton copula, which tends to assign higher probability to joint downside movements. In contrast, the Frank copula provides slightly lower tail-risk estimates, reflecting its zero asymptotic tail dependence and better overall fit to the empirical dependence structure.

Although the FGM copula produces risk estimates close to those of the Gaussian and Frank copulas, this does not imply that it is the most appropriate model. As shown in the model selection analysis, the FGM copula cannot fully capture the observed dependence level because its implied parameter exceeds the admissible range. Therefore, the FGM results should be interpreted only as a benchmark rather than as the main risk model.

4.3.1 Stress testing

To evaluate the behavior of the BTC–USDT portfolio under an extreme downside scenario, we impose a BTC shock equal to its empirical 1% quantile:

$$R_{BTC}^{stress} = -6.2330\%. \tag{19}$$

For each fitted copula, the conditional distribution of USDT returns given this BTC stress event is simulated. The results are reported in Table 3.

Table 3: Stress testing results conditional on BTC 1% lower-tail shock

Copula	Mean USDT Return	USDT 5% Quantile	Mean Portfolio Return	Portfolio 5% Quantile
Frank	-0.0255%	-0.0864%	-3.1292%	-3.1597%
Gaussian	-0.0354%	-0.1136%	-3.1342%	-3.1733%
FGM	-0.0191%	-0.0800%	-3.1260%	-3.1565%
Clayton	-0.0553%	-0.1296%	-3.1441%	-3.1813%

The stress testing results show that, when BTC experiences a severe negative shock of approximately -6.23%, the conditional response of USDT remains relatively small in magnitude. This is expected because USDT is a stablecoin and its return variability is substantially lower than that of BTC.

Among the fitted copulas, the Clayton copula generates the strongest conditional downside response for USDT, with a mean conditional return of -0.0553% and a 5% conditional quantile of -0.1296%. This reflects the lower-tail dependence structure of the Clayton copula. The FGM copula produces the weakest conditional downside response, which is consistent with its limited dependence structure and absence of tail dependence.

The Frank copula, selected as the preferred model based on goodness-of-fit criteria, produces an intermediate stress response. Under the Frank copula, the mean stressed portfolio return is -3.1292%, while the 5% stressed portfolio quantile is -3.1597%.

The relatively small difference between the mean stressed portfolio return and the 5% stressed portfolio quantile reflects the limited variability of USDT returns and the weak dependence structure between BTC and USDT. Since the portfolio shock is primarily driven by the imposed BTC downside event, the conditional portfolio distribution remains relatively concentrated around its mean under stress.

It should be emphasized that the Frank copula has zero asymptotic tail dependence. Therefore, although the stress scenario is mathematically valid, the probability of joint extreme losses decreases as the threshold becomes more extreme. This point is particularly important when interpreting the 1% BTC shock scenario: the copula can describe conditional dependence at this finite threshold, but it does not imply persistent asymptotic tail dependence.

Overall, these results confirm that copula choice affects risk estimates, especially under stress scenarios. The FGM copula is useful only as a benchmark, while the Frank copula provides the most statistically appropriate basis for the main risk analysis of the BTC–USDT portfolio. Empirical analysis demonstrates that the dependence structure between BTC and USDT is characterized by moderate association and weak tail dependence. Among the considered models, the Frank copula achieved the highest log-likelihood and the lowest AIC and BIC values among the fitted models.

5 Conclusion

This study investigated the dependence structure between Bitcoin (BTC) and Tether (USDT) returns over the period from May 2024 to May 2025 within a copula-based modeling framework. The analysis was conducted as a case study, and therefore the findings are inherently specific to the selected asset pair and sample period.

The empirical results indicate the presence of a moderate positive dependence between BTC and USDT returns, with Spearman's rank correlation estimated at approximately $\rho_s = 0.4099$. However, graphical diagnostics based on pseudo-observations and contour plots reveal that tail dependence in both lower and upper regions is negligible, with no significant clustering of joint extreme events. This suggests asymptotic independence in the tails and a dependence structure predominantly concentrated in the central region of the distribution.

A key finding of this study is that the Farlie–Gumbel–Morgenstern (FGM) copula, despite its analytical simplicity and zero tail dependence property, lacks sufficient flexibility to adequately capture the observed dependence structure. In particular, the implied dependence parameter exceeds the admissible range, indicating a structural incompatibility of this copula family with the empirical data. Consequently, the FGM copula is retained solely as a baseline benchmark model and is not suitable as a primary model for risk analysis.

To address this limitation, alternative copula families, including Gaussian, Frank, and Clayton copulas, were estimated and systematically compared using likelihood-based criteria such as AIC and BIC. The results provide strong evidence that the Frank copula offers the best overall fit to the BTC–USDT dependence structure. Although both the Frank and Gaussian copulas exhibit zero asymptotic tail dependence, the Frank copula demonstrates superior flexibility in capturing nonlinear dependence patterns within the central region of the distribution.

The risk analysis further highlights that the choice of copula model has a substantial impact on the estimated Value-at-Risk (VaR) and Conditional Value-at-Risk (CVaR). Under the selected Frank copula, the one-day portfolio loss is expected to exceed approximately 1.90% at the 95% confidence level and

3.11% at the 99% confidence level. In contrast, the Clayton copula yields more conservative tail risk estimates due to its lower-tail dependence structure, whereas the FGM copula produces comparable but less reliable estimates owing to its restricted dependence range.

The stress testing results corroborate that copula selection also plays a critical role in shaping portfolio behavior under adverse scenarios. When BTC is subjected to a severe downside shock corresponding to its empirical 1% quantile, the conditional response of USDT remains limited, reflecting its stablecoin characteristics. In this context, the Frank copula provides an intermediate and statistically robust stress response, while the Clayton copula generates more pessimistic downside scenarios.

Overall, the findings underscore the fundamental importance of appropriate copula selection in financial risk modeling. While simple copulas such as FGM may serve as useful baseline benchmarks, their application as primary models can lead to misleading risk assessments when the observed dependence exceeds their theoretical limitations. For the BTC–USDT case study, the Frank copula achieves an effective balance between statistical goodness-of-fit, structural flexibility, and practical relevance in risk estimation.

Finally, this study is limited to a single cryptocurrency pair and a one-year sample period. Therefore, the results should not be generalized without caution to other assets, longer time horizons, or different market conditions. Future research may extend this framework by incorporating dynamic copulas, regime-switching dependence structures, alternative marginal specifications, and multi-asset cryptocurrency portfolios to provide a more comprehensive understanding of dependence and risk in emerging financial markets.

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Conflict of Interest

The authors declare no competing financial interests or personal relationships that could have influenced this work.

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