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Fractional-order optimal control of a tuberculosis model with exogenous reinfection

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Abstract. This paper presents a general formulation of a fractional-order tuberculosis model with exogenous reinfection, based on the susceptible–exposed–infected–treated (SEIT) epidemiological framework. A control representing the prevention of exogenous reinfection is employed to effectively minimize the number of infectious individuals, including both actively contagious and latently infected populations. The Caputo fractional derivative is used to model the system, and the resulting fractional-order optimal control problem is theoretically analyzed using Pontryagin's maximum principle. A forward-backward sweep algorithm, using a generalized Euler method, is applied to numerically solve the optimality system. Numerical simulations demonstrate the efficiency and effectiveness of the proposed optimization procedure.

Keywords: Optimal control, tuberculosis model with exogenous reinfection, fractional derivatives.
AMS Subject Classification 2010: 34A08, 49J15, 49M25.

1 Introduction

Tuberculosis (TB) is a long-lasting infectious disease caused by the bacterium Mycobacterium tuberculosis. It affects primarily the lungs, but can also spread to and damage other organs in the body. Common symptoms include haemoptysis, chest pain, shortness of breath, fever, weight loss, and night sweats. Transmission occurs mainly through airborne droplets released when infectious individuals cough, sneeze, speak, or laugh, allowing susceptible individuals to inhale the bacteria [7].

Despite significant advances in diagnosis and treatment, TB remains one of the leading causes of death worldwide. Although global TB incidence has shown a gradual decline in recent years, the disease

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continues to pose a major public health challenge, particularly in regions with high HIV prevalence. In 2017, approximately 10 million new TB cases were reported globally, resulting in about 1.6 million deaths, including 0.3 million among HIV-positive individuals [33]. These statistics underscore the urgent need for effective and sustainable control strategies.

Following infection, the host immune system often limits bacterial replication. Only about 10% of infected individuals develop active TB, while the majority remain in a latent stage. The incubation period for latent tuberculosis can vary widely, from months to several decades [8]. An accurate classification of tuberculosis is essential, as treatment and prevention strategies differ significantly between active and latent infections. Although drug therapy is effective for drug-sensitive TB, with recovery rates exceeding 90%, multidrug-resistant TB treatment is prolonged (18–24 months) and achieves cure rates of only about 50% [33]. In many high-burden regions, prevention still relies heavily on the Bacillus Calmette–Guérin (BCG) vaccine, whose protective efficacy remains variable [30]. These limitations motivate the exploration of complementary, non-pharmaceutical intervention strategies supported by mathematical modeling.

In recent years, fractional calculus has attracted considerable attention in modeling complex dynamical systems. Fractional derivatives generalize classical integer-order derivatives to arbitrary real or complex orders [11, 21, 34]. Among the various definitions, the fractional derivatives Riemann-Liouville and Caputo are the most widely used [4]. The Caputo fractional derivative is particularly suitable for time-dependent biological systems because it allows initial conditions to be expressed in terms of physically interpretable integer-order derivatives [27]. Fractional differential equations have been successfully applied in physics, biology, economics, and epidemiology [13, 22, 23], especially in systems where historical effects influence present dynamics.

Mathematical modeling is fundamental for analyzing the spread of infectious diseases and informing the development of effective public health strategies [17, 19]. Numerous compartmental TB models have been developed to study susceptible, exposed, infectious, and treated populations. Optimal control theory has also been widely applied in epidemiological modeling to identify efficient intervention strategies for reducing disease spread. For example, Akbari [3] investigated the optimal control of communicable infectious diseases through horizontal and vertical transmission mechanisms. In addition, numerical approaches for solving optimal control problems governed by differential equations have been extensively studied; see, for instance, Benalia et al. [6], who developed a numerical method for boundary optimal control problems modeled by heat transfer equations. In particular, models incorporating exogenous reinfection have received substantial attention, since reinfection significantly influences disease persistence in high-burden regions. Hattaf et al. [14] and Abimbade et al. [1] analyzed TB models with exogenous reinfection under optimal control frameworks, highlighting the importance of treatment and contact reduction strategies.

The TB is marked by extended latency, recurrence, and reinfection, suggesting that its progression depends not only on the present epidemiological status but also on an individual's prior exposure history. Classical integer-order models assume Markovian dynamics and therefore neglect such memory effects. Fractional derivatives, particularly in the Caputo sense, naturally incorporate memory and hereditary properties, making them well suited for modeling diseases with long latent periods such as TB. Recent studies demonstrate that fractional epidemiological models provide improved flexibility and realism by capturing long-range temporal dependence and complex transmission behavior [5, 9, 24, 29].

While classical integer-order models have provided valuable insights, fractional-order epidemiological models offer enhanced capability to describe long-term memory effects in disease transmission. Recent works [12, 25, 31, 32] demonstrate that fractional formulations can produce richer and more realistic dynamical behavior compared to their integer-order counterparts.

Motivated by these considerations, this study develops a fractional-order TB model incorporating exogenous reinfection and formulates an associated optimal control problem aimed at reducing disease transmission by limiting interactions between infectious and exposed individuals. The main contributions of this work are outlined as follows: We formulate a novel fractional-order TB model with exogenous reinfection using the Caputo fractional derivative. We establish the existence of solutions and derive the necessary optimality conditions for the associated fractional-order optimal control problem. We provide numerical simulations to confirm the influence of fractional order effects on disease dynamics and to evaluate the effectiveness of the proposed control plan.

To the best of our knowledge, a rigorous integration of fractional-order optimal control with a TB model explicitly incorporating exogenous reinfection has not been thoroughly studied in the existing literature. The proposed framework therefore extends classical TB models by incorporating memory-dependent dynamics within an optimal control setting.

This paper is organized as follows. In Section 2, we introduce the necessary preliminary concepts from fractional calculus and formulate the corresponding fractional-order optimal control problem. Section 3 establishes the existence of solutions and derives necessary conditions for optimality. Numerical simulations are provided in Section 4 to illustrate the effectiveness of the proposed method. Finally, conclusions are drawn in Section 5.

2 Preliminaries and fractional-order optimal control formulation

In this section, we briefly review the main concepts of fractional calculus and the fractional-order optimal control framework that will be used to formulate the proposed fractional-order TB model. Among the various definitions, the Riemann–Liouville and Caputo fractional derivatives are the most commonly used ones [26]. In this work, we adopt the Caputo fractional derivative since it allows the use of standard (integer-order) initial conditions, which is particularly suitable for epidemiological applications.

2.1 Fractional derivatives

Definition 1. *Let $\gamma > 0$, and let $r \in \mathbb{N}$ be the smallest integer such that $r-1 < \gamma \leq r$. The Riemann–Liouville fractional derivatives of order γ are defined as follows:*

$$(left) \quad {}^{RL}_{a}D^{\gamma}_{t} f(t) = \frac{1}{\Gamma(r-\gamma)} \frac{d^r}{dt^r} \int_a^t (t-p)^{r-\gamma-1} f(p)\, dp,$$

$$(right) \quad {}^{RL}_{t}D^{\gamma}_{b} f(t) = \frac{(-1)^r}{\Gamma(r-\gamma)} \frac{d^r}{dt^r} \int_t^b (p-t)^{r-\gamma-1} f(p)\, dp,$$

where $\Gamma(\cdot)$ denotes Euler's gamma function.

Definition 2. *Let $\gamma > 0$, and let $r \in \mathbb{N}$ be the smallest integer such that $r-1 < \gamma \leq r$. The Caputo fractional derivatives of order γ are defined as follows:*

$$(left) \quad {}^{C}_{a}D^{\gamma}_{t} f(t) = \frac{1}{\Gamma(r-\gamma)} \int_a^t (t-p)^{r-\gamma-1} \frac{d^r f(p)}{dp^r}\, dp,$$

$$(right) \quad {}_{t}^{C}D_{b}^{\gamma}f(t) = \frac{(-1)^{r}}{\Gamma(r-\gamma)} \int_{t}^{b} (p-t)^{r-\gamma-1} \frac{d^{r}f(p)}{dp^{r}}\, dp,$$

where $\Gamma(\cdot)$ *denotes Euler's gamma function.*

2.2 Fractional-order optimal control problem

Following the approach of Agrawal [2], we consider the following fractional-order optimal control problem defined on the time interval $[0, t_f]$.

Determine a control function ϕ that minimizes the objective functional

$$J(\phi) = \int_{0}^{t_f} M\left(x(t), \phi(t), t\right) dt,$$

subject to the Caputo fractional differential equation

$${}_{0}^{C}D_{t}^{\alpha}x(t) = V\left(x(t), \phi(t), t\right), \qquad 0 < \alpha \leq 1,$$

with the initial condition

$$x(0) = x_0.$$

Here, ${}_{0}^{C}D_{t}^{\alpha}$ is the Caputo fractional derivative of order α. The state variable $x(t) \in \mathbb{R}^n$ represents the state vector, while $\phi \in \mathbb{R}^m$ denotes the admissible control vector. The function $M(x, \phi, t)$ is the running cost functional, and $V(x, \phi, t)$ characterizes the system dynamics.

Define the Hamiltonian function as

$$\mathscr{H}(x, \lambda, \phi, t) = M(x, \phi, t) + \lambda^{\top} V(x, \phi, t),$$

where $\lambda(t) \in \mathbb{R}^n$ is the adjoint (co-state) vector, and the superscript $\top$ denotes the transpose.

Using variational arguments together with the fractional integration-by-parts formula (see [2]), one derives the necessary conditions for optimality, which are expressed through the following system, commonly referred to as the fractional Pontryagin-type necessary optimality system

$$\begin{aligned}
{}_{0}^{C}D_{t}^{\alpha}x(t) &= \frac{\partial \mathscr{H}}{\partial \lambda}(x, \lambda, \phi, t), \\
{}_{t}^{C}D_{t_f}^{\alpha}\lambda(t) &= -\frac{\partial \mathscr{H}}{\partial x}(x, \lambda, \phi, t), \\
0 &= \frac{\partial \mathscr{H}}{\partial \phi}(x, \lambda, \phi, t),
\end{aligned}$$

together with the boundary conditions

$$x(0) = x_0, \quad \lambda(t_f) = 0.$$

In the subsequent sections, this framework will be applied to the fractional-orde TB model.

3 Fractional-order optimal control of TB model

In this section, we formulate and analyze a fractional-order optimal control problem for the TB model. The control function ϕ represents the prevention of exogenous reinfection and aims to reduce the number of infectious individuals by decreasing interactions between exposed and infectious persons. The fundamental objective is to find an optimal control ϕ^* that minimizes the objective functional while satisfying the fractional TB dynamics.

3.1 Controlled TB model with fractional-order derivatives

Let $t_f > 0$ be the final time and $0 < \alpha \leq 1$. We consider the objective functional

$$J(\phi) = \int_0^{t_f} \left(I(t) + A\,\phi^2(t) \right) dt,$$

subject to the fractional Caputo dynamics

$${}_0^C D_t^\alpha X(t) = V(X(t), \phi(t), t), \quad X(0) = X_0,$$

where

$$X(t) = (S(t), E(t), I(t), T(t))^\top, \quad X_0 = (S_0, E_0, I_0, T_0)^\top, \quad N(t) = S(t) + E(t) + I(t) + T(t),$$

and the state system is given by

$$\begin{aligned}
{}_0^C D_t^\alpha S(t) &= \Lambda - \mu S - \frac{\beta c I S}{N}, \\
{}_0^C D_t^\alpha E(t) &= \frac{\beta c I S}{N} - \frac{\beta p c I (1-\phi) E}{N} - (\mu + k) E + \frac{\beta \sigma c I T}{N}, \\
{}_0^C D_t^\alpha I(t) &= \frac{\beta p c I (1-\phi) E}{N} + kE - (\mu + r + d) I, \\
{}_0^C D_t^\alpha T(t) &= rI - \mu T - \frac{\beta \sigma c I T}{N},
\end{aligned} \tag{1}$$

with the initial conditions

$$S(0) = S_0, \quad E(0) = E_0, \quad I(0) = I_0, \quad T(0) = T_0.$$

The admissible control set is

$$\mathscr{U} = \left\{ \phi : [0, t_f] \to [0, 1] \text{ measurable} \right\}.$$

3.2 Existence and uniqueness of solutions

The right-hand side of system (1) is continuous and locally Lipschitz with respect to the state variables (S, E, I, T). Moreover, for any admissible control $\phi \in \mathscr{U}$, the system satisfies the standard growth conditions.

Therefore, by the classical existence and uniqueness theorem for Caputo fractional differential equations (see Diethelm [11, Theorem 6.5, p. 92]), the fractional-order TB model (1) admits a unique solution for given non-negative initial conditions.

The model variables and parameters are described in Tables 1 and 2.

Table 1: Variable and control functions

Variables	Description
S	Susceptible individuals
E	Exposed individuals
I	Infectious individuals
T	Treated individuals
ϕ	Control variable representing distancing between infected and exposed individuals

Table 2: Description of model parameters

Parameters	Definition
$\sigma\beta$ $(0 \leq \sigma \leq 1)$	The mean number of secondary infections among treated individuals caused by one infectious individual per contact per unit time
β	The mean number of new infections among susceptible individuals caused by one contagious individual per contact per unit time
μ	Natural mortality rate
k	Progression rate to active TB
r	Per-capita treatment rate
c	Per-capita contact rate
d	TB-induced mortality rate
p	Level of exogenous reinfection
N	Total population
Λ	Constant recruitment rate

The parameter p represents the level of exogenous reinfection, reflecting the increased risk faced by latently infected individuals upon repeated exposure in high-burden settings. The control variable ϕ models interventions aimed at reducing effective contact between infectious and exposed individuals. Epidemiologically, ϕ may represent a combination of public health measures such as contact tracing, chemoprophylaxis for latently infected individuals, infection control in healthcare settings, and behavioral interventions that reduce repeated exposure.

Theorem 1. *Consider the fractional-order TB model* (1) *with* $0 < \alpha \leq 1$. *If*

$$S_0 \geq 0, \quad E_0 \geq 0, \quad I_0 \geq 0, \quad T_0 \geq 0,$$

then the solutions satisfy

$$S(t),\ E(t),\ I(t),\ T(t) \geq 0 \quad \textit{for all } t \geq 0.$$

Proof. Let the initial conditions be non-negative. We prove that the set $\mathbb{R}_+^4$ remains positively invariant under the flow generated by system (1). Consider a fractional-order system of order $0 < \alpha \leq 1$, given by the following formulation:

$$ {}_0^C D_t^\alpha x_i(t) = f_i(x), \quad i = 1, \ldots, 4. $$

According to standard results for Caputo fractional-order systems, if $f_i(x) \geq 0$ whenever $x_i = 0$ and $x_j \geq 0$ for $j \neq i$, then the non-negative region is invariant.

- Positivity of $S(t)$.

  When $S = 0$ and $E, I, T \geq 0$, from system (1),

  $$ {}_0^C D_t^\alpha S = \Lambda \geq 0. $$

  Hence, the vector field points inward on the boundary $S = 0$.

- Positivity of $E(t)$.

  When $E = 0$ and $S, I, T \geq 0$,

  $$ {}_0^C D_t^\alpha E = \frac{\beta c I S}{N} + \frac{\beta \sigma c I T}{N} \geq 0. $$

- Positivity of $I(t)$.

  When $I = 0$ and $S, E, T \geq 0$,

  $$ {}_0^C D_t^\alpha I = kE \geq 0. $$

- Positivity of $T(t)$.

  When $T = 0$ and $S, E, I \geq 0$,

  $$ {}_0^C D_t^\alpha T = rI \geq 0. $$

Since on each boundary hyperplane the fractional derivative is non-negative, the vector field is directed into $\mathbb{R}_+^4$. Therefore, by the invariance property of Caputo fractional-order systems, the solutions remain non-negative for all $t \geq 0$. □

Theorem 2. *Let $0 < \alpha \leq 1$ and assume $S_0,\ E_0,\ I_0,\ T_0 \geq 0$. Define $N(t) = S(t) + E(t) + I(t) + T(t)$. Then the solutions of* (1) *are bounded and satisfy*

$$ 0 \leq N(t) \leq \max\left\{ N(0), \frac{\Lambda}{\mu} \right\} \quad \text{for all } t \geq 0. $$

Consequently,

$$ 0 \leq S(t), E(t), I(t), T(t) \leq \max\left\{ N(0), \frac{\Lambda}{\mu} \right\}, \quad t \geq 0, $$

and the region

$$ \Omega = \left\{ X \in \mathbb{R}_+^4 : N \leq \max\left\{ N(0), \frac{\Lambda}{\mu} \right\} \right\} $$

is positively invariant.

Proof. Let

$$N(t) = S(t) + E(t) + I(t) + T(t).$$

Adding all equations in system (1) gives

$${}_0^C D_t^\alpha N(t) = \Lambda - \mu N(t) - dI(t).$$

Since $I(t) \geq 0$, we obtain

$${}_0^C D_t^\alpha N(t) \leq \Lambda - \mu N(t).$$

To estimate $N(t)$, consider the auxiliary fractional differential equation

$${}_0^C D_t^\alpha y(t) = \Lambda - \mu y(t), \qquad y(0) = N(0).$$

Its explicit solution is

$$y(t) = \frac{\Lambda}{\mu} + \left(N(0) - \frac{\Lambda}{\mu} \right) E_\alpha(-\mu t^\alpha),$$

where $E_\alpha(\cdot)$ is the Mittag–Leffler function.

Because the Mittag–Leffler function satisfies

$$0 < E_\alpha(-\mu t^\alpha) \leq 1, \qquad t \geq 0,$$

it follows that

$$0 \leq y(t) \leq \max\left\{ N(0), \frac{\Lambda}{\mu} \right\}, \qquad t \geq 0.$$

Applying the comparison principle for Caputo fractional differential equations yields

$$N(t) \leq y(t), \qquad t \geq 0.$$

Hence,

$$0 \leq N(t) \leq \max\left\{ N(0), \frac{\Lambda}{\mu} \right\}.$$

Therefore, the total population $N(t)$ remains uniformly bounded for all $t \geq 0$. Since each compartment satisfies

$$S(t), E(t), I(t), T(t) \geq 0$$

and

$$S(t), E(t), I(t), T(t) \leq N(t),$$

all state variables are nonnegative and bounded. □

3.3 Existence of fractional-order optimal control

Theorem 3. *Let $t_f > 0$ and $0 < \alpha \leq 1$. Then there exists an optimal control $\phi^* \in \mathscr{U}$ such that*

$$J(\phi^*) = \min_{\phi \in \mathscr{U}} J(\phi),$$

subject to the controlled fractional TB system (1), where

$$\mathscr{U} = \left\{ \phi : [0,t_f] \to [0,1] \text{ Lebesgue measurable} \right\}.$$

Proof. We employ the direct method in the calculus of variations in the spirit of Filippov–Cesari, adapted to the fractional-order setting.

Existence of an optimal control. The admissible control set

$$\mathscr{U} = \{\phi \in L^\infty(0,t_f) : 0 \leq \phi(t) \leq 1 \text{ a.e. on } [0,t_f]\}$$

is nonempty, convex, bounded in $L^2(0,t_f)$, and weakly closed; hence it is weakly sequentially compact in $L^2(0,t_f)$.

State system and compactness. For each $\phi \in \mathscr{U}$, the fractional system (1) admits a unique solution $X = (S,E,I,T) \in C([0,t_f],\mathbb{R}^4)$ due to the local Lipschitz continuity of the right-hand side with respect to the state variables and standard results for Caputo fractional differential equations. Moreover, the solution is positive and uniformly bounded in the positively invariant set Ω.

Let $\{\phi_n\} \subset \mathscr{U}$ be a minimizing sequence and X_n the corresponding solutions. Since X_n is uniformly bounded and equicontinuous on $[0,t_f]$, the Arzelà–Ascoli theorem implies the existence of a subsequence (not relabeled) and a function $X^* \in C([0,t_f],\mathbb{R}^4)$ such that

$$X_n \to X^* \quad \text{uniformly on } [0,t_f].$$

Moreover, by weak compactness, there exists $\phi^* \in \mathscr{U}$ such that $\phi_n \rightharpoonup \phi^*$ in $L^2(0,t_f)$, and X^* is the corresponding state trajectory associated with ϕ^*.

Optimality. The cost functional

$$J(\phi) = \int_0^{t_f} \left(I(t) + A\phi^2(t)\right) dt$$

satisfies that the control term is convex and weakly lower semicontinuous in $L^2(0,t_f)$, while the state component satisfies $I_n \to I^*$ strongly in $L^2(0,t_f)$ due to uniform convergence of X_n. Hence,

$$J(\phi^*) \leq \liminf_{n\to\infty} J(\phi_n).$$

Since $\{\phi_n\}$ is minimizing, we conclude that $J(\phi^*) = \inf_{\phi\in\mathscr{U}} J(\phi)$, and therefore ϕ^* is an optimal control.

□

3.4 Hamiltonian and adjoint system

To formulate the optimal control problem for the fractional-order TB model, we define the Hamiltonian function $\mathscr{H}$ by combining the objective functional and the state dynamics using the adjoint (co-state) variables λ_i:

$$\mathscr{H}(X,\lambda,\phi,t) = M(X,\lambda,\phi,t) + \sum_{i=1}^{4} \lambda_i^{\top}(t) V_i(X,\phi,t),$$

where $X(t) = (S,E,I,T)^{\top}$ represents the state variables, ϕ is the control variable and λ_i are the adjoint variables. The adjoint system is obtained using the fractional Pontryagin's maximum principle.

Theorem 4. *There exist adjoint variables λ_i satisfying the adjoint system together with terminal conditions $\lambda_i(t_f) = 0$. Moreover, the optimal control ϕ^* is given by*

$$\phi^*(t) = \min\left\{1, \max\left\{0, \frac{\beta pcI^*(t)E^*(t)(\lambda_3(t) - \lambda_2(t))}{2AN}\right\}\right\}.$$

Proof. We apply Pontryagin's maximum principle for systems involving Caputo fractional derivatives to obtain the necessary optimality conditions for the fractional-order control problem. The Hamiltonian is given by

$$\begin{aligned}
\mathscr{H}(X,\phi,\lambda) = I + A\phi^2 &+ \lambda_1\left(\Lambda - \mu S - \frac{\beta cIS}{N}\right) \\
&+ \lambda_2\left(\frac{\beta cIS}{N} - \frac{\beta pcI(1-\phi)E}{N} - (\mu+k)E + \frac{\beta\sigma cIT}{N}\right) \\
&+ \lambda_3\left(\frac{\beta pcI(1-\phi)E}{N} + kE - (\mu+r+d)I\right) \\
&+ \lambda_4\left(rI - \mu T - \frac{\beta\sigma cIT}{N}\right).
\end{aligned}$$

According to the fractional Pontryagin framework, each adjoint variable satisfies a right-sided Caputo fractional differential equation of the form

$${}_{t}^{C}D_{t_f}^{\alpha}\lambda_i(t) = -\frac{\partial \mathscr{H}}{\partial X_i}, \quad \lambda_i(t_f) = 0, \quad i = 1,2,3,4.$$

These arise because the terminal state is free and the performance index contains no terminal cost.

To obtain the explicit adjoint system, we compute derivatives of the Hamiltonian with respect to the state variables. Substituting these derivatives into the adjoint equations leads to the following system:

$$\begin{aligned}
{}_{t}^{C}D_{t_f}^{\alpha}\lambda_1 &= \lambda_1\left(\mu + \frac{\beta cI}{N}\right) - \lambda_2\frac{\beta cI}{N}, \\
{}_{t}^{C}D_{t_f}^{\alpha}\lambda_2 &= \lambda_2\left(\mu + k + \frac{\beta pcI(1-\phi)}{N}\right) - \lambda_3\frac{\beta pcI(1-\phi)}{N}, \\
{}_{t}^{C}D_{t_f}^{\alpha}\lambda_3 &= -1 + \lambda_1\frac{\beta cS}{N} - \lambda_2\left(\frac{\beta cS}{N} - \frac{\beta pc(1-\phi)E}{N} - \frac{\beta\sigma cT}{N}\right)
\end{aligned}$$

$$+\lambda_3(\mu+r+d)-\lambda_4\left(r-\frac{\beta\sigma cT}{N}\right),$$

$$ {}_{t}^{C}D_{t_f}^{\alpha}\lambda_4=-\lambda_2\frac{\beta\sigma cI}{N}+\lambda_4\left(\mu+\frac{\beta\sigma cI}{N}\right),$$

The optimal control ϕ^* is obtained by minimizing the Hamiltonian with respect to ϕ

$$\frac{\partial\mathscr{H}}{\partial\phi}=0 \quad \Longrightarrow \quad 2A\phi-\frac{\beta pcIE}{N}(\lambda_3-\lambda_2)=0.$$

Solving for the control gives

$$\phi(t)=\frac{\beta pcIE(\lambda_3-\lambda_2)}{2AN}.$$

Since the control is constrained to the admissible set $0\leq\phi(t)\leq 1$, we project this expression onto the interval $[0,1]$. Therefore, the optimal control takes the form

$$\phi^*(t)=\min\left\{1,\max\left\{0,\frac{\beta pcI^*(t)E^*(t)(\lambda_3(t)-\lambda_2(t))}{2AN}\right\}\right\}.$$

Consequently, the coupled system consisting of the state equations, the adjoint system, the optimality condition, and the control constraints provides a complete set of necessary conditions for the optimal fractional control of the TB model. This completes the proof. □

4 Numerical simulations

Lenhart and Workman [15] introduced the forward-backward sweep method for solving optimal control problems in biological systems. In this study, we adapt this approach to the fractional-order TB model by combining the forward–backward sweep framework with a fractional Euler discretization for Caputo fractional derivatives.

Numerical scheme. The optimality system is composed of the fractional-order state equations, the right-sided fractional-order adjoint equations, and the optimality condition for the control variable. To numerically approximate the Caputo fractional derivative

$$ {}_{0}^{C}D_{t}^{\alpha}x(t)=f(t,x(t)),$$

we employ a fractional Euler forward scheme of the form

$$x_{n+1}=x_n+\frac{h^{\alpha}}{\Gamma(\alpha+1)}f(t_n,x_n),$$

where h denotes the time-step size and $\Gamma(\cdot)$ represents the Gamma function. For $\alpha=1$, this formulation reduces to the classical Euler method.

The fractional forward–backward sweep algorithm has the following steps:

(i) Initialize the control function $\phi^{(0)}(t)$ over the interval $[0,t_f]$.

(ii) **Forward sweep:** Solve the fractional-order state system forward in time using the fractional Euler discretization

$$X_{n+1} = X_n + \frac{h^\alpha}{\Gamma(\alpha+1)} V(X_n, \phi_n, t_n).$$

(iii) **Backward sweep:** Solve the fractional-order adjoint system backward in time according to

$$\lambda_{n-1} = \lambda_n - \frac{h^\alpha}{\Gamma(\alpha+1)} \frac{\partial H}{\partial X}(X_n, \lambda_n, \phi_n, t_n).$$

(iv) **Control update:** Update the control using the characterization

$$\phi^{(m+1)}(t) = \min\left\{1, \max\left\{0, \frac{\beta p c I^{(m)}(t) E^{(m)}(t)}{2AN^{(m)}(t)} (\lambda_3^{(m)}(t) - \lambda_2^{(m)}(t))\right\}\right\}.$$

(v) Iterate steps (ii)–(iv) until the convergence criterion

$$\|\phi^{(m+1)} - \phi^{(m)}\|_\infty < 10^{-6}$$

is satisfied.

This procedure yields a numerical approximation of the coupled state–adjoint–control system for different fractional orders α.

The simulations are performed on the time interval $[0, t_f]$ with $t_f = 1$ year and time step $h = 8.333 \times 10^{-5}$. The initial conditions are

$$S_0 = 7600, \quad E_0 = 3800, \quad I_0 = 500, \quad T_0 = 100,$$

with total population $N_0 = 12000$. The parameter values are chosen as

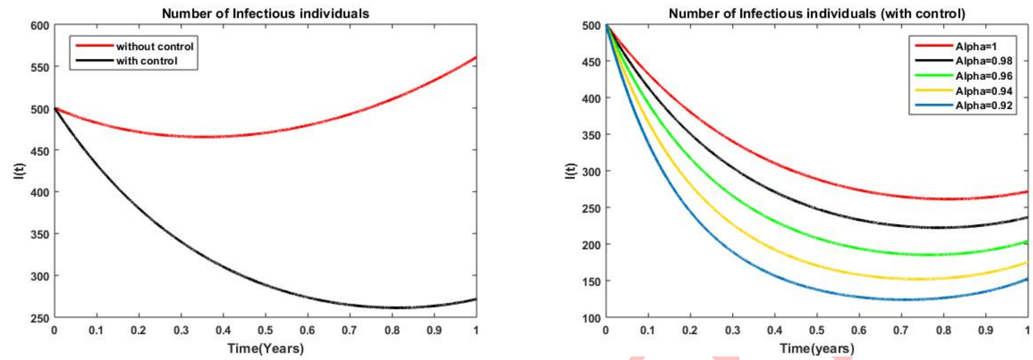
$$A = 400,\ \Lambda = 192,\ \mu = 0.016,\ \beta = 13,\ c = 1,\ p = 0.4,\ k = 0.005,\ \sigma = 0.9,\ d = 0.1,\ r = 2.$$

These parameter values are consistent with those reported in the TB modeling literature [10, 14, 16, 18]. The control weight A balances intervention cost and infection reduction as commonly adopted in optimal control formulations [20]. The control variable and model parameters are summarized in Table 1 and Table 2, respectively.

Figure 1 illustrates the temporal evolution of the infected population under optimal control for different fractional orders α. For fractional orders $\alpha < 1$, the infected population declines more rapidly and remains at lower levels.

This enhanced performance arises from the nonlocal memory property of the Caputo fractional derivative, whereby the current rate of change depends on the entire history of the infection process. The fractional-order dynamics effectively incorporate long-term memory effects, leading to smoother transitions and stronger attenuation of infection peaks compared to the integer-order case.

To highlight the advantage of the fractional-order optimal control approach, we compare the results obtained for the integer-order case ($\alpha = 1$) with several fractional orders ($\alpha < 1$). The simulations indicate that fractional-order dynamics result in a faster and more sustained reduction in the infectious population. This improvement can be attributed to the memory effect embedded in fractional derivatives, which allows prior states to impact its current dynamics. Hence, the proposed method offers improved control efficiency compared to classical integer-order optimal control models.



(a) **(b)**

Figure 1: (a) Plot of infected with and without control. (b) Number of Infectious individuals under fractional-order optimal control with various values of fractional parameter

5 Conclusion

In this work, we formulated and analyzed a fractional-order TB model incorporating exogenous reinfection and an optimal control strategy. The model was constructed using the Caputo fractional derivative to account for the memory effects inherent in the dynamics of TB transmission, particularly prolonged latency and repeated exposure.

We established the positivity and boundedness of solutions, ensuring the biological feasibility of the model. The existence of an optimal control was proved, and the corresponding fractional Pontryagin optimality system was derived. The resulting coupled state–adjoint system was solved numerically. Numerical simulations demonstrate that fractional-order dynamics ($\alpha < 1$) produce a more sustained reduction in the infectious population compared to the classical integer-order case ($\alpha = 1$). This behavior is attributed to the nonlocal memory structure of fractional derivatives, which incorporates past disease states into present transmission dynamics. Consequently, the fractional-order optimal control framework provides enhanced flexibility in modeling long-term epidemiological effects.

The present study assumes homogeneous mixing and a single TB strain. Future work may extend the model to incorporate multi-strain dynamics, drug resistance, spatial heterogeneity, time delays, or alternative fractional operators such as Atangana–Baleanu or Caputo–Fabrizio fractional derivatives. From a public health perspective, the findings emphasize the importance of persistent and well-timed intervention strategies in high reinfection settings, where memory effects may significantly influence disease dynamics.

Conflict of Interest

The authors declare no competing financial interests or personal relationships that could have influenced this work.

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