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Lyapunov-type inequalities for nonlinear fractional differential equations involving Caputo-type operators

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Abstract. This manuscript is devoted to deriving Lyapunov-type inequalities through a new approach for nonlinear fractional differential problems involving Caputo fractional operators subject to two boundary conditions. The problem under consideration includes multiple ψ-Laplacian operators of the form

$$^{C}D^{\alpha}_{\eta^+}\left[\psi_2\left(\frac{d}{dx}(\psi_1\left(\frac{d}{dx}w\right))\right)\right]+p(x)g(w)=0,$$

where ψ_2 and ψ_1 are odd and increasing functions, ψ_1 is submultiplicative, $1/\psi_1$ is convex, and g is continuous. Our results utilize p_+ and p_- instead of $|p|$, which is commonly used in most existing results in the literature.

Keywords: Riemann-Liouville fractional operator, Caputo derivative, Lyapunov-type inequality, convex function.
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1 Introduction

Nowadays, inequalities and their applications play a crucial role in the mathematical sciences. Among these, Lyapunov-type inequalities have attracted considerable attention. For instance, these inequalities have important applications in classical analysis and in broader areas of mathematics.

In 1892, Lyapunov-type inequalities were studied and investigated, and the results were presented as follows [27]:
Let $p:[a,b]\longrightarrow\mathbb{R}$ be a continuous function. The following differential problem is known as the Hill differential model:

$$\begin{cases} y''+p(x)y(x)=0, & x\in(a,b)\\ y(a)=y(b)=0, \end{cases} \tag{1}$$

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and under the conditions imposed in model (1), the function given p satisfies the following inequality:

$$\int_a^b |p(z)|dz > 4(b-a)^{-1}. \tag{2}$$

The inequality expressed in Eq. (2) is sharp when a relatively large number is not replaced instead of "4". Further in Eq. (1), if the function $p(x)$ is a function and this function equals zero at two separate points on the interval $[a, b]$, thus $p^+(x) = \max\{0, p(x)\}$ which is the non-negative part of $p(x)$, satisfies:

$$\int_a^b p^+(z)dz > 4(b-a)^{-1}. \tag{3}$$

Other important roles of Lyapunov inequalities can be found in oscillation theory, disconjugacy, impulsive differential models, and Hamiltonian systems. Juan P. Pinasco in [33] displayed the eigenvalue problems for differential models. Sui-Sun Cheng and Aydın Tiryaki in the Refs. [12,40] studied the Lyapunov inequalities for differential models. Authors Chen [10], Cheng [11], Hochstadt [23], Kwong [26], Nehari [30,31], Reid [35,36], and Singh [39], studied the Lyapunov inequality.
In recent years, numerous studies have been devoted to the improvement and development of Lyapunov inequalities. Although we will not present them here, some can be found in the references [1–5, 7–9, 13, 14, 18, 19, 21, 24, 37].

Yang [41] investigated the Lyapunov inequality for a second-order differential problem subject to the following boundary condition:

$$(\varphi_\beta(w'))' + p(x)\varphi_\beta(w) = 0,\ \beta > 0 \tag{4}$$

in which $\varphi_\beta(w) = |w|^{\beta-1}w$ and $w(x) = 0$ for $x \in \partial([a,b])$. The inequality associated with Eq. (4) can be written as follows:

$$\int_a^b p^+(x)dx > 2(2^{-1}(b-a))^{-\beta}. \tag{5}$$

By substituting β into Eq. (5), we obtain Eq. (3).
In 2022, Behrens [6] derived Lyapunov inequalities for the third-order nonlinear equations:

$$\frac{d}{dx}\Big[\psi_2\Big(\frac{d}{dx}\big(\psi_1(\frac{d}{dx}w)\big)\Big)\Big] + p(x)g(w) = 0 \tag{6}$$

subject to one of the following two boundary condition:

$$w(a) = w(b) = 0,\ w(x) \neq 0,\ x \in (a,b)\ \frac{d}{dx}\big(\psi_1(\frac{d}{dx}w)\big)(\xi) = 0, \text{ for some } \ \xi \in [a,b] \tag{7}$$

and

$$w(a) = w(b) = w(c) = 0,\ w(x) \neq 0,\ x \in (a,b) \cup (b,c). \tag{8}$$

This naturally raises the question of whether these results can be extended to fractional-order equations. While several researchers have studied various differential equations of different fractional orders under diverse boundary conditions and obtained notable results on Lyapunov inequalities [15, 20, 22, 28, 29]. For instance, Ferreira [19, 21] computed Lyapunov inequality for a fractional system.

Theorem 1. *([21]) Let z be a solution of*

$$\begin{cases} {}^{C}D_{a^+}^{\alpha} z(t) + q(x)z(t) = 0, \quad t \in [a,\, b] \\ z(a) = z(b) = 0, \end{cases} \tag{9}$$

in which $a < b$, $1 < \alpha \leq 2$ and $q \in C([a,\, b], \mathbb{R})$. If $z(x) \neq 0$ for all $x \in (a,\, b)$, then

$$\int_a^b \Big| q(t) \Big| dt > \frac{\Gamma(\alpha)\alpha^{\alpha}}{[(\alpha-1)(b-a)]^{\alpha-1}} \tag{10}$$

holds, that ${}^{C}D_{a^+}^{\alpha}$ is defined in the next section.

Due to the model determined in Eq. (6) for the classical case, we consider its non-linear fractional model including the Caputo fractional operator as follows:

$${}^{C}D_{\eta^+}^{\alpha} \Big[\psi_2 \Big(\frac{d}{dx} \big(\psi_1 (\frac{d}{dx} w) \big) \Big) \Big] + p(x) g(w) = 0 \tag{11}$$

subject to the two boundary condition

$$w(a) = w(b) = 0,\, w(x) \neq 0,\, x \in (a,\, b),\; \psi_2'(0) = 0,\; \Big(\psi_1 (w') \Big)' (\eta) = 0 \text{ for some } \eta \in [a,\, b] \tag{12}$$

and

$$w(a) = w'(a) = w(b) = w'(b) = w(c) = 0,\; w(x) \neq 0, x \in (a,b) \cup (b,c). \tag{13}$$

Definition 1. *A map $\theta : V \to [0,\infty]$ is said a sub-multiplicative map if the inequality $\theta(zy) \leq \theta(z)\theta(y)$ holds for every $z, y \in V$. Further, the map $\theta : V \to [0,\infty]$ is said a super-multiplicative map if the inequality $\theta(zy) \geq \theta(z)\theta(y)$ holds for every $z, y \in V$.*

Throughout this paper, unless explicitly mentioned otherwise, we will assume the following:

(A1) ψ_1 and ψ_2 are odd and increasing functions.

(A2) $\psi_1(s)$ is sub-multiplicative on $[0,\, \infty)$ and $\frac{1}{\psi_1(s)}$ is a convex function for $s > 0$.

(A3) $\psi_2(s)$ is super-multiplicative on $[0,\, \infty)$

(A4) $p \in C\big([a,\, b],\, \mathbb{R}\big)$ and can change the sign in $[a,\, b]$.

(A5) $g : \mathbb{R} \to \mathbb{R}$ is an odd continuous function and satisfies $xg(x) > 0$ for $x \neq 0$.

The outline of this manuscript is displayed as follows. In Section 2, we state some important definitions and lemmas in fractional calculus that we need to prove our main results. In Section 3, we will display the Lyapunov inequalities for Eq. (11) under the conditions (12) and (13).

2 Preliminaries

In this section, we present several key notations, definitions, and lemmas that will be utilized in the subsequent sections.

Definition 2. *A function* $f : [a,\ b] \to \mathbb{R}$ *is absolutely continuous on* $[a, b]$ *if for every* $\varepsilon > 0$ *there exists* $\delta > 0$ *such that for any finite collection of pairwise disjoint subintervals* $(a_k, b_k) \subset [a, b]$ *with* $\sum(b_k - a_k) < \delta$*, we have*

$$\sum_k |f(b_k) - f(a_k)| < \varepsilon.$$

Let $f^{(n)}$ be the nth derivative of a real-valued function f on $[a,\ b]$, which is assumed to exist and $AC[a,\ b]$ denotes the space of absolutely continuous functions on the interval [a, b]. Define $\|f\| = \max_{a \le x \le b} f(x)$. Let $AC^n[a,\ b]$ denote the class of functions $h : [a,\ b] \to \mathbb{R}$ such that $h^{(n-1)} \in AC[a,\ b]$. Furthermore,

$$L_p[a,\ b] = \left\{ h : [a,\ b] \to \mathbb{R}\ :\ \left(\int_a^b |h(t)|^p\, dt \right)^{\frac{1}{p}} < \infty \right\}, \quad 1 \le p < \infty.$$

A set $K \subseteq \mathbb{R}^n$ is a convex set if, for all values $x, y \in K$ and $\mu \in [0,\ 1]$, we have $\mu x + (1-\mu)y \in K$. Let $K \subseteq \mathbb{R}^n$ that $K \neq \emptyset$. A function $h : K \to \mathbb{R}$ is convex if for every $x, y \in K$ and $\mu \in [0,\ 1]$, $h(\mu x + (1-\mu)y) \le \mu h(x) + (1-\mu)h(y)$. Also, $h : K \to \mathbb{R}$ is called α-convex if $h(\mu^\alpha x + (1-\mu)^\alpha y) \le \mu^\alpha h(x) + (1-\mu)^\alpha h(y)$ for all $\mu \in [0,\ 1]$. If we change the direction of the inequality, then h is called an α-concave function.

In recent years, considerable attention in mathematical sciences and their applications has been devoted to fractional calculus [25, 32, 34, 38].

Definition 3. *([38])* The Riemann-Liouville fractional integral $I^{\beta}_{a^+} w(z)$ is given as:

$$I^{\beta}_{a^+} w(z) = \Gamma(\beta)^{-1} \int_a^z w(\nu)(z-\nu)^{\beta-1} d\nu \tag{14}$$

in which $m - 1 < \beta \le m$, $m \in \{1, 2, \cdots\}$.

Definition 4. *The Riemann–Liouville fractional differential operator of order* β *is defined by*

$$D^{\beta}_{a} f(z) = D^m \left(I^{\beta - m} f(z) \right) \tag{15}$$

in which $m = [\beta]$.

Definition 5. *([34]) The Caputo fractional derivative of* ω *is displayed by*

$${}^{C}D^{\beta}_{a^+} w(z) = I^{m-\beta}_{a^+} w^m(z) \tag{16}$$

in which $\beta \in (m-1, m]$, $m \in \{1, 2, \cdots\}$.

Lemma 1. *([38]) Let $\beta \in (m-1, m]$, that $m \in N$. Also, let $w(z) \in L_1[a, b]$, then*

$$I_{a^+}^{\beta}\left({}^{C}D_{a^+}^{\beta} w(z)\right) = w(z) - \sum_{k=0}^{m-1} \frac{w^{(k)}(a)(z-a)^k}{k!}. \tag{17}$$

Also, if ${}^{C}D_{a^+}^{\beta} w(z) = 0$, then

$$w(z) = \sum_{k=0}^{m-1} \frac{w^{(k)}(a)(z-a)^k}{k!}. \tag{18}$$

It should be noted that $I_{a^+}^{\beta}\left(z-a\right)^{\nu} = \frac{\Gamma(\nu+1)}{\Gamma(\nu+\beta+1)}\left(z-a\right)^{\nu+\beta}$ and ${}^{C}D_{a^+}^{\beta}\, c = 0$, where c is a constant.

Lemma 2. *([25]) Let $\nu \in \mathbb{R}_+$ and $p \geq 1$, then the fractional integration operator $I_{a^+}^{\nu}$ and $I_{b^-}^{\nu}$ are bounded in $L_p(a,b)$, that is*

$$\left\|I_{a^+}^{\nu} h\right\|_p \leq C\|h\|_p \text{ and } \left\|I_{b^-}^{\nu} h\right\|_p \leq C\|h\|_p, \qquad \left(C = \frac{(b-a)^{\nu}}{\Gamma(\nu+1)}\right). \tag{19}$$

Theorem 2. *([16, 17]) Let $\beta > 0$ and assume $g \in C^{\lceil\beta\rceil-1}[a,b]$, so that $D_{*a}^{\beta} g \in C[a,b]$. Then*

$$\frac{g(b) - T_{\lceil\beta\rceil-1}[g;a](b)}{(b-a)^{\beta}} = \frac{1}{\Gamma(\beta)} D_{*a}^{\beta} g(\xi), \exists a < \xi < b, \tag{20}$$

where $D_{*a}^{\beta} g(z) = D^{\beta}[g(z) - T_{[\beta]-1}[g;a](z)]$ and

$$T_{[\beta]-1}[g;a](z) = g(a) + \sum_{j=1}^{[\beta]-1} \frac{g^j(a)}{j!}(z-a)^j.$$

Proposition 1. *Let $h : [0, \infty) \to \mathbb{R}$ be a concave function and $h(0) \geq 0$, then h is subadditive, that is*

(i) $h(tz) \geq th(z),\ for\ all\ z \in [0, \infty)\ and\ t \in [0, 1]$,

(ii) $h(y_1 + y_2) \leq h(y_1) + h(y_2),\ for\ all\ y_1,\ y_2 \in [0, \infty)$.

Proof. Since h is concave on $[0, \infty)$ and $h(0) \geq 0$, for any $z \in [0, \infty)$ and any $t \in [0, 1]$, we have

$$h(tz) = h\big(tz + (1-t)0\big) \geq t\,h(z) + (1-t)h(0) \geq t\,h(z)$$

which proves part (i).

Now let $y_1, y_2 \in [0, \infty)$. If $y_1 + y_2 = 0$, then $y_1 = y_2 = 0$, and there is nothing to prove. Assume that $y_1 + y_2 > 0$, and set

$$t = \frac{y_1}{y_1 + y_2} \in [0, 1].$$

Then

$$y_1 = t(y_1 + y_2), \qquad y_2 = (1-t)(y_1 + y_2).$$

Applying (i) with $z = y_1 + y_2$, we obtain

$$h(y_1) = h\big(t(y_1 + y_2)\big) \geq t\,h(y_1 + y_2)$$

and similarly,

$$h(y_2) = h\big((1-t)(y_1+y_2)\big) \geq (1-t)\,h(y_1+y_2).$$

Adding the above inequalities, we get

$$h(y_1)+h(y_2) \geq \big(t+(1-t)\big)h(y_1+y_2) = h(y_1+y_2).$$

Therefore,

$$h(y_1+y_2) \leq h(y_1)+h(y_2)$$

which proves part (ii). $\square$

3 Main results

In this section, we state key results concerning Lyapunov inequalities in the form of theorems.

Theorem 3. *Suppose that $1<\alpha\leq 2$. If (12) holds for some $\eta\in[a,b]$, then there is a constant as $a<c<b$ that*

$$\frac{\frac{2}{(b-a)^\alpha}\psi_1(w(c))}{\psi_1(\frac{b-a}{2})} \leq \frac{(\eta-a)^\alpha}{(b-a)^\alpha}\psi_2^{-1}\Big\{\frac{1}{\Gamma(\alpha)}\int_a^\eta p_-(\nu)g(\omega(\nu))d\nu\Big\} + \frac{(b-\eta)^\alpha}{(b-a)^\alpha}\psi_2^{-1}\Big\{\frac{1}{\Gamma(\alpha)}\int_\eta^b p_+(\nu)g(\omega(\nu))d\nu\Big\} \tag{21}$$

and as a result, the following inequality is obtained from (21):

$$\psi_2\Big(\frac{\frac{2}{(b-a)^\alpha}}{\psi_1(\frac{b-a}{2})}\Big) \leq \frac{\|G\|}{(b-a)^\alpha\Gamma(\alpha)}\Big\{(\eta-a)^{\alpha+1}\|p_-\|+(b-\eta)^{\alpha+1}\|p_+\|\Big\} \tag{22}$$

in which

$$p_-(x) = \max_{x\in[a,b]}\Big\{-p(x),0\Big\},\ \ p_+(x) = \max_{x\in[a,b]}\Big\{p(x),0\Big\},\ \text{and } G(w) = \frac{g(w)}{\psi_2(\psi_1(w(c)))}. \tag{23}$$

Proof. Assume that for all $a<x<b$, we have $w(x)>0$. Since $w(x)$ satisfy $w(a)=w(b)=0$, then there is $c\in(a,b)$ that $w(c)=\max_{x\in[a,b]}w(x)$. Using the mean value theorem for w we obtain

$$w'(s_1) = \frac{w(c)-w(a)}{c-a} = \frac{w(c)}{c-a} > 0,\ \exists\, s_1\in(a,c) \tag{24}$$

and

$$w'(s_2) = \frac{w(b)-w(c)}{b-c} = \frac{-w(c)}{b-c} < 0,\ \exists\, s_2\in(c,b). \tag{25}$$

Since $\frac{1}{\psi_1(x)}$ is a convex function, we obtain

$$\frac{2}{\psi_1(\frac{b-a}{2})} \leq \frac{1}{\psi_1(c-a)}+\frac{1}{\psi_1(b-c)} = \frac{1}{\psi_1(w(c))}\Big[\frac{\psi_1(w(c))}{\psi_1(c-a)}+\frac{\psi_1(w(c))}{\psi_1(b-c)}\Big]. \tag{26}$$

Eq. (26) can be rewritten as

$$\frac{2\psi_1(w(c))}{\psi_1(\frac{b-a}{2})} \leq \frac{\psi_1\left((c-a)\frac{w(c)}{c-a}\right)}{\psi_1(c-a)} + \frac{\psi_1\left((b-c)\frac{w(c)}{b-c}\right)}{\psi_1(b-c)}. \tag{27}$$

Since ψ_1 is the sub-multiplicative, Eq. (27) yields

$$\frac{2\psi_1(w(c))}{\psi_1(\frac{b-a}{2})} \leq \psi_1\left(\frac{w(c)}{c-a}\right) + \psi_1\left(\frac{w(c)}{b-c}\right). \tag{28}$$

Applying Eqs. (24) and (25) in (28), then with the help of Definition 3 and under the assumptions of the theorem, it follows that

$$\frac{2\psi_1(w(c))}{\psi_1(\frac{b-a}{2})} \leq \psi_1(w'(s_1)) - \psi_1(w'(s_2)) = -\int_{s_1}^{s_2} \Big(\psi_1(w'(s))\Big)' ds. \tag{29}$$

Integrating (11) by Riemann-Liouville fractional integral of order α from η to x and using this fact, we have

$$I^{\alpha}_{\eta^+}\left\{{}^{C}D^{\alpha}_{\eta^+}\left[\psi_2\Big(\psi_1(w'(x))\Big)'\right] + I^{\alpha}_{\eta^+}\Big\{p(x)g(w)\Big\} = 0. \tag{30}$$

Using Lemma 1, Eq. (30) yields

$$\begin{aligned} \psi_2\Big(\psi_1(w'(x))\Big)' - \psi_2\Big(\psi_1(w'(\eta))\Big)' - \psi_2'\Big(\psi_1(w'(\eta))\Big)' + I^{\alpha}_{\eta^+}\Big\{p(x)g(w)\Big\} \\ = \psi_2\Big(\psi_1(w'(x))\Big)' - 0 + I^{\alpha}_{\eta^+}\Big\{p(x)g(w)\Big\} = 0. \end{aligned} \tag{31}$$

Hence, we get

$$\psi_2\Big(\psi_1(w'(x))\Big)' = -I^{\alpha}_{\eta^+}\Big\{p(x)g(w)\Big\}. \tag{32}$$

Since ψ_2 is odd and increasing, it follows that

$$\Big(\psi_1(w'(x))\Big)' = -\psi_2^{-1}\Big\{I^{\alpha}_{\eta^+}\Big(p(x)g(w)\Big)\Big\}. \tag{33}$$

Substituting (33) in Eq. (29), we get

$$\begin{aligned} \frac{2\psi_1(w(c))}{\psi_1(\frac{b-a}{2})} &\leq \int_{s_1}^{s_2} \psi_2^{-1}\Big\{I^{\alpha}_{\eta^+}\Big(p(s)g(w(s))\Big)\Big\} ds \\ &\leq \int_{a}^{b} \psi_2^{-1}\Big\{I^{\alpha}_{\eta^+}\Big(p(s)g(w(s))\Big)\Big\} ds \\ &= \int_{a}^{\eta} \psi_2^{-1}\Big\{I^{\alpha}_{\eta^+}\Big(p(s)g(w(s))\Big)\Big\} ds + \int_{\eta}^{b} \psi_2^{-1}\Big\{I^{\alpha}_{\eta^+}\Big(p(s)g(w(s))\Big)\Big\} ds \\ &= \int_{a}^{\eta} \psi_2^{-1}\Big\{\frac{1}{\Gamma(\alpha)}\int_{s}^{\eta} -p(\nu)(s-\nu)^{\alpha-1} g(\omega(\nu)) d\nu\Big\} ds \end{aligned}$$

$$+\int_{\eta}^{b}\psi_2^{-1}\Big\{\frac{1}{\Gamma(\alpha)}\int_{\eta}^{s}(s-\nu)^{\alpha-1}p(\nu)g(\omega(\nu))d\nu\Big\}ds. \tag{34}$$

Then, due to the increasing of ψ_2, from Eq. (34) we have

$$\begin{aligned}\frac{2\psi_1(w(c))}{\psi_1(\frac{b-a}{2})} \leq & \int_{a}^{\eta}\psi_2^{-1}\Big\{\frac{1}{\Gamma(\alpha)}\int_{a}^{\eta}p_-(\nu)(s-\nu)^{\alpha-1}g(\omega(\nu))d\nu\Big\}ds\\ &+\int_{\eta}^{b}\psi_2^{-1}\Big\{\frac{1}{\Gamma(\alpha)}\int_{\eta}^{b}(s-\nu)^{\alpha-1}p_+(\nu)g(\omega(\nu))d\nu\Big\}ds.\end{aligned} \tag{35}$$

It is worth noting that, in the first integral mentioned above, we have $a \leq \nu \leq s \leq \eta$ and $1 < \alpha \leq 2$. Hence, it is clear that $f(x) = x^{\alpha-1}$ is an increasing function. Thus, $f(s-\nu) = (s-\nu)^{\alpha-1} \leq (\eta-a)^{\alpha-1} = f(\eta-a)$. In the second above integral, since $a \leq \eta \leq \nu \leq s \leq b$, $f(s-\nu) = (s-\nu)^{\alpha-1} \leq (b-\eta)^{\alpha-1} = f(b-\eta)$. Therefore

$$\begin{aligned}\frac{2\psi_1(w(c))}{\psi_1(\frac{b-a}{2})} \leq & (\eta-a)^{\alpha}\psi_2^{-1}\Big\{\frac{1}{\Gamma(\alpha)}\int_{a}^{\eta}p_-(\nu)g(\omega(\nu))d\nu\Big\}\\ &+(b-\eta)^{\alpha}\psi_2^{-1}\Big\{\frac{1}{\Gamma(\alpha)}\int_{\eta}^{b}p_+(\nu)g(\omega(\nu))d\nu\Big\}.\end{aligned} \tag{36}$$

Dividing both sides of Eq. (36) By $(b-a)^{\alpha}$, we obtain

$$\begin{aligned}\frac{\frac{2}{(b-a)^{\alpha}}\psi_1(w(c))}{\psi_1(\frac{b-a}{2})} \leq & \frac{(\eta-a)^{\alpha}}{(b-a)^{\alpha}}\psi_2^{-1}\Big\{\frac{1}{\Gamma(\alpha)}\int_{a}^{\eta}p_-(\nu)g(\omega(\nu))d\nu\Big\}\\ &+\frac{(b-\eta)^{\alpha}}{(b-a)^{\alpha}}\psi_2^{-1}\Big\{\frac{1}{\Gamma(\alpha)}\int_{\eta}^{b}p_+(\nu)g(\omega(\nu))d\nu\Big\}.\end{aligned} \tag{37}$$

Thus, inequality (21) has been obtained. To obtain inequality (22), we use the properties of the function ψ_2. We consider two cases based on the properties of ψ_2.
Case 1: Assume ψ_2 is α - convex. Applying ψ_2 to (37), we have

$$\begin{aligned}\psi_2\Big(\frac{\frac{2}{(b-a)^{\alpha}}\psi_1(w(c))}{\psi_1(\frac{b-a}{2})}\Big) \leq & \psi_2\Big(\Big(\frac{\eta-a}{b-a}\Big)^{\alpha}\psi_2^{-1}\Big(\frac{1}{\Gamma(\alpha)}\int_{a}^{\eta}p_-(\nu)g(\omega(\nu))d\nu\Big)\\ &+\Big(\frac{b-\eta}{b-a}\Big)^{\alpha}\psi_2^{-1}\Big(\frac{1}{\Gamma(\alpha)}\int_{\eta}^{b}p_+(\nu)g(\omega(\nu))d\nu\Big)\Big).\end{aligned} \tag{38}$$

It should be noted that $0 \leq \frac{\eta-a}{b-a} \leq 1,\ \ 0 \leq \frac{b-\eta}{b-a} \leq 1$ and $\frac{\eta-a}{b-a}+\frac{b-\eta}{b-a} = 1$. We have

$$\begin{aligned}\psi_2\Big(\frac{\frac{2}{(b-a)^{\alpha}}\psi_1(w(c))}{\psi_1(\frac{b-a}{2})}\Big) \leq & \Big(\frac{\eta-a}{b-a}\Big)^{\alpha}\frac{1}{\Gamma(\alpha)}\int_{a}^{\eta}p_-(\nu)g(\omega(\nu))d\nu\\ &+\Big(\frac{b-\eta}{b-a}\Big)^{\alpha}\frac{1}{\Gamma(\alpha)}\int_{\eta}^{b}p_+(\nu)g(\omega(\nu))d\nu.\end{aligned} \tag{39}$$

Case 2: Assume ψ_2 is α - concave function over the interval $[0,\infty)$, Therefore, using Proposition 1 and applying ψ_2, from (37) we obtain

$$\psi_2\Big(\frac{\frac{2}{(b-a)^{\alpha}}\psi_1(w(c))}{\psi_1(\frac{b-a}{2})}\Big) \leq \frac{1}{\Gamma(\alpha)}\Big\{\Big(\frac{\eta-a}{b-a}\Big)^{\alpha}\int_{a}^{\eta}p_-(\nu)g(\omega(\nu))d\nu$$

$$+\Big(\frac{b-\eta}{b-a}\Big)^{\alpha}\int_{\eta}^{b} p_{+}(\nu)g(\omega(\nu))d\nu\Big\}. \tag{40}$$

In each case, we have

$$\psi_2\Big(\frac{\frac{2}{(b-a)^{\alpha}}\psi_1(w(c))}{\psi_1(\frac{b-a}{2})}\Big) \le \frac{1}{\Gamma(\alpha)}\Big\{\Big(\frac{\eta-a}{b-a}\Big)^{\alpha}\int_{a}^{\eta} p_{-}(\nu)g(\omega(\nu))d\nu$$
$$+\Big(\frac{b-\eta}{b-a}\Big)^{\alpha}\int_{\eta}^{b} p_{+}(\nu)g(\omega(\nu))d\nu\Big\} \tag{41}$$

which is equivalent to inequality (39). Since ψ_2 is the super-multiplicative, we have

$$\psi_2(\psi_1(w(c)))\psi_2\Big(\frac{\frac{2}{(b-a)^{\alpha}}}{\psi_1(\frac{b-a}{2})}\Big) \le \Big(\frac{\eta-a}{b-a}\Big)^{\alpha}\frac{1}{\Gamma(\alpha)}\int_{a}^{\eta} p_{-}(\nu)g(\omega(\nu))d\nu$$
$$+\Big(\frac{b-\eta}{b-a}\Big)^{\alpha}\frac{1}{\Gamma(\alpha)}\int_{\eta}^{b} p_{+}(\nu)g(\omega(\nu))d\nu. \tag{42}$$

Thus, Eq. (42) yields

$$\psi_2\Big(\frac{\frac{2}{(b-a)^{\alpha}}}{\psi_1(\frac{b-a}{2})}\Big) < \Big(\frac{\eta-a}{b-a}\Big)^{\alpha}\frac{1}{\Gamma(\alpha)}\int_{a}^{\eta} p_{-}(\nu)G(w)d\nu$$
$$+\Big(\frac{b-\eta}{b-a}\Big)^{\alpha}\frac{1}{\Gamma(\alpha)}\int_{\eta}^{b} p_{+}(\nu)G(w)d\nu \tag{43}$$

in which $G(w)=\frac{g(w)}{\psi_2(\psi_1(w(c)))}$. Hence, we get

$$\psi_2\Big(\frac{\frac{2}{(b-a)^{\alpha}}}{\psi_1(\frac{b-a}{2})}\Big) \le \Big(\frac{\eta-a}{b-a}\Big)^{\alpha}\frac{\|p_{-}\|\|G\|}{\Gamma(\alpha)}(\eta-a)$$
$$+\Big(\frac{b-\eta}{b-a}\Big)^{\alpha}\frac{\|p_{+}\|\|G\|}{\Gamma(\alpha)}(b-\eta)$$
$$=\frac{\|G\|}{(b-a)^{\alpha}\Gamma(\alpha)}\Big\{(\eta-a)^{\alpha+1}\|p_{-}\|+(b-\eta)^{\alpha+1}\|p_{+}\|\Big\}. \tag{44}$$

Therefore, the result is obtained. □

Example 1. As a direct consequence of Theorem 3, we present the following illustrative example. Let $a=0$, $b=1$ and choose $\alpha=\frac{3}{2}$. Define the functions

$$\psi_1(s)=s^2+1, \qquad \psi_2(s)=s^3.$$

Clearly, ψ_1 is positive, increasing and sub-multiplicative on $[0,\infty)$, while ψ_2 is odd, increasing and super-multiplicative on $\mathbb{R}$. Consider the function

$$w(x)=x(1-x)e^{-x}, \qquad x\in[0,1].$$

Then

$$w(0) = w(1) = 0, \qquad w(x) > 0 \quad \text{for } x \in (0,1),$$

and w attains its maximum at some point $c \in (0,1)$. Choose

$$p(x) = x - \tfrac{1}{2}, \qquad g(w) = 20e^{w},$$

and let $\eta = \frac{1}{2}$. The positive and negative parts of p are given by

$$p_-(x) = \max\{-p(x), 0\} = \begin{cases} \frac{1}{2} - x, & 0 \le x \le \frac{1}{2}, \\ 0, & \frac{1}{2} < x \le 1, \end{cases}$$

and

$$p_+(x) = \max\{p(x), 0\} = \begin{cases} 0, & 0 \le x < \frac{1}{2}, \\ x - \frac{1}{2}, & \frac{1}{2} \le x \le 1. \end{cases}$$

Since $\alpha = \frac{3}{2}$, we have

$$\Gamma(\alpha) = \Gamma\left(\tfrac{3}{2}\right) = \frac{\sqrt{\pi}}{2}.$$

Moreover, the nonlinear integrals appearing in Theorem 3 take the form

$$\int_0^{\eta} p_-(\nu) g(w(\nu))\, d\nu = \int_0^{1/2} \left(\tfrac{1}{2} - \nu\right) e^{\nu(1-\nu)e^{-\nu}}\, d\nu$$

and

$$\int_{\eta}^{1} p_+(\nu) g(w(\nu))\, d\nu = \int_{1/2}^{1} \left(\nu - \tfrac{1}{2}\right) e^{\nu(1-\nu)e^{-\nu}}\, d\nu.$$

Define

$$G(w) = \frac{g(w)}{\psi_2(\psi_1(w(c)))}.$$

Since w is continuous on $[0,1]$, it follows that $\|G\| < \infty$. Furthermore,

$$\|p_-\| = \|p_+\| = \tfrac{1}{2}.$$

Therefore, all the hypotheses of Theorem 3 are satisfied. Consequently, there exists $c \in (0,1)$ such that inequality (22) holds, that is

$$\psi_2\left(\frac{\frac{2}{(b-a)^{\alpha}}}{\psi_1\left(\frac{b-a}{2}\right)}\right) \le \frac{\|G\|}{(b-a)^{\alpha}\Gamma(\alpha)} \left\{(\eta - a)^{\alpha+1}\|p_-\| + (b-\eta)^{\alpha+1}\|p_+\|\right\}.$$

For the present choice of parameters, this inequality reduces to

$$\left(\frac{2}{\psi_1\left(\frac{1}{2}\right)}\right)^3 \le \frac{\left(\frac{1}{2}\right)^{5/2}}{\Gamma\left(\frac{3}{2}\right)} \|G\|.$$

Now, to calculate $\|G\|$, we must first determine the value of $w(c)$, where $c \in (0,1)$ is the point at which the function w attains its maximum. By setting $w'(x) = 0$, we find

$$x = \frac{3-\sqrt{5}}{2}.$$

Since $x \in [0,1]$, we have

$$c = \frac{3-\sqrt{5}}{2} \approx 0.381966$$

and consequently, $w(c) = c(1-c)e^{-c} \in (0,1)$, for $c \in (0,1)$.

Therefore, the function G can be written as:

$$G(w(x)) = \frac{g(w(x))}{\psi_2(\psi_1(w(c)))} = \frac{20e^{w(x)}}{((w(c))^2+1)^3}.$$

Thus, the norm of G is obtained by

$$\|G\| = \sup_{x\in[0,1]} |G(w(x))| = \sup_{\substack{x\in[0,1]\\ c\in(0,1)}} \left| \frac{20e^{w(x)}}{((w(c))^2+1)^3} \right| = \frac{20e^{w(c)}}{((w(0))^2+1)^3} \approx 23.497$$

which satisfies the following inequality:

$$23.497 \geq 20.534.$$

This example shows that the conclusion of Theorem 3 is valid for nonlinear fractional models with exponential nonlinearity and sign-changing coefficients, thereby confirming the effectiveness and applicability of the theoretical result.

Theorem 4. *Suppose $\omega(x)$ for Eq. (11) be a nontrivial solution that $w(a) = w'(a) = w(b) = w'(b) = w(c) = 0$, $x \in (a,\ b) \cup (b,\ c)$, and $\alpha \in (1,2]$. Then either*

$$\psi_2\left(\frac{\frac{2}{(b-a)^\alpha}}{\psi_1(\frac{b-a}{2})}\right) \leq \frac{\|G\|}{(b-a)^\alpha \Gamma(\alpha)} \left\{ (\eta-a)^{\alpha+1}\|p_-\| + (b-\eta)^{\alpha+1}\|p_+\| \right\} \tag{45}$$

or

$$\psi_2\left(\frac{\frac{2}{(c-b)^\alpha}}{\psi_1(\frac{c-b}{2})}\right) \leq \frac{\|G\|}{(c-b)^\alpha \Gamma(\alpha)} \left\{ (\eta-b)^{\alpha+1}\|p_-\| + (c-\eta)^{\alpha+1}\|p_+\| \right\}. \tag{46}$$

Also from Eqs. (45) and (46), we have

$$\psi_2\left(\frac{\frac{2}{(c-a)^\alpha}}{\psi_1(\frac{c-a}{2})}\right) \leq \frac{\|G\|}{(c-a)^\alpha \Gamma(\alpha)} \left\{ (\eta-a)^{\alpha+1}\|p_-\| + (c-\eta)^{\alpha+1}\|p_+\| \right\} \tag{47}$$

in which $G(w) = \frac{g(w)}{\psi_2(\psi_1(w))}$.

Proof. Since the function w is continuous and differentiable on the interval $x_1 \in [a,b]$ and $x_2 \in [b,c]$, by Rolle's theorem there exist points $a < x_1 < b$ and $b < x_2 < c$ such that

$$w'(x_1) = 0,\; w'(x_2) = 0. \tag{48}$$

Using Theorem 2 shows that there exists $\eta \in (x_1,\, x_2) \subset (a,c)$ such that

$$\begin{aligned}
\frac{1}{\Gamma(\alpha)}\,{}^{C}D^{\alpha}_{a^+}\Big(\psi_1\big(\frac{d}{dx}w\big)\Big)(\eta) &= \frac{\psi_1\Big(\frac{d}{dx}w\Big)(x_2) - T_{\lceil\alpha\rceil-1}\Big[\psi_1\big(\frac{d}{dx}\omega\big);x_1\Big](x_2)}{(x_2-x_1)^{\alpha}} \\
&= \frac{\psi_1(0) - T_1\Big[\psi_1\big(\frac{d}{dx}\omega\big);x_1\Big](x_2)}{(x_2-x_1)^{\alpha}} \\
&= \frac{\psi_1(0) - 0}{(x_2-x_1)^{\alpha}} = 0.
\end{aligned} \tag{49}$$

Hence, using Lemma 1 we have

$$\Big(\psi_1\big(\frac{d}{dx}w\big)\Big)(\eta) = \Big(\psi_1\big(\frac{d}{dx}\omega(a)\big)\Big) + \Big(\psi_1\big(\frac{d}{dx}w(a)\big)\Big)'(\eta - a)$$

and with assuming for $\omega'(a) = 0$, we obtain

$$\Big(\psi_1\big(\frac{d}{dx}w\big)\Big)(\eta) = 0. \tag{50}$$

Therefore, by the above proof and the fact that ψ_2 is an odd function, it follows that

$$\psi_2\Big(\frac{d}{dx}\Big(\psi_1\big(\frac{d}{dx}w\big)\Big)(\eta)\Big) = 0. \tag{51}$$

Thus, two inequalities can be concluded from Theorem 3 as follows:

$$\begin{aligned}
\psi_2\Big\{\frac{\frac{2}{(b-a)^{\alpha}}}{\psi_1\big(\frac{b-a}{2}\big)}\Big\} &< \frac{1}{\Gamma(\alpha)}\Big\{\Big(\frac{\eta-a}{b-a}\Big)^{\alpha}\int_a^{\eta} p_-(\nu)\mathbf{G}(w)d\nu \\
&\quad + \Big(\frac{b-\eta}{b-a}\Big)^{\alpha}\int_{\eta}^{b} p_+(\nu)\mathbf{G}(w)d\nu\Big\} \\
&\le \frac{\|G\|}{(b-a)^{\alpha}\Gamma(\alpha)}\Big\{(\eta-a)^{\alpha+1}\|p_-\| + (b-\eta)^{\alpha+1}\|p_+\|\Big\}, \quad \text{if } a < \eta \le b.
\end{aligned} \tag{52}$$

With a similar behavior for $\eta \in [b,c)$, we have

$$\psi_2\Big\{\frac{\frac{2}{(c-b)^{\alpha}}}{\psi_1\big(\frac{c-b}{2}\big)}\Big\} \le \frac{\|G\|}{(c-b)^{\alpha}\Gamma(\alpha)}\Big\{(\eta-b)^{\alpha+1}\|p_-\| + (c-\eta)^{\alpha+1}\|p_+\|\Big\}. \tag{53}$$

Using a similar argument, we obtain Eq. (47). As a result, the relation (45) is computed. □

Example 2. As a direct consequence of Theorem 4, we present the following illustrative example. Let $a = 0$, $b = 1$, $c = 2$ and choose $\alpha = \frac{3}{2}$. Define the functions

$$\psi_1(s) = s^2 + 1, \quad \psi_2(s) = s.$$

Clearly, ψ_1 is positive, increasing and sub-multiplicative on $[0,\infty)$, while ψ_2 is odd, increasing and super-multiplicative on $\mathbb{R}$. Consider the function

$$w(x) = x^2(1-x)^2(2-x)e^{-x}, \quad x \in [0,2].$$

Then

$$w(0) = w'(0) = w(1) = w'(1) = w(2) = 0, \quad w(x) > 0 \text{ for } x \in (0,1) \cup (1,2),$$

and w attains its maximum at some point $x^* \in (0,2)$. Choose

$$p(x) = x - 1, \quad g(w) = e^w,$$

and let $\eta = 1$. The positive and negative parts of p are given by

$$p_-(x) = \max\{-p(x), 0\} = \begin{cases} 1-x, & 0 \le x \le 1, \\ 0, & 1 < x \le 2, \end{cases}$$

and

$$p_+(x) = \max\{p(x), 0\} = \begin{cases} 0, & 0 \le x < 1, \\ x-1, & 1 \le x \le 2. \end{cases}$$

Since $\alpha = \frac{3}{2}$, we have

$$\Gamma(\alpha) = \Gamma\left(\frac{3}{2}\right) = \frac{\sqrt{\pi}}{2}.$$

Moreover, the nonlinear integrals appearing in Theorem 4 take the form

$$\int_0^1 p_-(v)g(w(v))dv = \int_0^1 (1-v)e^{v^2(1-v)^2(2-v)e^{-v}}dv$$

and

$$\int_1^2 p_+(v)g(w(v))dv = \int_1^2 (v-1)e^{v^2(1-v)^2(2-v)e^{-v}}dv.$$

Define

$$G(w) = \frac{g(w)}{\psi_2(\psi_1(w(x^*)))}.$$

Since w is continuous on $[0,2]$, it follows that $\|G\| < \infty$. Furthermore,

$$\|p_-\| = \|p_+\| = 1.$$

Therefore, all the hypotheses of Theorem 4 are satisfied. Consequently, inequality (47) holds, that is

$$\psi_2\left(\frac{2}{(c-a)^\alpha \psi_1\left(\frac{c-a}{2}\right)}\right) \le \frac{\|G\|}{(c-a)^\alpha \Gamma(\alpha)} \left\{(\eta - a)^{\alpha+1}\|p_-\| + (c-\eta)^{\alpha+1}\|p_+\|\right\}.$$

For the present choice of parameters, this inequality reduces to

$$0.353 \leq \frac{\|G\|}{2^{3/2}\Gamma(3/2)} \left\{(1)^{5/2}(1)+(1)^{5/2}(1)\right\}.$$

By using the above relations, and similarly to Example 1, we obtain $\|G\| = 1.06$, which satisfies the following inequality:

$$0.353 \leq 0.845$$

which is clearly a true relation.

Corollary 1. *Suppose that the relation (11) with the stated assumptions under the conditions (12) for ψ_1 and ψ_2 hold. Also, suppose that it has a non-trivial solution. Then, we have*

$$\psi_2\left(\frac{\frac{2}{(b-a)^\alpha}}{\psi_1(\frac{b-a}{2})}\right) \leq \frac{(b-a)}{\Gamma(\alpha)}\left\{\|G\|\|p\|\right\}. \tag{54}$$

Under the assumptions stated above, suppose that Eq. (11) under the conditions (13) has a non-trivial solution. Then, we have

$$\psi_2\left(\frac{\frac{2}{(b-a)^\alpha}}{\psi_1(\frac{b-a}{2})}\right) \leq \frac{(b-a)}{\Gamma(\alpha)}\left\{\|G\|\|p\|\right\} \tag{55}$$

and

$$\psi_2\left(\frac{\frac{2}{(c-b)^\alpha}}{\psi_1(\frac{c-b}{2})}\right) \leq \frac{(c-b)}{\Gamma(\alpha)}\left\{\|G\|\|p\|\right\}. \tag{56}$$

Thus from Eqs. (55) and (56), we have

$$\psi_2\left(\frac{\frac{2}{(c-a)^\alpha}}{\psi_1(\frac{c-a}{2})}\right) \leq \frac{(c-a)}{\Gamma(\alpha)}\left\{\|G\|\|p\|\right\}. \tag{57}$$

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Conflict of interest

The authors declare that they have no conflict of interest.

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