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The generators of total multiplication group of Cheban loop

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Abstract. A Cheban loop $(G, \circ)$ is characterized by the identities $(z \circ yx)x = zx \circ xy$ and $x(xy \circ z) = yx \circ xz$ for all $x, y, z \in G$. It was established that the left, right, and middle nuclei of a Cheban loop coincide, and the nucleus of a Cheban loop is the set of elements a whose middle inner mappings T_a are automorphisms. The generators of the inner mapping group of a Cheban were refined in terms of one of the generators of the total inner mapping group of a Cheban loop. Necessary and sufficient conditions regarding the inner mapping group (associators) for a loop to be a Cheban loop were established. It was shown that, in a Cheban loop, the mapping $a \mapsto T_a$ is an endomorphism if and only if the left (right) inner mapping is a left (right) regular mapping. Additionally, a Cheban loop was proved to be a left and right automorphic loop and that the left and right inner mappings belong to its middle inner mapping group. Furthermore, a Cheban loop was shown to be an automorphic loop (A-loop) if and only if it is a middle automorphic loop (middle A-loop). Some interesting relations involving the generators of the total multiplication group and total inner mapping group of a Cheban loop were derived, and based on these, the generators of the total inner mapping group of a Cheban loop were fine-tuned. Finally, it was shown that a Cheban loop is a totally automorphic loop (TA-loop) if and only if it is a commutative and flexible loop. These results above were used to give a partial answer to a 2013 question and an apparent solution to the 2015 problem in the case of a Cheban loop.

Keywords: Cheban loop, Automorphic loop, Inner mapping group, Total multiplication group.
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1 Introduction

1.1 Quasigroup and Loop

Let G be a non-empty set. Define a binary operation " $\circ$ " on G. If $x \circ y \in G$ for all $x, y \in G$, then the pair $(G, \circ)$ is called a groupoid. If $a \circ x = b$ and $y \circ a = b$ have unique solutions $x, y \in G$ for all $a, b \in G$ then $(G, \circ)$ is called a quasigroup. Let $(G, \circ)$ be a quasigroup and let there exist a unique element $e \in G$ called the identity element such that for all $x \in G, x \circ e = e \circ x = x$, then $(G, \circ)$ is called a loop. At times, we shall write xy instead of $x \circ y$ and stipulate that " $\circ$ " has lower priority than juxtaposition among factors to be multiplied. Let $(G, \circ)$ be a groupoid and a be a fixed element in G, then the left and right translations L_a and R_a of a are respectively defined by $xL_a = a \circ x$ and $xR_a = x \circ a$ for all $x \in G$. It can now be seen that a groupoid $(G, \circ)$ is a quasigroup if its left and right translation mappings are permutations. Since the left and right translation mappings of a quasigroup are bijective, then the inverse mappings L_a^{-1} and R_a^{-1} exist.

Let

$$a\backslash b = bL_a^{-1} = aM_b \qquad \text{and} \qquad a/b = aR_b^{-1} = bM_a^{-1}$$

and note that

$$a\backslash b = c \Longleftrightarrow a \circ c = b \qquad \text{and} \qquad a/b = c \Longleftrightarrow c \circ b = a.$$

Thus, for any quasigroup $(G, \circ)$, we have two new binary operations; right division $(/)$ and left division $(\backslash)$. M_a is the middle translation for any fixed $a \in G$. Consequently, $(G, \backslash)$ and $(G, /)$ are also quasigroups. Using the operations $(\backslash)$ and $(/)$, the definition of a loop can be restated as follows.

Definition 1. *A loop $(G, \circ, /, \backslash, e)$ is a set G together with three binary operations $(\circ)$, $(/)$, $(\backslash)$ and one nullary operation e such that*

(i) $a \circ (a\backslash b) = b$, $(b/a) \circ a = b$ *for all* $a, b \in G$.

(ii) $a\backslash(ab) = b$, $(ba)/a = b$ *for all* $a, b \in G$.

(iii) $a\backslash a = b/b$ *or* $e \circ a = a \circ e = a$ *for all* $a, b \in G$.

We also stipulate that $(/)$ and $(\backslash)$ have higher priority than $(\circ)$ among factors to be multiplied. For instance, $a \circ b/c$ and $a \circ b\backslash c$ stand for $a(b/c)$ and $a(b\backslash c)$ respectively.

In a loop $(G, \circ)$ with identity element e, the *left inverse element* of $x \in G$ is the element $xJ_\lambda = x^\lambda \in G$ such that

$$x^\lambda \circ x = e$$

while the *right inverse element* of $x \in G$ is the element $xJ_\rho = x^\rho \in G$ such that

$$x \circ x^\rho = e.$$

Let $(G, \circ)$ be a loop and let $a, b, c \in G$. Then, associator of a, b and c is the unique element of G that satisfies $\{a \circ bc\}(a, b, c) = ab \circ c$. The commutator of a loop defined for a and b is the unique element in G that satisfies $(ab) = ba(a, b)$.

For further study on quasigroups and loops, the reader may consult [6–8, 11, 14–17, 19, 22, 26]

Definition 2 ([20]). *A quasigroup $(G, \circ)$ is said to have the*

1. *left inverse property (LIP) if there exists a mapping $J_\lambda : x \mapsto x^\lambda$ such that $x^\lambda \circ (x \circ y) = y$ for all $x, y \in G$.*

2. *right inverse property (RIP) if there exists a mapping $J_\rho : x \mapsto x^\rho$ such that $(y \circ x) \circ x^\rho = y$ for all $x, y \in G$.*

3. *inverse property (IP) if it has both the LIP and RIP.*

4. *right alternative property (RAP) if $y \circ xx = yx \circ x$ for all $x, y \in G$.*

5. *left alternative property (LAP) if $xx \circ y = x \circ xy$ for all $x, y \in G$.*

6. *flexible or elasticity property if $(x \circ y) \circ x = x \circ (y \circ x)$ for all $x, y \in G$.*

7. *cross inverse property (CIP) if there exist mapping $J_\lambda : x \mapsto x^\lambda$ or $J_\rho : x \mapsto x^\rho$ such that $xy \circ x^\rho = y$ or $x \circ yx^\rho = y$ or $x^\lambda \circ yx = y$ or $x^\lambda y \circ x = y$ for all $x, y \in G$.*

8. *weak inverse property loop (WIPL) if and only if it obeys $x(yx)^\rho = y^\rho$ or $(xy)^\lambda x = y^\lambda$ for all $x, y \in G$.*

Definition 3 ([20]). *A loop $(G, \circ)$ is said to be:*

1. *an automorphic inverse property loop (AIPL) if $(xy)^{-1} = x^{-1}y^{-1}$ for all $x, y \in G$.*

2. *an anti- automorphic inverse property loop (AAIPL) if $(xy)^{-1} = y^{-1}x^{-1}$ for all $x, y \in G$.*

3. *a semi-automorphic inverse property loop (SAIPL) if it obeys any of the identities $(xy \circ x)^\rho = x^\rho y^\rho \circ x^\rho$ or $(xy \circ x)^\lambda = x^\lambda y^\lambda \circ x^\lambda$ for all $x, y \in G$.*

4. *a power associative loop if $< x >$ is a subgroup for all $x \in G$ and a diassociative loop if $< x, y >$ is a subgroup for all $x, y \in G$.*

Definition 4 ([20]). *Let $(G, \circ)$ be a loop:*

1. *$\mathbf{N}_\lambda = \{v \in G : (v \circ x) \circ y = v \circ (x \circ y) \ \forall \ x, y \in G\}$ is called the left nucleus of G .*

2. *$\mathbf{N}_\rho = \{v \in G : y \circ (x \circ v) = (y \circ x) \circ v \ \forall \ x, y \in G\}$ is called the right nucleus of G.*

3. *$\mathbf{N}_\mu = \{v \in G : (y \circ v) \circ x = y \circ (v \circ x) \ \forall \ x, y \in G\}$ is called the middle nucleus of G .*

4. *The centrum or commutant of G is $C(G, \circ) = C(G) = \{a \in G : ax = xa \ \forall x \in G\}$.*

5. *The centre of G is $Z(G, \circ) = \mathbf{N}(G, \circ) \cap C(G, \circ)$.*

A group of all permutations on G is called the permutation group of a loop G and is denoted by $SYM(G)$. In loop theory, it is well known that $\mathcal{M}(G, \circ) = \big\langle \{L_a, R_a | a \in G\} \big\rangle$ is called a multiplication group and $\mathcal{M}(G, \circ) \leq SYM(G)$. Now, suppose that $e\phi = e$ in a loop $(G, \circ)$, such that $\phi \in \mathcal{M}(G, \circ)$, then ϕ is called the inner mapping of G and it forms a group $Inn(G, \circ)$ known as the inner mapping group.

Let $(G, \circ)$ be a loop, the right, left, and middle inner mappings of $(G, \circ)$ are defined as follows:

$$R_{x,y} = R_x R_y R_{xy}^{-1},\ L_{x,y} = L_x L_y L_{yx}^{-1},\ T_x = R_x L_x^{-1}$$

and they are respectively generate the right inner mapping group $Inn_\rho(G)$, the left inner mapping group $Inn_\lambda(G)$ and the middle inner mapping group $Inn_\mu(G)$.

Definition 5. *Let G be a non-empty set. Let $(G, \circ)$ be a loop and let $A, B, C \in SYM(G)$. If*

$$xA \circ yB = (x \circ y)C\ \forall\ x, y \in G$$

then the triple (A, B, C) called an autotopism and such set of triples form a group $AUT(G, \circ)$ called the autotopism group of $(G, \circ)$ under componentwise multiplication . If $A = B = C$, then A is called an automorphism of $(G, \circ)$ whose set forms a group $AUM(G, \circ)$ called the automorphism group of $(G, \circ)$.

If

$$Inn_\rho(G) \leq AUM(G), \qquad Inn_\lambda(G) \leq AUM(G), \qquad Inn_\mu(G) \leq AUM(G),$$

then G is called a right A-loop (A_ρ-loop), left A-loop (A_λ-loop) and middle A-loop (A_μ-loop) respectively.

Definition 6. *Let $(G, \circ)$ be a quasigroup. Then,*

1. *$A \in SYM(G)$ is called $\lambda-$regular if $(A, I, A) \in AUT(G, \circ)$; the set of such mappings forms a group $\Lambda(G, \circ)$.*
2. *$A \in SYM(G)$ is called $\rho-$regular if $(I, A, A) \in AUT(G, \circ)$; the set of such mappings forms a group $\mathcal{P}(G, \circ)$.*

It is well known that

$$Inn(G, \circ) = \left\langle L_{(a,b)},\ R_{(a,b)},\ T_a : a, b \in G \right\rangle.$$

This group was later rediscovered and improved by Vojtěchovský [27] in 2015 as

$$Inn(G, \circ) = \left\langle R_{(a,b)},\ T_a : a, b \in G \right\rangle = \left\langle L_{(a,b)},\ T_a : a, b \in G \right\rangle.$$

If

$$Inn(G, \circ) \leq AUM(G, \circ),$$

then $(G, \circ)$ is called an *automorphic loop* (A-loop).

The group

$$\mathcal{TM}(G, \circ) = \left\langle \{L_a,\ R_a,\ M_a : a, b \in G\} \right\rangle$$

is called the total multiplication group of $(G, \circ)$. A permutation $\phi \in \mathcal{TM}(G, \circ)$ satisfying $e\phi = e$ is called a *total inner mapping* of $(G, \circ)$ and the group $TInn(G, \circ)$ formed by them is called the total inner mapping group.

In 2013, Stanovský and Vojtěchovský [23] established that the total inner mapping group is

$$TInn(G,\circ) = \big\langle R_{(a,b)},\, L_{(a,b)},\, T_a,\, M_{(a,b)},\, D_a : a,b \in G \big\rangle,$$

where

$$M_{(a,b)} = M_b M_a M_{b\backslash a}^{-1}, \qquad D_a = M_a R_a^{-1}.$$

In 2017, Syrbu [25] reported that

$$TInn(G,\circ) = \big\langle R_{(a,b)},\, L_{(a,b)},\, T_a,\, D_a^{-1},\, E_a,\, P_{(a,b)},\, P'_{(a,b)} : a,b \in G \big\rangle,$$

where

$$P'_{(a,b)} = M_b^{-1} M_a^{-1} R_b L_a^{-1}, \qquad E_a = R_a M_a, \qquad P_{(a,b)} = M_a M_b L_a R_b^{-1}.$$

In 2019, Syrbu and Grecu [24] improved the result to

$$TInn(G,\circ) = \big\langle R_{(a,b)},\, L_{(a,b)},\, T_a,\, E_a,\, P_{(a,b)} : a,b \in G \big\rangle.$$

If $(G,\circ)$ is an inverse property loop, then

$$\mathcal{TM}(G,\circ) = \big\langle L_a J_\lambda : a \in G \big\rangle = \big\langle M_a : a \in G \big\rangle,$$

and

$$TInn(G,\circ) = \big\langle Inn(G,\circ),\, J : a,b \in G \big\rangle = \big\langle M_{(a,b)} : a,b \in G \big\rangle.$$

A loop $(G,\circ)$ is called a *totally automorphic loop* (TA-loop) if

$$TInn(G,\circ) \leq AUM(G,\circ).$$

Note that all totally automorphic loops are commutative.

Theorem 1 (Kinyon [13])**.** *Let $(G,\circ)$ be a loop. Then $(G,\circ)$ is a TA-loop if and only if it is a commutative Moufang loop.*

The insight behind the above theorem is given below.

Theorem 2 (Kinyon [13])**.** *In a loop $(G,\circ)$, the following are equivalent:*

(a)
$$\big\langle L_{a\backslash b}^{-1} M_b M_a,\;\; R_{b/a}^{-1} M_b^{-1} M_a^{-1} : a,b \in G \big\rangle \leq AUM(G,\circ).$$

(b) $(G,\circ)$ is an A-loop and a Moufang loop.

The author posed the following question:

Question 1.1 (Kinyon [13], (2013))**.** What other interesting varieties of loops can be characterized by specifying that some group of total inner mappings acts as automorphisms?

Question 1.2 (Stanovský and Vojtěchovský [23], (2015))**.** Are the generating sets of the total inner mapping group minimal? Can any of the five types of inner mappings be removed? Is there a generating set for the total multiplication group using only two types of inner mappings?

In 2019, Syrbu and Grecu [24], in order to address Question 1.2, established the following:
(1) Let $(G, \circ)$ a power associative loop, then

$$TInn(G, \circ) = \left\langle R_{(a,b)}, L_{(a,b)}, E_a, P_{(a,b)} : a, b \in G \right\rangle.$$

(2) If $(G, \circ)$ is a middle Bol loop, then

$$TInn(G, \circ) = \left\langle R_{(a,b)}, E_a, P_{(a,b)} : a, b \in G \right\rangle.$$

It was further established by the author that if $(G, \circ)$ is a middle Bol loop, then $Inn(G, \circ) \trianglelefteq TInn(G, \circ)$ and reported that the result is not necessarily true for a right Bol or left Bol loop. In 2021, Jaiyéọlá and Effiong [9] presented characterizations of Basarab loops and identified conditions under which a Basarab loop is an automorphic loop. This result provides a partial solution to the underlying problem in the case of Basarab loops. In 2025, Jaiyéọlá et al. [12] studied a new class of power associative loop and addressed Question 1.2 relative to it.

1.2 Cheban Loops

The central aim of this paper is to address two fundamental questions concerning total inner mappings in loops, namely:

- **Question 1.** Under what conditions do total inner mappings act as automorphisms of a loop?
- **Question 2.** What are minimal and effective generating sets for the total inner mapping group and the total multiplication group?

Cheban loops provide a particularly suitable and natural framework for investigating these questions. Structurally, Cheban loops lie between generalized Bol-Moufang type loops inheriting strong algebraic regularities such as power associativity, conjugacy closedness, weak inverse property and the coincidence of left and right inverses. These properties significantly simplify the behavior of translation mappings and inner mappings, while still allowing enough flexibility to detect nontrivial phenomena. Cheban loops were recently found to belong to two newly found varieties of loops in Jaiyéọlá and George [10].

From the perspective of **Question 1**, Cheban loops offer a setting in which the interaction between middle translations T_a, inverse mappings J_λ, J_ρ, and classical inner mappings can be analyzed explicitly. In particular, the Cheban identities force strong relations among these mappings, making it possible to characterize precisely when total inner mappings belong to the automorphism group. This leads to clean necessary and sufficient conditions, most notably, that the loop must be flexible and an A-loop for total inner mappings to act by automorphisms. Such a characterization is generally inaccessible in broader classes of loops without imposing much stronger assumptions.

Regarding **Question 2**, the structural constraints of Cheban loops allow the total multiplication group $\mathcal{TM}(G, \circ)$ and the total inner mapping group $TInn(G, \circ)$ to be generated by comparatively small and transparent sets of mappings. The collapse of left and right inverses, together with explicit formulas for middle inner mappings, makes it possible to express $TInn(G, \circ)$ entirely in terms of translations and conjugations by inverse mappings. This not only yields concrete generating sets, but also permits a careful analysis of redundancy and minimality, thereby providing meaningful answers to the question of efficient generators.

In summary, Cheban loops form an ideal testbed for Questions 1 and 2: they are sufficiently structured to allow precise algebraic control of total inner mappings, yet sufficiently general to produce results that extend and refine what is known for Moufang, Bol, and automorphic loops. For these reasons, the restriction to Cheban loops is not merely technical, but essential for obtaining sharp and informative answers to both questions.

Definition 7. *A loop* $(G, \circ)$ *is called*

1. *RChL if it satisfies the identity* $\underbrace{(z \circ yx) \circ x = zx \circ xy}_{RChL}$ *for all* $x, y, z \in G$.
2. *LChL if it satisfies the identity* $\underbrace{x(xy \circ z) = yx \circ xz}_{LChL}$ *for all* $x, y, z \in G$.

The questions 1.1 and 1.2 will be the focus of a detailed investigation for Cheban loops. Our primary objectives are (i) to determine a minimal generating set for the total inner mapping group $TInn(G, \circ)$ in the Cheban setting, and (ii) to characterize necessary and sufficient conditions under which every total inner mapping is an automorphism in a Cheban loop.

To achieve these goals, we analyze the structural identities that define Cheban loops, study the interaction between the standard inner mappings and the middle translations M_a, and derive criteria expressed as identities or inclusion relations in $AUM(G, \circ)$ that guarantee $TInn(G, \circ) \leq AUM(G, \circ)$. Along the way, we highlight differences with previously studied varieties (e.g., Moufang, middle Bol, and power-associative loops), and we identify instances in which known generating sets are either minimal or redundant.

The concept of Cheban loops was introduced by Cheban in [1] as a generalization of the Bol–Moufang type. Cheban loops occur in two variants: left Cheban loops (LChL) and right Cheban loops (RChL), both of which fall within the broader class of generalized Moufang loops. The characterization of left Cheban loops was developed by Côté et al. in 2010 [5], while Phillips and Shcherbacov [21], (2011) described structural properties of Cheban loops, noting in particular that they form power-associative and conjugacy closed loops (CCL). Further developments appeared in 2022, when Chinaka et al. [2] constructed a right Cheban loop of small order, followed by an investigation of its holomorphic structure in 2023 by the same author [3]. More recent work by Osoba et al. (2025) [18] established several algebraic features of Cheban loops, showing that a Cheban loop $(G, \circ)$ may be: (i) a weak inverse property loop when it is flexible and its middle inner mapping lies in a permutation group; (ii) an automorphic inverse property loop when it satisfies a semi-commutative law and its middle inner mapping is contained in a permutation group; (iii) an anti-automorphic inverse property loop when every element has a two-sided inverse and the middle inner mapping is contained in a permutation group. It was also reported in [4] that, in any Cheban loop, $x^\lambda = x^\rho = x^{-1}$.

In the present study, we focus on the characterization of the nuclei of Cheban loops in terms of middle inner mappings. We express associators in Cheban loops using total inner mappings and provide necessary and sufficient conditions on the generator(s) of the inner mapping group for a Cheban loop to be an A-loop. Furthermore, we establish results describing the generators of both the inner mapping group and the total inner mapping group of a Cheban loop, demonstrating that a specific class of total inner mappings acts on a Cheban loop $(G, \circ)$ by automorphisms if and only if $(G, \circ)$ is an A-loop and flexible.

Theorem 3 (Osoba et al. [18], 2025)**.** *Let $(G, \circ)$ be a Cheban loop. The following are equivalent:*

1. *$(G, \circ)$ is a cross inverse property.*
2. *$(G, \circ)$ is commutative.*
3. *$(G, \circ)$ is an abelian group.*
4. *$R_x \in \Lambda(G, \circ)\ \forall\ x \in G$.*
5. *$R_x \in \mathcal{M}_\lambda(G, \circ)\ \forall\ x \in G$.*

Remark 1. *In this study, our focus is on left and right Cheban loops, specifically the identities listed in Definition 7. When results apply to both types, we simply refer to "Cheban loops." If a result concerns only one of the two varieties, this will be explicitly indicated.*

2 Main Results

2.1 Algebraic Properties of Cheban Loop

We begin by deriving a family of identities that hold in every Cheban loop. Starting from the left and right Cheban identities, we perform a series of elementary substitutions to obtain relations involving products, divisions, inverses, and the middle inner mapping T_a. These identities clarify the algebraic behavior of translations in Cheban loops and provide several equivalent expressions for T_a and T_a^{-1}. The results of this section will play a central role in the sequel, when we examine the minimal generating sets for total inner mappings and establish criteria under which these mappings are automorphisms.

Lemma 1. *In a Cheban loop $(G, \circ)$, the following hold:*

1. $(c \circ b)a = ca \circ a(b/a)$.
2. $a(b \circ c) = (a\backslash b)a \circ ac$.
3. $ba = a \circ a(b/a)$.
4. $ab = (a\backslash b)a \circ a$.
5. $a = b^\rho a \circ a(b/a)$.
6. $a = (a\backslash b)a \circ ab^\lambda$.
7. $a = ca \circ a(c^\rho/a)$.
8. $a = (a\backslash c^\lambda)a \circ ac$.
9. $(a^\lambda b)a = a(b/a)$.
10. $a(ba^\rho) = (a\backslash b)a$.
11. $ba = a((a^\lambda \circ b) \circ a)$.
12. $ab = (a \circ (b \circ a^\rho))a$.
13. $T_a^{-1} = L_a^{-1} R_a$.
14. $T_a = R_a^{-1} L_a$.
15. $T_a = L_{a^\lambda} R_a$.
16. $T_a^{-1} = R_{a^\rho} L_a$.
17. $[L_a^{-1} R_a,\ R_a^{-1} L_a] = I$.
18. $T_a = R_a^{-1} L_a = L_{a^\lambda} R_a$.
19. $T_a^{-1} = L_a^{-1} R_a = R_{a^\rho} L_a$.

Proof. We shall apply substitutions in the RChL and LChL identities.

Doing $b \mapsto b/a$ in RChL and $b \mapsto a\backslash b$ in LChL yield

$$(cb) \circ a = ca \circ (a(b/a)) \qquad \text{and} \qquad a(bc) = (a\backslash b)a \circ ac,$$

establishing 1 and 2. Setting $c = e$ in 1–2 gives 3 and 4:

$$b \circ a = a \circ (a(b/a)) \quad \text{and} \quad a \circ b = (a\backslash b)a \circ a.$$

Setting $c = b^\rho$ in 1 and $c = b^\lambda$ in 2 yields 5 and 6 respectively. Similarly, setting $b = c^\rho$ and $b = c^\lambda$ in 1 and 2 produces 7 and 8 respectively.

Putting $c = a^\lambda$ in 1 gives 9, and putting $c = a^\rho$ in 2 gives 10. Using 9 in 3 and 10 in 4 yields 11 and 12.

To obtain 13, note from 3 that $bR_a^{-1}L_aL_a = bR_a$, hence $R_a^{-1}L_a = R_aL_a^{-1}$, so $R_a^{-1}L_a = T_a$. From 4, we get $L_aR_a^{-1} = L_a^{-1}R_a$, so $T_a^{-1} = L_a^{-1}R_a$, proving 14. Using 9 gives 15 as $L_{a^\lambda}R_a = R_a^{-1}L_a = T_a$, and using 10 gives 16 as $R_{a^\rho}L_a = L_a^{-1}R_a = T_a^{-1}$.

The remaining identities 17–19 follow from combinations of 13–16 and basic group relations for permutations. □

Lemma 2. *Let $(G, \circ)$ be a loop. $(G, \circ)$ is a Cheban loop if and only if the following hold:*

1. $(R_a, R_a^{-1}L_a, R_a) = (R_a, T_a, R_a) \in AUT(G, \circ)\ \forall\ a \in G$.
2. $(L_a^{-1}R_a, L_a, L_a) = (T_a^{-1}, L_a, L_a) \in AUT(G, \circ)\ \forall\ a \in G$.

Proof. Use Definition 7, the result follows by rewriting the Cheban identities as autotopisms. □

2.2 Nuclei of Cheban Loop

The following theorem highlights a key structural property of Cheban loops. Unlike general loops, in Cheban loops the left, right, and middle nuclei coincide and are closely tied to inner mappings. Several conditions including the characterization of nucleus, certain associator identities, the action of middle inner mappings as automorphisms, and associativity are equivalent. Consequently, the nucleus can be identified as the set of elements whose middle inner mappings are automorphisms.

Theorem 4. *Let $(G, \circ)$ be a Cheban loop.*

1. *The following are equivalent for any fixed $a \in G$:*
   - *(a)* $a \in \mathbf{N}_\lambda$.
   - *(b)* $a \in \mathbf{N}_\rho$.
   - *(c)* $a \in \mathbf{N}_\mu$.
   - *(d)* $T_a \in AUM(G, \circ)$.
2. *The following are equivalent:*
   - *(a)* $a \in \mathbf{N}_\lambda\ \forall\ a \in G$.

(b) $a \in \mathbf{N}_\rho \ \forall \ a \in G$.

(c) $a \in \mathbf{N}_\mu \ \forall \ a \in G$.

(d) $T_a \in AUM(G, \circ)$ *for all* $a \in G$.

(e) $Inn_\mu(G, \circ) \leq AUM(G, \circ)$.

(f) $(G, \circ)$ *is an* $A_\mu-$*loop.*

3. $\mathbf{N}(G, \circ) = \mathbf{N}_\lambda(G, \circ) = \mathbf{N}_\rho(G, \circ) = \mathbf{N}_\mu(G, \circ)$.

4. *For all* $a \in G$, $\mathcal{T}_a = T_a|_{N(G,\circ)} \in AUM\,(N(G, \circ))$.

5. $\mathcal{T}_a : (G, \circ) \to AUM\,(N(G, \circ))$ *is an homomorphism for all* $a \in G$.

Proof.

1. We shall prove using both RChL and LChL.

(a) $\iff$ **(d)** Let $(G, \circ)$ be a RChL. Now, $a \in \mathbf{N}_\lambda \Leftrightarrow (L_a, I, L_a) \in AUT(G, \circ) \Leftrightarrow (L_a^{-1}, I, L_a^{-1}) \in AUT(G, \circ)$. So, $a \in \mathbf{N}_\lambda \Leftrightarrow (R_a, R_a^{-1}L_a, R_a)(L_a^{-1}, I, L_a^{-1}) = (R_aL_a^{-1}, T_a, R_aL_a^{-1}) \in AUT(G, \circ)$ $\Leftrightarrow (T_a, T_a, T_a) \in AUT(G, \circ) \Leftrightarrow T_a \in AUM(G, \circ)$. Thus, in right Cheban loop, $a \in \mathbf{N}_\lambda$ if and only if T_a is an automorphism of $(G, \circ)$.

(b) $\iff$ **(d)** Let $(G, \circ)$ be a LChL. Now, $a \in \mathbf{N}_\rho \Leftrightarrow (I, R_a, R_a) \in AUT(G, \circ) \Leftrightarrow (I, R_a^{-1}, R_a^{-1}) \in AUT(G, \circ)$. Since $(G, \circ)$ is LChL, then $(T_a^{-1}, L_a, L_a) \in AUT(G, \circ)$. So, $a \in \mathbf{N}_\rho \Leftrightarrow$ $(T_a^{-1}, L_a, L_a)(I, R_a^{-1}, R_a^{-1}) = (T_a^{-1}, L_aR_a^{-1}, L_aR_a^{-1}) \in AUT(G, \circ) \Leftrightarrow (T_a^{-1}, T_a^{-1}, T_a^{-1}) \in AUT(G, \circ) \Leftrightarrow T_a \in AUM(G, \circ)$. Thus, in LChL, $a \in \mathbf{N}_\rho$ if and only if T_a is an automorphism of $(G, \circ)$.

(b) $\iff$ **(c)** Let $(G, \circ)$ be a RChL. Now, $a \in \mathbf{N}_\mu$ if and only if $(R_a, L_a^{-1}, I) \in AUT(G, \circ) \Leftrightarrow$ $(R_a^{-1}, L_a, I) \in AUT(G, \circ) \Leftrightarrow (R_a, T_a, R_a)(R_a^{-1}, L_a, I) = (I, R_a, R_a) \in AUT(G, \circ) \Leftrightarrow a \in \mathbf{N}_\rho(G, \circ)$. In a RChL, $a \in \mathbf{N}_\mu(G, \circ)$ if and only if $a \in \mathbf{N}_\rho(G, \circ)$.

2. Use 1.

3. Use 2(a) to 2(c).

4. Use 2 and 3.

5. Use 4.

□

2.3 Cheban Loop and its inner mappings

In this section, we make use of two related results describing how the middle inner map T_a correspond to commutator translations in Cheban loops. For a right Cheban loop, the identity $T_{ab} = T_aT_b$ is shown to be equivalent to the condition that the right inner mapping $R_{(a,b)}$ lies in the permutation group $\Lambda(G, \circ)$. Dually, for a left Cheban loop, the relation $T_bT_a = T_{ba}$ holds precisely when the left inner mapping $L_{(a,b)}$ belongs to the group $\mathcal{P}(G, \circ)$. These equivalences reveal the close connection between the multiplicative structure of the inner mappings and their membership of autotopic-derived subgroups.

Lemma 3. *Let $(G, \circ)$ be a loop. Then, the following hold:*

1. $cR_{(a,b)} \circ ab = ca \circ b$ *and* $(c, b, a) = (c \circ ab)M_{[cR_{(a,b)} \circ ab]}$.
2. $cL_{(a,b)} = (ba)M_{(b \circ ac)}$ *and* $(b, a, c) = \big[(ba \circ cL_{(a,b)}\big]M_{(b \circ ac)}$.
3. $cT_a = aM_{(ca)}$ *and* $(b, a) = (ab)M_{a \circ (bT_a)}$.

Proof. 1. Follow from the definition of right inner mapping,

$$R_{(a,b)} = R_a R_b R_{ab}^{-1} \Rightarrow cR_{(a,b)} = cR_a R_b R_{ab}^{-1} \Rightarrow$$
$$cR_{(a,b)} \circ ab = ca \circ b \Rightarrow cR_{(a,b)} = (ab)M^{-1}_{(ca \circ b)}.$$

Recall that

$$\underbrace{c \circ ab(c, b, a) = ca \circ b}_{\text{using the associators of } a, b \text{ and } c} = cR_{(a,b)} \circ ab \Rightarrow$$
$$(c \circ ab)(c, b, a) = cR_{(a,b)} \circ ab \Rightarrow (c, b, a) = (c \circ ab)M_{[cR_{(a,b)} \circ ab]}.$$

2. $cL_{(a,b)} = cL_a L_b L_{ba}^{-1} \Rightarrow ba \circ cL_{(a,b)} = (b \circ ac) \Rightarrow cL_{(a,b)} = (ba)M_{(b \circ ac)}$. Recall that

$$ba \circ c = (b \circ ac)(b, a, c) \Rightarrow (b, a, c) = (ba \circ c)L_{(a,b)} \backslash (ba \circ c)$$
$$\Rightarrow (b, a, c) = [(ba) \circ cL_{(a,b)}]M_{(ba \circ c)}.$$

3. $T_a = R_a L_a^{-1} \Rightarrow bT_a = bR_a L_a^{-1} \Rightarrow bT_a = aM_{(ba)}$. Recall that $\underbrace{(ba) = ab \circ (b, a)}_{\text{Using the commutators of } a \text{ and } b} \Rightarrow$ $(b, a) = (ab) \backslash (ba) \Rightarrow (b, a) = (ab) \backslash (a \circ bT_a) = (ab)M_{a \circ (bT_a)}$.

□

Theorem 5. *The following are equivalent in a loop $(G, \circ)$:*

1. $(G, \circ)$ *is a Cheban loop.*
2. $(a, b, c) = \Big[(a \backslash b)a \circ (ac)\Big] \backslash ((ab) \circ c)$ *and* $(a, b, c) = (c \circ ba) \backslash \Big[ca \circ a(b/a)\Big]$.
3. $Inn(G, \circ) = \Big\langle L_a L_{(a \backslash b)a} L_{ab}^{-1}, T_a \mid a, b \in G \Big\rangle$ *and*
   $Inn(G, \circ) = \Big\langle R_a R_{(a(b/a))} R_{ba}^{-1}, T_a \mid a, b \in G \Big\rangle$.

Proof. **1** $\iff$ **2** . Recall that in a loop, LChL holds $\Leftrightarrow a \circ bc = (a\backslash b)a \circ ac$ by item 2 of Lemma 1. In Lemma 3, $ab \circ c = (a \circ bc)(a,b,c)$. So, $(G,\circ)$ is a LChL $\Leftrightarrow \Big[(a\backslash b)a \circ (ac)\Big](a,b,c) = ab \circ c \Leftrightarrow (a,b,c) = \Big[(a\backslash b)a \circ (ac)\Big]\backslash(ab \circ c)$. Also, $(G,\circ)$ is a RChL, $\Leftrightarrow cb \circ a = (c \circ ba)(a,b,c) \Leftrightarrow (a,b,c) = (c \circ ba)\backslash\Big[ca \circ a(b/a)\Big]$.

2 $\iff$ **3** . $(G,\circ)$ is a LChL if and only if,

$$L_bL_a = L_aL_{(a\backslash b)a)} \Leftrightarrow L_bL_aL_{ab}^{-1} = L_aL_{(a\backslash b)a)}L_{ab}^{-1}$$
$$\Leftrightarrow L_{(b,a)} = L_aL_{(a\backslash b)a)}L_{ab}^{-1} \Leftrightarrow Inn(G,\circ) = \Big\langle L_aL_{(a\backslash b)a}L_{ab}^{-1}, T_a | a,b \in G\Big\rangle.$$

And RChL holds if and only if

$$R_bR_a = R_aR_{(a(b/a))} \Leftrightarrow R_bR_aR_{ba}^{-1} = R_aR_{(a(b/a))}R_{ba}^{-1} \Leftrightarrow$$
$$R_{(b,a)} = R_aR_{(a(b/a))}R_{ba}^{-1} \Leftrightarrow Inn(G,\circ) = \Big\langle R_aR_{(a(b/a))}R_{ba}^{-1}, T_a \mid a,b \in G\Big\rangle.$$

□

Corollary 1. *Let $(G,\circ)$ be a Cheban loop. Then*

1. $(a,b,c) = \Big[(a(ba^\rho) \circ (ac)\Big]\backslash((ab) \circ c)$ *and* $(a,b,c) = (c \circ ba)\backslash\Big[ca \circ (a^\lambda b)a)\Big]$
2. $Inn(G,\circ) = \Big\langle L_aL_{(bT_a^{-1})}L_{ab}^{-1}, T_a|a,b \in G\Big\rangle$ *and*
   $Inn(G,\circ) = \Big\langle R_aR_{(bT_a)}R_{ba}^{-1}, T_a|a,b \in G\Big\rangle$

Proof. Consequence of Theorem 5 and 9, 10 of Lemma 1. □

Corollary 2. *Let $(G,\circ)$ be a Cheban loop. Then, the following hold for all $a,b,c \in G$:*

1. $(b,a,c) = \Big[\big(cL_{ba}\big)J_\lambda\big[(ba \circ c)L_{(a,b)}\big]\Big]J_\lambda.$
2. $(c,b,a) = (cR_{ba})J_\lambda[cR_{(a,b)}R_{ab}].$
3. $(a,b,c) = \Big[(bT_a^{-1}R_{ac}\Big]J_\lambda \circ cL_{(ab)}.$
4. $(a,b,c) = (c \circ ba)^\lambda \circ \Big[ca \circ (bT_a)\Big].$

Proof. 1. Recall in item 2 of Lemma 3 , $(b,a,c)=\big[(ba\circ c)L_{(a,b)}\big]\backslash(b\circ ac)\Rightarrow\big[(ba\circ c)L_{(a,b)}\big]$ $(b,a,c)=(b\circ ac)\Rightarrow\big[(ba\circ c)L_{(a,b)}\big]=(ba\circ c)(b,a,c)^{\rho}\Rightarrow(b,a,c)^{\rho}=\big(ba\circ c\big)^{\lambda}\big[(ba\circ c)L_{(a,b)}\big]$ $\Rightarrow(b,a,c)=\Big[\big(cL_{ba}\big)J_{\lambda}\big[(ba\circ c)L_{(a,b)}\big]\Big]J_{\lambda}$.

2. Recall in 1(b) of Lemma 3, $(c,b,a)=(c\circ ab)\backslash[cR_{(a,b)}\circ ab]\Rightarrow(c\circ ab)(c,a,b)=[cR_{(a,b)}\circ ab]$ $\Rightarrow(c,b,a)=(c\circ ba)\circ J_{\lambda}[cR_{(a,b)}\circ ab]\Rightarrow(c,b,a)=(cR_{ba})J_{\lambda}\circ[cR_{(a,b)}R_{ab}]$.

3. Recall in Theorem 5, item 2, $(a,b,c)=\Big[(a\backslash b)a\circ(ac)\Big]\backslash((ab)\circ c)\Rightarrow(a,b,c)=\Big[(bL_a^{-1}R_a)\circ(ac)\Big]\backslash((ab)\circ c)\Rightarrow(a,b,c)=\Big[bT_a^{-1}R_{ac}\Big]J_{\lambda}\circ cL_{(ab)}$.

4. Recall in Theorem 5, item 2(b), $(a,b,c)=(c\circ ba)\backslash\Big[ca\circ a(b/a)\Big]\Rightarrow(a,b,c)=(c\circ ba)^{\lambda}\circ\Big[ca\circ a(b/a)\Big]\Rightarrow(a,b,c)=(c\circ ba)^{\lambda}\circ\Big[ca\circ(bT_a)\Big]$.

□

Theorem 6. *In a Cheban loop $(G,\circ)$, the following are equivalent for all $a,b\in G$:*

1. $T_{ab}=T_aT_b$.
2. $R_{(a,b)}\in\Lambda(G,\circ)$.
3. $L_{(a,b)}\in\mathcal{P}(G,\circ)$.

Proof. Let $(G,\circ)$ be a right Cheban loop (RChL). For each $x\in G$ set

$$P(x)=(R_x,T_x,R_x)\in\mathrm{AUT}(G,\circ).$$

Then

$$P(a)P(b)P(ab)^{-1}=(R_a,T_a,R_a)(R_b,T_b,R_b)(R_{ab}^{-1},T_{ab}^{-1},R_{ab}^{-1})$$
$$=(R_aR_bR_{ab}^{-1},\ T_aT_bT_{ab}^{-1},\ R_aR_bR_{ab}^{-1})=(R_{(a,b)},\ T_aT_bT_{ab}^{-1},\ R_{(a,b)})\in\mathrm{AUT}(G,\circ).$$

We have

$$(R_{(a,b)},T_aT_bT_{ab}^{-1},R_{(a,b)})\in\mathrm{AUT}(G,\circ)\text{ and }R_{(a,b)}\in\Lambda(G,\circ)\Leftrightarrow(R_{(a,b)},I,R_{(a,b)})\in\mathrm{AUT}(G,\circ),$$

and hence,

$$R_{(a,b)}\in\Lambda(G,\circ)\Leftrightarrow T_aT_bT_{ab}^{-1}=I\Leftrightarrow T_aT_b=T_{ab}.$$

Similarly, let $(G,\circ)$ be a left Cheban loop (LChL). For each $x\in G$, set

$$Q(x)=(T_x^{-1},L_x,L_x)\in\mathrm{AUT}(G,\circ).$$

Then

$$Q(a)Q(b)Q(ba)^{-1}=(T_a^{-1},L_a,L_a)(T_b^{-1},L_b,L_b)(T_{ba},L_{ba}^{-1},L_{ba}^{-1})$$

$$= (T_a^{-1}T_b^{-1}T_{ba},\ L_aL_bL_{ba}^{-1},\ L_aL_bL_{ba}^{-1}) = (T_a^{-1}T_b^{-1}T_{ba},\ L_{(b,a)},\ L_{(b,a)}) \in AUT(G,\circ).$$

Thus,

$$(T_a^{-1}T_b^{-1}T_{ba}, L_{(a,b)}, L_{(a,b)}) \in \mathrm{AUT}(G,\circ) \text{ and } L_{(a,b)} \in \mathcal{P}(G,\circ) \Leftrightarrow (I, L_{(a,b)}, L_{(a,b)}) \in \mathrm{AUT}(G,\circ)$$

$$\Leftrightarrow T_a^{-1}T_b^{-1}T_{ba} = I \Leftrightarrow T_bT_a = T_{ba}$$

This completes the proof. □

2.4 Cheban loop and its automorphic loop.

In this section, we provide structural characterizations of Cheban loops by analyzing their translation mappings, inverse mappings, and total inner mappings. We first show that the cross inverse property (CIP), commutativity, and the abelian group property are equivalent, with $R_a \in \Lambda(G,\circ)$ and $L_a \in \mathcal{P}(G,\circ)$ fully capturing these structures. We also define specific mappings to describe the algebraic properties of Cheban loops. Next, we prove that a Cheban loop is Moufang if and only if $L_aR_a = R_aL_a$ for all $a \in G$, and that it is a TA–loop precisely when it is flexible and commutative. We further investigate the total inner mapping group $TInn(G,\circ)$ and the total multiplication group $\mathcal{TM}(G,\circ)$, providing explicit generating sets in terms of translations and conjugations by J_λ and J_ρ. Finally, we characterize when the subgroup $\langle T_a : a \in G\rangle$ lies in the automorphism group, showing this occurs exactly when the Cheban loop is an A–loop and flexible. These results give partial answers to Question 1.1 and an apparent solution to Question 1.2 regarding the conditions under which total inner mappings act by automorphisms and the minimality of their generating sets.

Theorem 7. *Let $(G,\circ)$ be a Cheban loop such that a, b are arbitrary elements in G. Then*

1. *$(G,\circ)$ is an $A_\rho - loop \Leftrightarrow R_{(a,b)} = T_aT_bT_{ab}^{-1} \Leftrightarrow T_b^{-1}L_aR_b = L_{(ab)} \Leftrightarrow Inn_\rho(G,\circ) = Inn_\mu(G,\circ)$.*
2. *$(G,\circ)$ is an $A_\lambda - loop \Leftrightarrow L_{(a,b)} = T_a^{-1}T_b^{-1}T_{ba} \Leftrightarrow T_bR_aL_b = R_{(ba)} \Leftrightarrow Inn_\lambda(G,\circ) = Inn_\mu(G,\circ)$.*
3. *$(G,\circ)$ is an $A - loop \Leftrightarrow Inn(G,\circ) = \left\langle T_a | a \in G \right\rangle \leq AUM(G,\circ) \Leftrightarrow L_aR_bL_{(ab)}^{-1} = T_b = R_{ba}L_b^{-1}R_a^{-1} \in AUM(G,\circ) \Leftrightarrow Inn(G,\circ) = Inn_\mu(G,\circ) \leq AUM(G,\circ)$.*

Proof. 1. In a RChL,

$$P(a) = (R_a, T_a, R_a) \in AUT(G,\circ), \quad P(b) = (R_b, T_b, R_b) \in AUT(G,\circ),$$

$$P(ab)^{-1} = (R_{ab}^{-1}, T_{ab}^{-1}, R_{ab}^{-1}) \in AUT(G,\circ).$$

Thus,

$$P(a)P(b)P(ab)^{-1} = (R_{(a,b)},\ T_aT_bT_{ab}^{-1},\ R_{(a,b)}) \in AUT(G,\circ),$$

and therefore in a RChL,

$$(G,\circ) \text{ is an } A_\rho - loop \Leftrightarrow R_{(a,b)} = T_aT_bT_{ab}^{-1} \Leftrightarrow R_aR_bR_{ab}^{-1} = T_aT_bT_{ab}^{-1}$$

$$\Leftrightarrow R_aR_bR_{ab}^{-1} = T_aT_bL_{(ab)}R_{ab}^{-1} \Leftrightarrow R_aR_b = T_aT_bL_{(ab)}$$

$$\Leftrightarrow R_aR_b = R_aL_a^{-1}T_bL_{(ab)} \Leftrightarrow R_b = L_a^{-1}T_bL_{(ab)} \Leftrightarrow T_b^{-1}L_aR_b = L_{(ab)}.$$

2. In a LChL, $Q(x) = (T_x^{-1}, L_x, L_x) \in AUT(G, \circ)$ for all $x \in G$.

$$Q(a)Q(b)Q(ba)^{-1} = (\, T_a^{-1}T_b^{-1}T_{ba},\, L_{(a,b)},\, L_{(a,b)}\,) \in AUT(G, \circ).$$

   Hence, in a LChL,

$$(G, \circ) \text{ is an } A_\lambda \Leftrightarrow L_{(a,b)} = T_a^{-1}T_b^{-1}T_{ba} \Leftrightarrow L_aL_bL_{ba}^{-1} = T_a^{-1}T_b^{-1}T_{ba}$$

$$\Leftrightarrow L_aL_bL_{ba}^{-1} = T_a^{-1}T_b^{-1}R_{(ba)}L_{ba}^{-1} \Leftrightarrow L_aL_b = T_a^{-1}T_b^{-1}R_{(ba)}$$

$$\Leftrightarrow L_aL_b = L_aR_a^{-1}T_b^{-1}R_{(ba)} \Leftrightarrow L_b = R_a^{-1}T_b^{-1}R_{(ba)} \Leftrightarrow T_bR_aL_b = R_{(ba)}.$$

3. Combine (1) and (2).

□

Corollary 3. *Let $(G, \circ)$ be a Cheban loop and let a, b be arbitrary elements of G.*

1. *$(G, \circ)$ is an $A_\rho-$loop, $R(a, b) = T_aT_bT_{ab}^{-1}, T_b^{-1}L_aR_b = L_{(ab)}$, and $Inn_\rho(G, \circ) = Inn_\mu(G, \circ)$.*
2. *$(G, \circ)$ is an $A_\lambda-$loop, $L(a, b) = T_a^{-1}T_b^{-1}T_{ab}, T_bL_aR_b = R_{(ba)}$, and $Inn_\lambda(G, \circ) = Inn_\mu(G, \circ)$.*
3. *$(G, \circ)$ is an $A-$loop $\Leftrightarrow L_aR_bL_{(ab)}^{-1} = T_b = R_{ba}L_b^{-1}R_a^{-1} \in AUM(G, \circ) \in G \Leftrightarrow Inn(G, \circ) = Inn_\mu(G, \circ) \leq AUM(G, \circ)$.*

Proof. In a Cheban loop, we have

$$(R_{(a,b)},\, T_aT_bT_{ab}^{-1},\, R_{(a,b)}) \in AUT(G, \circ) \quad \text{and} \quad (T_a^{-1}T_b^{-1}T_{ba},\, L_{(b,a)},\, L_{(b,a)}) \in AUT(G, \circ).$$

Recall that in a loop G, if $(A, B, C) \in AUT(G, \circ)$ with

$$eA = eB = e,$$

then $A = B = C$. Since, in particular,

$$eR_{(a,b)} = e = eL_{(a,b)},$$

it follows that the components of each autotopism coincide. Consequently,

$$R_{(a,b)},\ L_{(b,a)} \in AUM(G, \circ).$$

This completes the proof. □

Theorem 8. *Let $(G, \circ)$ be a Cheban loop and let $D_a = M_aR_a^{-1}, E_a = M_a^{-1}L_a^{-1}, F_a = R_aM_a$ for any fixed $a \in G$.*

1. *$T_a = J_\rho R_aM_a$.*

2. $T_a^{-1} = J_\lambda L_a M_a^{-1}$.

3. *The following are true:*

   (a) $F_a E_a = T_a$.
   (b) $M_a^2 E_a = D_a T_a$.
   (c) $T_a F_a = D_a T_a^2$.
   (d) $R_a = L_a \Leftrightarrow T_a J_\lambda^2 T_a = M_a^2$.
   (e) $|M_a| = 2 \Leftrightarrow J_\lambda T_a^2 = T_a J_\rho$.
   (f) $|T_a| = 2$ *if and only if* $J_a^2 = F_a D_a T_a$.
   (g) $J_\lambda D_a T_a = F_a J_\lambda$.
   (h) $J_\lambda = L_a J_\lambda R_a$.
   (i) $J_\lambda T_a = T_a J_\lambda \Leftrightarrow D_a T_a = F_a$.
   (j) $R_a^n T_a J_\lambda L_a^n = R_a^{n-1} J_\lambda T_a L_a^{n-1}$ *for all* $\in \mathbb{N}$.
   (k) $J_\rho T_a = T_a E_a$.
   (l) $J_\lambda T_a = F_a$.
   (m) $D_a T_a = T_a J_\lambda$.
   (n) $D_a F_a = M_a^2$.
   (o) $F_a D_a R_a = R_a D_a F_a$.
   (p) $M_a D_a R_a = D_a F_a$.
   (q) $D_a T_a = M_a^2 E_a$.

4. *The following are true:*

   (a) $J_\rho = J_\lambda$.
   (b) $T_a^2 = D_a T_a F_a$.
   (c) $D_a T_a R_a M_a = T_a^2$.
   (d) $D_a T_a^2 = T_a^2 E_a$.
   (e) $\mathcal{TM}(G, \circ) = \langle \{L_a, R_a, D_a \mid a \in G\} \rangle = \langle \{L_a, R_a, E_a \mid a \in G\} \rangle = \langle \{L_a, R_a, F_a \mid a \in G\} \rangle$

5. $T_a^{-2} = D_a^n T_a^{-2} E_a^n$ *and* $T_a^2 = D_a^n T_a^2 E_a^n$ *for all* $n \in \mathbb{N}$. $D_a^n E_a^n = I$ *if* $|T_a| = 2$.

6. *The following hold for all* $n \in \mathbb{N}$:

   (a) $J_\lambda^n = T_a^{-1} D_a^n T_a$.
   (b) $|J_\lambda| = n \Leftrightarrow |D_a| = n$.
   (c) $J_\rho^n = T_a E_a^n T_a^{-1}$.
   (d) $|J_\rho| = n \Leftrightarrow |E_a| = n$.
   (e) $D_a^n T_a^2 = T_a^2 D_a^n$; $D_a^n = E_a^n$ *if* $|T_a| = 2$.

Proof. 1. Use Lemma 1(5), $(b^\rho a)\backslash a = a(b/a) \Rightarrow J_\rho R_a M_a = R_a^{-1} L_a \Rightarrow J_\rho R_a M_a = T_a$.

2. Follow from Lemma 1(6), $a/(ab^\lambda) = (a\backslash b)a \Rightarrow J_\lambda L_a M_a^{-1} = L_a^{-1} R_a \Rightarrow J_\lambda L_a M_a^{-1} = T_a^{-1}$

3. (a) $F_a E_a = R_a M_a M_a^{-1} L_a^{-1} = R_a L_a^{-1} = T_a$.

   (b) $M_a^2 E_a T_a^{-1} = M_a^2 M_a^{-1} L_a^{-1} T_a^{-1} = M_a^2 M_a^{-1} L_a^{-1} L_a R_a^{-1} = M_a R_a^{-1} = D_a \Rightarrow M_a^2 E_a = D_a T_a$.

   (c) From 1 and 2, $J_\rho = T_a M_a^{-1} R_a^{-1}$ and $J_\rho = L_a M_a^{-1} T_a$. Then, $T_a M_a^{-1} R_a^{-1} = L_a M_a^{-1} T_a \Rightarrow L_a M_a^{-1} T_a = T_a (R_a M_a)^{-1} = T_a (F_a)^{-1}$ Thus, $L_a M_a^{-1} T_a = T_a (F_a)^{-1} \Rightarrow L_a M_a M_a^{-2} T_a = T_a (F_a)^{-1} \Rightarrow (M_a^{-1} L_a^{-1})^{-1} M_a^{-2} T_a = T_a (F_a)^{-1} \Rightarrow E_a^{-1} M_a^{-2} T_a = T_a F_a^{-1} \Rightarrow M_a^{-2} T_a F_a = E_a T_a \Rightarrow M_a^2 E_a T_a = T_a F_a \Rightarrow T_a F_a = D_a T_a^2$.

   (d) From 1 and 2, we have $R_a = J_\lambda T_a M_a^{-1}$ and $L_a = J_\rho T_a^{-1} M_a$. Then, $R_a = L_a \Leftrightarrow J_\lambda T_a M_a^{-1} = J_\rho T_a^{-1} M_a \Leftrightarrow T_a J_\lambda^2 T_a = M_a^2$.

   (e) Use 1 and 2, $L_a^{-1} J_\rho T_a^{-1} = M_a^{-1}$ and $M_a = R_a^{-1} J_\lambda T_a$ So, $|M_a| = 2 \Leftrightarrow M_a = L_a^{-1} J_\rho T_a^{-1} = R_a^{-1} J_\lambda T_a \Leftrightarrow L_a R_a^{-1} J_\lambda T_a^2 = J_\rho \Leftrightarrow J_\lambda T_a^2 = T_a J_\rho$.

   (f) From 1 and 2, we have $|T_a| = 2$ if and only if $T_a = T_a^{-1}$ if and only if $J_\rho R_a M_a = J_\lambda L_a M_a^{-1} \Leftrightarrow J_\lambda^2 = R_a M_a M_a L_a^{-1} \Leftrightarrow J_\lambda^2 = F_a M_a L_a^{-1} = F_a M_a^2 M_a^{-1} L_a^{-1} = F_a M_a^2 E_a \Leftrightarrow J_\lambda^2 = F_a D_a T_a$.

   (g) From 1 and 2, we have $T_a T_a^{-1} = J_\rho R_a M_a J_\lambda L_a M_a^{-1} = I \Rightarrow R_a^{-1} J_\lambda M_a = M_a J_\lambda L_a \Rightarrow J_\lambda M_a L_a^{-1} = R_a M_a J_\lambda \Rightarrow J_\lambda M_a^2 M_a^{-1} L_a^{-1} = F_a J_\lambda \Rightarrow J_\lambda M_a^2 E_a = F_a J_\lambda \Rightarrow J_\lambda D_a T_a = F_a J_\lambda$.

   (h) From 1 and 2, we have $M_a = T_a J_\lambda L_a$ and $M_a = R_a^{-1} J_\lambda T_a \Rightarrow R_a^{-1} J_\lambda T_a = T_a J_\lambda L_a \Rightarrow R_a^{-1} J_\lambda T_a = R_a^{-1} L_a J_\lambda L_a \Rightarrow J_\lambda T_a = L_a J_\lambda L_a \Rightarrow J_\lambda R_a^{-1} L_a L_a^{-1} = L_a J_\lambda \Rightarrow J_\lambda = L_a J_\lambda R_a$.

   (i) From 1 and 2, $J_\rho T_a^{-1} = L_a M_a^{-1}$ and $T_a^{-1} J_\rho = M_a^{-1} R_a^{-1}$. So, $T_a^{-1} J_\rho = J_\rho T_a^{-1} \Leftrightarrow M_a^{-1} R_a^{-1} = L_a M_a^{-1} \Leftrightarrow (R_a M_a)^{-1} = L_a M_a^{-1} \Leftrightarrow L_a M_a M_a^{-2} = F_a^{-1} \Leftrightarrow E_a^{-1} M_a^{-2} = F_a^{-1} \Leftrightarrow M_a^2 E_a = F_a \Leftrightarrow D_a T_a = F_a$.

   (j) From 1 and 2, we have $R_a^{-1} J_\lambda T_a = M_a$ and $L_a^{-1} J_\rho T_a^{-1}$. Then, $R_a^{-1} J_\lambda T_a L_a^{-1} J_\rho T_a^{-1} = I \Rightarrow R_a T_a J_\lambda = J_\lambda T_a L_a^{-1} \Rightarrow R_a T_a J_\lambda L_a = J_\lambda T_a \Rightarrow R_a^2 T_a J_\lambda L_a^2 = R_a J_\lambda T_a L_a \Rightarrow R_a^n T_a J_\lambda L_a^n = R_a^{n-1} J_\lambda T_a L_a^{n-1}$ for all $\in \mathbb{N}$.

   (k) From 1, $T_a^{-1} J_\rho R_a = M_a^{-1}$. Then, $T_a^{-1} J_\rho R_a = M_a^{-1} \Rightarrow T_a^{-1} J_\rho R_a L_a^{-1} = M_a^{-1} L_a^{-1} \Rightarrow T_a^{-1} J_\rho T_a = E_a \Rightarrow J_\rho T_a = T_a E_a$.

   (l) From 1, $M_a = R_a^{-1} J_\lambda T_a \Rightarrow J_\lambda T_a = F_a$.

   (m) From 2, $M_a = T_a J_\lambda L_a \Rightarrow M_a R_a^{-1} = D_a = T_a J_\lambda L_a R_a^{-1} \Rightarrow D_a = T_a J_\lambda T_a^{-1} \Rightarrow D_a T_a = T_a J_\lambda$.

   (n) $D_a F_a = M_a R_a^{-1} R_a M_a = M_a^2 \Rightarrow D_a F_a = M_a^2$.

   (o) $F_a D_a = R_a M_a M_a R_a^{-1} = R_a M_a^2 R_a^{-1} \Rightarrow F_a D_a R_a = R_a M_a^2$
   $\Rightarrow F_a D_a R_a = R_a D_a F_a$.

   (p) $M_a D_a = M_a^2 R_a^{-1} \Rightarrow D_a F_a R_a^{-1} = M_a D_a \Rightarrow M_a D_a R_a = D_a F_a$.

   (q) $D_a T_a = T_a J_\lambda$. Note that $D_a T_a = M_a^2 E_a \Rightarrow T_a J_\lambda = M_a^2 E_a$.

4. $J_\lambda = J_\rho$ if and only if

$$T_aM_a^{-1}R_a^{-1} = T_a^{-1}M_aL_a^{-1} \Leftrightarrow T_aM_a^{-1} = T_a^{-1}M_aL_a^{-1}R_a \Leftrightarrow$$
$$T_a^2M_a^{-1} = M_aL_a^{-1}R_a \Leftrightarrow T_a^2M_a^{-1} = M_aR_a^{-1}R_aL_a^{-1}R_a \Leftrightarrow T_a^2 = D_aT_aR_aM_a.$$

$J_\rho = J_\lambda$ if and only if $T_a^2 = D_aT_aR_aM_a \Leftrightarrow T_a^2 = D_aT_aF_a$.

$$\Leftrightarrow T_a^2M_a^{-1}R_a^{-1}T_a^{-1} = D_a \Leftrightarrow T_a^2M_a^{-1}L_a^{-1}L_aR_a^{-1}T_a^{-1} = D_a$$
$$\Leftrightarrow T_a^2E_aL_aR_a^{-1}T_a^{-1} = D_a \Leftrightarrow T_a^2E_a = D_aT_a^2.$$
$$\text{Hence, } \mathcal{TM} = \Big\langle \{L_a, R_a, D_a\} : a \in G \Big\rangle = \Big\langle \{L_a, R_a, E_a\} : a \in G \Big\rangle =$$
$$\Big\langle \{L_a, R_a, F_a\} : a \in G \Big\rangle.$$

5. From 1, 2, we get $J_\lambda = T_a^{-1}M_aL_a^{-1}$ and $J_\rho = T_aM_a^{-1}R_a^{-1}$. Then,

$$J_\rho J_\lambda = I \Leftrightarrow T_a^{-1}M_aL_a^{-1}T_aM_a^{-1}R_a^{-1} = I \Leftrightarrow T_a^{-1}M_aR_a^{-1}R_aL_a^{-1}T_aM_a^{-1}L_a^{-1}L_aR_a^{-1} = I$$
$$\Leftrightarrow T_a^{-1}D_aT_a^2E_aT_a^{-1} = I \Leftrightarrow D_aT_a^2E_a = T_a^2 \tag{1}$$

And

$$J_\lambda J_\rho = I \Leftrightarrow T_aM_a^{-1}R_a^{-1}T_a^{-1}M_aL_a^{-1} = I$$
$$\Leftrightarrow T_aM_a^{-1}L_a^{-1}L_aR_a^{-1}T_a^{-1}M_aR_a^{-1}R_aL_a^{-1} = I \Leftrightarrow$$
$$T_aE_aT_a^{-2}D_aT_a = I \Leftrightarrow T_a^{-2} = E_aT_a^{-2}D_a \tag{2}$$

Use (1) and (2) to get

$$D_aT_a^2E_aE_aT_a^{-2}D_a = D_aT_a^2E_a^2T_a^{-2}D_a = I \Leftrightarrow D_a^{-1}T_a^2 = D_aT_a^2E_a^2 \Leftrightarrow T_a^2 = D_a^2T_a^2E_a^2 \tag{3}$$

Substituting (3) in (1), we obtain $T_a^2 = D_aD_a^2T_a^2E_a^2E = D_a^3T_a^2E_a^3$. By induction, we get $T_a^2 = D_a^nT_a^2E_a^n$, where $n \in \mathbb{N}$. On the other hand, we use similar steps, to get $T_a^{-2} = D_a^nT_a^{-2}E_a^n$. We see that $D_a^nE_a^n = I$, if $|T_a| = 2$ for all $a \in G$.

6. (a) From 1 and 2,

$$J_\lambda^2 = T_a^{-1}M_aL_a^{-1}T_a^{-1}M_aL_a^{-1} = T_a^{-1}M_aL_a^{-1}L_aR_a^{-1}M_aL_a^{-1} =$$
$$T_a^{-1}M_aR_a^{-1}M_aL_a^{-1} = T_a^{-1}D_aM_aL_a^{-1} = T_a^{-1}D_aM_a^2M_a^{-1}L_a^{-1} = T_a^{-1}D_aM_a^2E_a.$$

Note that $M_a^2E_a = D_aT_a$, then $J_\lambda^2 = T_a^{-1}D_a^2T_a$. Therefore, $J_\lambda^3 = T_a^{-1}D_a^2T_aT_a^{-1}D_aT_a = T_a^{-1}D_a^3T_a$. By induction, we get $J_\lambda^n = T_a^{-1}D_a^nT_a$ for all natural numbers n.
(b) Follow from (a).
(c)

$$J_\rho^2 = T_aM_a^{-1}R_a^{-1}T_aM_a^{-1}R_a^{-1} = T_aM_a^{-1}R_a^{-1}R_aL_a^{-1}M_a^{-1}R_a^{-1}$$

$$= T_a M_a^{-1} L_a^{-1} M_a^{-1} R_a^{-1} = T_a E_a F_a^{-1}.$$

Note that $T_a = F_a E_a$. So, $J_\rho^3 = T_a E_a F_a^{-1} F_a E_a F_a^{-1} = T_a E_a^2 F_a^{-1} = T_a E_a^2 \left(T_a E_a^{-1}\right)^{-1} = T_a E_a^3 T_a^{-1}$. By induction, we get $J_\rho^n = T_a E_a^n T_a^{-1}$.
(d) Follows from (c).
(e) $J_\rho^n = J_\lambda^n \Leftrightarrow T_a^{-1} D_a^n T_a = T_a E_a^n T_a^{-1} \Leftrightarrow D_a^n T_a^2 = T_a^2 D_a^n$. Hence, $D_a^n = E_a^n$ if $|T_a| = 2$.

□

Lemma 4. *Let $(G, \circ)$ be a Cheban loop. The following are equivalent:*

1. *$(G, \circ)$ has the cross inverse property.*
2. *$(G, \circ)$ is commutative.*
3. *$(G, \circ)$ is an abelian group.*
4. *$R_a \in \Lambda(G, \circ)$ for all $a \in G$.*
5. *$L_a \in \mathcal{P}(G, \circ)$ for all $a \in G$.*

Proof. From Lemma 1(11), we have

$$ba = a \circ (a^\lambda b \circ a).$$

If $(G, \circ)$ satisfies the cross inverse property, then

$$a \circ b = b \circ a,$$

so the loop is commutative. The converse is also true: if $(G, \circ)$ is commutative, then $b = a^\lambda b \circ a$.

Next, using item 9 and item 1 of Lemma 1, we have

$$cb \circ a = ca \circ (a^\lambda b)a.$$

Thus $(G, \circ)$ has the CIP if and only if

$$(c \circ b)a = ca \circ b,$$

which is equivalent to $(G, \circ)$ being an abelian group.

Finally, by Lemma 2,

$$(R_a, T_a, R_a), \quad (T_a^{-1}, L_a, L_a) \in AUT(G, \circ) \quad \text{for all } a \in G.$$

Hence, $(G, \circ)$ commutativity if and only if

$$(T_a^{-1}, L_a, L_a) = (I, L_a, L_a) \in AUT(G, \circ) \quad \Leftrightarrow \quad L_a \in \mathcal{P}(G, \circ).$$

Also, $(G, \circ)$ is commutative if and only if

$$(R_a, T_a, R_a) = (R_a, I, R_a) \in AUT(G, \circ) \quad \Leftrightarrow \quad R_a \in \Lambda(G, \circ).$$

This completes the proof. □

Theorem 9. *A Cheban loop is Moufang if and only if it is flexible.*

Proof. Using Lemma 2, we have

$$(R_a, R_a^{-1}L_a, R_a) = (R_a, T_a, R_a) \in AUT(G, \circ)$$

and

$$(L_a^{-1}R_a, L_a, L_a) = (T_a^{-1}, L_a, L_a) \in AUT(G, \circ)$$

for all $a \in G$.

Thus,

$$(T_a^{-1}, L_a, L_a)(R_a, T_a, R_a) = (T_a^{-1}R_a,\ L_aT_a,\ L_aR_a) = (L_aR_a^{-1}R_a,\ L_aR_aL_a^{-1},\ L_aR_a) = (L_a, L_aR_aL_a^{-1},\ L_aR_a) \in AUT(G, \circ).$$

Therefore, $(G, \circ)$ is flexible if and only if $L_aR_a = R_aL_a \Leftrightarrow$

$$(L_a, R_a, L_aR_a) \in AUT(G, \circ),$$

which is equivalent to

$$ab \circ ca = (a(bc)) \circ a, \qquad \forall a, b, c \in G.$$

□

Theorem 10. *A Cheban loop is a TA-loop if and only if it is flexible and commutative.*

Proof. This follows from Theorem 9 together with Theorem 1. □

Corollary 4. *A Cheban loop $(G, \circ)$ is TA-loop if and only if flexibility any of the following hold:*

1. *$(G, \circ)$ is a cross inverse property.*
2. *$(G, \circ)$ is commutative.*
3. *$(G, \circ)$ is an abelian group.*
4. *$R_x \in \Lambda(G, \circ) \forall\ x \in G$.*
5. *$L_x \in \mathcal{P}(G, \circ)\ \forall\ x \in G$.*

Proof. Use Lemma 4 and Theorem 10. □

Theorem 11. *Let $(G, \circ)$ be a Cheban loop. Then*

$$\mathcal{TM}(G, \circ) = \left\langle \left\{\ L_a,\ R_a,\ R_a^{-1}J_\lambda T_a : a \in G \right\} \right\rangle = \langle \{\ L_a,\ R_a,\ T_aJ_\lambda L_a : a \in G\} \rangle,$$

$$TInn(G, \circ) = \left\langle \left\{\ T_a,\ T_aT_bT_{ab}^{-1},\ T_a^{-1}T_b^{-1}T_{ba},\ T_aJ_\lambda T_a^{-1},\ T_bJ_\lambda L_bR_a^{-1}J_\lambda T_aT_{b\backslash a}^{-1}J_\rho R_{b\backslash a} : a, b \in G \right\} \right\rangle$$
$$= \left\langle \left\{\ T_a,\ T_aT_bT_{ab}^{-1},\ T_a^{-1}T_b^{-1}T_{ba},\ T_aJ_\lambda T_a^{-1},\ T_bJ_\lambda L_bR_a^{-1}J_\lambda T_aL_{b\backslash a}^{-1}J_\rho T_{b\backslash a}^{-1} : a, b \in G \right\} \right\rangle.$$

Proof. The result follows from Corollary 3 and Theorem 8. We rely on the facts that

$\mathcal{TM}(G,\circ) = \big\langle \{\, L_a,\ R_a,\ M_a : a \in G\} \big\rangle, \qquad TInn(G,\circ) = \big\langle \{\, L_{(a,b)},\ R_{(a,b)},\ T_a,\ D_a,\ M_{(a,b)} : a,b \in G\} \big\rangle,$

where

$$D_a = M_a R_a^{-1} = T_a J_\lambda T_a^{-1}.$$

Note that in Theorem 8, item 1 and item 2, give $M_a = R_a^{-1} J_\lambda T_a$ and $M_a = T_a J_\lambda L_a$. So, substitute for M_a to obatin $\mathcal{TM}(G,\circ) = \big\langle \{\, L_a,\ R_a,\ R_a^{-1} J_\lambda T_a : a \in G\} \big\rangle = \big\langle \{\, L_a,\ R_a,\ T_a J_\lambda L_a : a \in G\} \big\rangle$.

For $M_{(a,b)}$ we have the two equivalent decompositions:

$$\begin{aligned} M_{(a,b)} = M_b M_a M_{b\backslash a}^{-1} &= T_b J_\lambda L_b R_a^{-1} J_\lambda T_a \big(T_{b\backslash a} J_\lambda L_{b\backslash a}\big)^{-1} \\ &= T_b J_\lambda L_b R_a^{-1} J_\lambda T_a L_{b\backslash a}^{-1} J_\rho T_{b\backslash a}^{-1}, \end{aligned}$$

and also

$$\begin{aligned} M_{(a,b)} = M_b M_a M_{b\backslash a}^{-1} &= T_b J_\lambda L_b R_a^{-1} J_\lambda T_a \big(R_{b\backslash a}^{-1} J_\lambda T_{b\backslash a}\big)^{-1} \\ &= T_b J_\lambda L_b R_a^{-1} J_\lambda T_a T_{b\backslash a}^{-1} J_\rho R_{b\backslash a}. \end{aligned}$$

In Corollary 3, we have $R_{(a,b)} = T_a T_b T_{ab}^{-1},\ L_{(b,a)} = T_a^{-1} T_b^{-1} T_{ba}$. Substitute for $D_a, R_{(a,b)}, L_{(b,a)}$ and $M_{(a,b)}$ to obtain

$$\begin{aligned} TInn(G,\circ) &= \Big\langle \Big\{\, T_a,\ T_a T_b T_{ab}^{-1},\ T_a^{-1} T_b^{-1} T_{ba},\ T_a J_\lambda T_a^{-1},\ T_b J_\lambda L_b R_a^{-1} J_\lambda T_a T_{b\backslash a}^{-1} J_\rho R_{b\backslash a} : a,b \in G \Big\} \Big\rangle \\ &= \Big\langle \Big\{\, T_a,\ T_a T_b T_{ab}^{-1},\ T_a^{-1} T_b^{-1} T_{ba},\ T_a J_\lambda T_a^{-1},\ T_b J_\lambda L_b R_a^{-1} J_\lambda T_a L_{b\backslash a}^{-1} J_\rho T_{b\backslash a}^{-1} : a,b \in G \Big\} \Big\rangle. \end{aligned}$$

This completes the proof. □

Remark 2.1. In Theorem 11, the expression for the total multiplication group and total inner mappings of a Cheban loop $(G,\circ)$ is provided in terms of finetuned generators. Therefore, this presents an apparent solution to Question 1.2.

Theorem 12. *Let $(G,\circ)$ be Cheban loop. Then, the following are equivalent:*

1. $\big\langle T_a : a \in G \big\rangle \le AUM(G,\circ) : R_a L_a = L_a R_a \forall\ a\ \in G.$
2. $\Big\langle \Big\{ L_{b/a}^{-1} T_b J_\lambda L_b R_a^{-1} J_\lambda T_a, R_{b/a}^{-1} T_b^{-1} J_\rho R_b L_a^{-1} J_\rho T_a^{-1} : a,b \in G \Big\} \Big\rangle$

Proof. Resting on the Theorem 2 and item 1 and item 2 of Theorem 8, $M_a = R_a^{-1} J_\lambda T_a$ and $M_a = T_a J_\lambda L_a$.
$M_b^{-1} M_a^{-1} = (M_a M_b)^{-1} = (T_a J_\lambda L_a R_b^{-1} J_\lambda T_b)^{-1} = T_b^{-1} J_\rho R_b L_a^{-1} J_\rho T_a^{-1}$
and $M_b M_a = T_b J_\lambda L_b R_a^{-1} J_\lambda T_a$. Using these results with the result obtained in Theorem 2, gives the final result. □

Remark 2.2. In Theorem 12, we showed that a class of total inner mappings acts on a Cheban loop G by automorphism if and only if G is an $A-$loop and flexible. Therefore, the combination of Theorem 10 and Theorem 12 give a partial answer to Question 1.1 in the case of a Cheban loop.

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