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The ranks of the conjugacy classes of the Symplectic group $Sp(6,2)$

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Abstract. Let G be a finite simple group, and X be a non-trivial conjugacy class of G. The *rank* of X in G, denoted by $rank(G : X)$, is defined to be the minimum number of elements of X generating G. In this paper, we investigate the ranks of the non-trivial classes of the symplectic simple group $Sp(6,2)$. We use the structure constants method to determine these ranks. The Groups, Algorithms and Programming (GAP) [12] and the Atlas of finite group representations [23] were used in our computations.

Keywords: Conjugacy classes, Rank, Generation, Symplectic simple group.
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1 Introduction

Definition 1. *Let G be a finite simple group, and nX be a non-trivial conjugacy class of elements of G. We define the rank of nX in G, denoted by $rank(G : nX)$, to be the minimum number of elements in nX that generate G.*

One of the applications of ranks of the conjugacy classes of a finite group is that they are involved in the computations of the covering number of the finite simple group (see Zisser [24]). Moori in various articles [17, 18] and [19]), computed the ranks of involution classes of the Fischer sporadic simple group Fi_{22} and his results were that $rank(Fi_{22} : 2A) \in \{5, 6\}$ and $rank(Fi_{22} : 2B) = 3 = rank(Fi_{22} : 2C)$. The work of Hall and Soicher [16] implies that

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$rank(Fi_{22} : 2A) = 6$. In determining the rank for each non-identity conjugacy class of a group G, we follow some of the methods used in the papers [2–8] and [20].

2 Preliminaries

Let G be a finite group and $C_1, C_2, \cdots, C_k$ (not necessarily distinct) and let $k \geq 3$ be conjugacy classes of G with $g_1, g_2, \cdots, g_k$ being representatives for these classes respectively.

For a fixed representative $g_k \in C_k$ and for $g_i \in C_i$, $1 \leq i \leq k-1$, denote by $\Delta_G = \Delta_G(C_1, C_2, \cdots, C_k)$ the number of distinct $(k-1)$-tuples $(g_1, g_2, \cdots, g_{k-1}) \in C_1 \times C_2 \times \cdots \times C_{k-1}$ such that $g_1 g_2 \cdots g_{k-1} = g_k$. This number is known as the *class algebra constant* or *the structure constant*. With $\mathrm{Irr}(G) = \{\chi_1, \chi_2, \cdots, \chi_r\}$, the number Δ_G is easily calculated from the character table of G through the formula

$$\Delta_G(C_1, C_2, \cdots, C_k) = \frac{\prod_{i=1}^{k-1} |C_i|}{|G|} \sum_{i=1}^{r} \frac{\chi_i(g_1)\chi_i(g_2)\cdots\chi_i(g_{k-1})\overline{\chi_i(g_k)}}{(\chi_i(1_G))^{k-2}}.$$

Also for a fixed $g_k \in C_k$ we denote by $\Delta_G^*(C_1, C_2, \cdots, C_k)$ the number of distinct $(k-1)$-tuples $(g_1, g_2, \cdots, g_{k-1})$ satisfying

$$g_1 g_2 \cdots g_{k-1} = g_k \quad \text{and} \quad G = \langle g_1, g_2, \cdots, g_{k-1} \rangle .$$

Definition 2. *If $\Delta_G^*(C_1, C_2, \cdots, C_k) > 0$, the group G is said to be $(C_1, C_2, \cdots, C_k)$-**generated**.*

Furthermore if H is any subgroup of G containing a fixed element $h_k \in C_k$, we let $\Sigma_H(C_1, \cdots, C_k)$ be the total number of distinct tuples $(h_1, h_2, \cdots, h_{k-1})$ such that

$$h_1 h_2 \cdots h_{k-1} = h_k \quad \text{and} \quad \langle h_1, h_2, \cdots, h_{k-1} \rangle \leq H.$$

The value of $\Sigma_H(C_1, C_2, \cdots, C_k)$ can be obtained as a sum of the structure constants $\Delta_H(c_1, \cdots, c_k)$ of H-conjugacy classes $c_1, c_2, \cdots, c_k$ such that $c_i \subseteq H \cap C_i$.

Theorem 1. *Let G be a finite group and H be a subgroup of G containing a fixed element g such that $\gcd(o(g), [N_G(H){:}H]) = 1$. Then the number $h(g, H)$ of conjugates of H containing g is $\chi_H(g)$, where $\chi_H(g)$ is the permutation character of G with action on the conjugates of H. In particular*

$$h(g, H) = \sum_{i=1}^{m} \frac{|C_G(g)|}{|C_{N_G(H)}(x_i)|},$$

where $x_1, x_2, \cdots, x_m$ are representatives of the $N_G(H)$-conjugacy classes fused to the G-class of g.

Proof. See [13] or [14, Theorem 2.1]. □

By [4], the above number $h(g,H)$ is useful in giving a lower bound for $\Delta^*_G(C_1,\cdots,C_k)$, namely $\Delta^*_G(C_1,\cdots,C_k)\geq\Theta_G(C_1,C_2,\cdots,C_k)$, where

$$\Theta_G(C_1,\cdots,C_k)=\Delta_G(C_1,\cdots,C_k)-\sum h(g_k,H)\Sigma_H(C_1,\cdots,C_k), \tag{1}$$

g_k is a representative of the class C_k and the sum is taken over all the representatives H of G-conjugacy classes of maximal subgroups of G containing elements of all the classes $C_1,C_2,\cdots,C_k$. Since we have all the maximal subgroups of the sporadic simple groups even for $G=\mathbb{M}$ the Monster group (see [10]), it is possible to build a small subroutine in GAP [12] to compute the values of $\Theta_G=\Theta_G(C_1,C_2,\cdots,C_k)$ for any collection of conjugacy classes and any finite simple group.

Lemma 1. *Let G be a finite centerless group. If $\Delta^*_G(C_1,C_2,\cdots,C_k)<|C_G(g_k)|,\ g_k\in C_k$, then $\Delta^*_G(C_1,C_2,\cdots,C_k)=0$ and therefore G is not $(C_1,C_2,\cdots,C_k)$-generated.*

Proof. See [3, Lemma 2.7]. □

Theorem 2 (Ree [21])**.** *Let G be a transitive permutation group generated by permutations $g_1,g_2,\cdots,g_s$ acting on a set of n elements such that $g_1g_2\cdots g_s=1_G$. If the generator g_i has exactly c_i cycles for $1\leq i\leq s$, then $\sum_{i=1}^{s}c_i\leq(s-2)n+2$.*

Theorem 3 (Scott [22])**.** *Let $g_1,g_2,\cdots,g_s$ be elements generating a group G with $g_1g_2\cdots g_s=1_G$ and $\mathbb{V}$ be an irreducible module for G with $\dim\mathbb{V}=n\geq 2$. Let $C_{\mathbb{V}}(g_i)$ denote the fixed point space of $\langle g_i\rangle$ on $\mathbb{V}$ and let d_i be the codimension of $C_{\mathbb{V}}(g_i)$ in $\mathbb{V}$. Then $\sum_{i=1}^{s}d_i\geq 2n$.*

With χ being the ordinary irreducible character afforded by the irreducible module $\mathbb{V}$ and $\mathbf{1}_{\langle g_i\rangle}$ being the trivial character of the cyclic group $\langle g_i\rangle$, the codimension d_i of $C_{\mathbb{V}}(g_i)$ in $\mathbb{V}$ can be computed using the following formula ([11]):

$$\begin{aligned} d_i &= \dim(\mathbb{V})-\dim(C_{\mathbb{V}}(g_i))=\dim(\mathbb{V})-\left\langle \chi{\downarrow}^G_{\langle g_i\rangle},\mathbf{1}_{\langle g_i\rangle}\right\rangle \\ &= \chi(1_G)-\frac{1}{|\langle g_i\rangle|}\sum_{j=0}^{o(g_i)-1}\chi(g_i^j). \end{aligned}\tag{2}$$

Theorem 4. *[3, Lemma 2.5] Let G be a $(2X,sY,tZ)$-generated simple group, then G is $(sY,sY,(tZ)^2)$-generated.*

The following results are in some cases useful in determining the ranks finite groups.

Lemma 2 ([1])**.** *Let G be a finite simple group such that G is (lX,mY,nZ)-generated. Then G is $(\underbrace{lX,lX,\ldots,lX}_{m-times},(nZ)^m)$-generated.*

Corollary 1 ([1])**.** *Let G be a finite simple group such that G is $(lX,mYnZ)$-generated. Then $rank(G:lX)\leq m$.*

Proof. The result follows immediately from Lemma 2. □

Theorem 5 ([15]). *Let G be a $(2X, sY, tZ)$-generated simple group, then G is $(sY, sY, (tZ)^2)$-generated.*

Corollary 2. *Let G be a finite simple group such that G is $(2X, mY, nZ)$-generated. Then $rank(G : mY) = 2$.*

Proof. Since G is $(2X, mY, nZ)$-generated group, it follows by Theorem 5 that G is $(mY, mY, (nZ)^2)$-generated. Therefore $rank(G : mY) \leq 2$. Since the rank of any non-trivial class in a finite simple group can not be 1, the result follows. □

3 The symplectic group $Sp(6, 2)$

In this section, we apply the results discussed in Section 2 to the group $Sp(6, 2)$. We determine the conjugacy class ranks of $Sp(6, 2)$. The symplectic group $Sp(6, 2)$ is a simple group of order $1451520 = 2^9 \times 3^4 \times 5 \times 7$. By the Atlas [9] the group $Sp(6, 2)$ has exactly 30 conjugacy classes of its elements and 8 conjugacy classes of its maximal subgroups. Representatives of conjugacy classes of the maximal subgroups can be taken as follows:

$$\begin{array}{lll} H_1 = U_4(2){:}2 & H_2 = S_8 & H_3 = 2^5{:}S_6 \\ H_4 = U_3(3){:}2 & H_5 = 2^6{:}L_3(2) & H_6 = (2^2 \times 2^{1+4}){:}(S_3 \times S_3) \\ H_7 = S_3 \times S_6 & H_8 = L_2(8){:}3. & \end{array}$$

Throughout the paper, we will use G instead of $Sp(6, 2)$, unless stated otherwise. For the sake of computations with GAP [12], we use a permutation presentation for G. By the electronic Atlas of Wilson [23], G can be generated in terms of permutations on 28 points. Generators g_1 and g_2 can be taken as follows:

$$\begin{aligned} g_1 &= (2,3)(6,7)(9,10)(12,14)(17,19)(20,22), \\ g_2 &= (1,2,3,4,5,6,8)(7,9,11,13,16,18,14)(10,12,15,17,20,19,21)(22,23,24,25,26,27,28), \end{aligned}$$

with $o(g_1) = 2$, $o(g_2) = 7$ and $o(g_1g_2) = 9$.

Table 1 gives all the values of $d_{nX} = \dim(\mathbb{V}/C_{\mathbb{V}}(nX))$, where V is the 7-dimesional irreducible complex module. This table will be referred to when we are proving the non-generation of a triple for the group G.

Table 2 is the same as Table 1, only that here we consider the vector space of dimension 15.

In Table 3, using the permutation action of G on 28 points, we list the values of the cyclic structure for each conjugacy of G which contains elements of prime order together with the values of both c_i and d_i obtained from Ree and Scott's theorems, respectively.

In Table 4 we list representatives of the conjugacy classes of the maximal subgroups of G, the orbits lengths of G on these subgroups and the decomposition of the permutation characters.

Table 5 gives us the partial fusion maps of classes of maximal subgroups into the classes of G.

In Table 6 we list the values of h over all the maximal subgroups and non-trivial conjugacy classes of G.

Table 1: $d_{nX} = \dim(\mathbb{V}/C_{\mathbb{V}}(nX)),\ nX$ is a non-trivial class of G and $\dim(\mathbb{V}) = 7$

nX	$2A$	$2B$	$2C$	$2D$	$3A$	$3B$	$3C$	$4A$	$4B$	$4C$	$4D$	$4E$	$5A$	$6A$	$6B$
d_{nX}	6	4	2	4	2	6	4	4	4	6	6	4	4	6	4
nX	$6C$	$6D$	$6E$	$6F$	$6G$	$7A$	$8A$	$8B$	$9A$	$10A$	$12A$	$12B$	$12C$	$15A$	
d_{nX}	6	4	6	6	6	6	6	6	6	6	6	6	6	6	

Table 2: $d_{nX} = \dim(\mathbb{V}/C_{\mathbb{V}}(nX)),\ nX$ is a non-trivial class of G and $\dim(\mathbb{V}) = 15$

nX	$2A$	$2B$	$2C$	$2D$	$3A$	$3B$	$3C$	$4A$	$4B$	$4C$	$4D$	$4E$	$5A$	$6A$	$6B$
d_{nX}	10	4	6	8	10	12	8	10	12	10	8	10	12	14	12
nX	$6C$	$6D$	$6E$	$6F$	$6G$	$7A$	$8A$	$8B$	$9A$	$10A$	$12A$	$12B$	$12C$	$15A$	
d_{nX}	12	12	12	10	12	12	12	12	14	14	14	14	14	14	

Table 3: Cycle structures of conjugacy classes of G

nX	Cycle Structure	c_i	d_i
$1A$	1^{28}	28	0
$2A$	$1^{16}2^{6}$	22	6
$2B$	$1^{4}2^{12}$	16	12
$2C$	$1^{8}2^{10}$	18	10
$2D$	$1^{4}2^{12}$	16	12
$3A$	$1^{10}3^{6}$	16	12
$3B$	$1\ \ 3^{9}$	10	18
$3C$	$1\ \ 3^{9}$	10	18
$4A$	$1^{4}4^{6}$	10	18
$4B$	$1^{2}2^{3}4^{5}$	10	18
$4C$	$1^{6}2\ \ 4^{5}$	12	16
$4D$	$2^{4}4^{5}$	9	19
$4E$	$1^{2}2^{3}4^{5}$	10	18
$5A$	$1^{3}5^{5}$	8	20
$6A$	$1^{4}2^{3}3^{4}6$	12	16
$6B$	$1^{4}2^{3}6^{3}$	10	18
$6C$	$1\ \ 3\ \ 6^{4}$	6	22
$6D$	$1^{2}2^{4}3^{2}6^{3}$	11	17
$6E$	$1\ \ 3^{5}6^{2}$	8	20
$6F$	$1\ \ 3\ \ 6^{4}$	6	22
$6G$	$1\ \ 3\ \ 6^{4}$	6	22
$7A$	7^{4}	4	24
$8A$	$1^{2}2\ \ 8^{3}$	6	22
$8B$	$4\ \ 8^{3}$	4	24
$9A$	$1\ \ 9^{3}$	4	24
$10A$	$1\ \ 2\ \ 5^{5}$	7	21

Table 3 continued

nX	Cycle Structure	c_i	d_i
$12A$	$1^2 4^2 6\ \ 12$	6	22
$12B$	$2\ \ 3^2 4^2 12$	6	22
$12C$	$1\ \ 3\ \ 12^2$	4	24
$15A$	$3\ \ 5^3$	4	24

Table 4: Maximal subgroups of $Sp(6,2)$

Maximal Subgroup	Order	Orbit Lengths	Character
$U_4(2){:}2$	$2^7 \cdot 3^4 \cdot 5$	[1,27]	$1a + 27a$
S_8	$2^7 \cdot 3^2 \cdot 5 \cdot 7$	[28]	$1a + 35b$
$2^5{:}S_6$	$2^9 \cdot 3^2 \cdot 5$	[12,16]	$1a + 27a + 35b$
$U_3(3){:}2$	$2^6 \cdot 3^3 \cdot 7$	[28]	$1a + 35a + 84a$
$2^6{:}L_3(2)$	$2^9 \cdot 3 \cdot 7$	[28]	$1a + 15a + 35b + 84a$
$(2 \cdot 2^6){:}(S_3 \times S_3)$	$2^9 \cdot 3^2$	[4,24]	$1a + 27a + 35b + 84a + 168a$
$S_3 \times S_6$	$2^5 \cdot 3^3 \cdot 5$	[10,18]	$1a + 27a + 35b + 105b + 168a$
$L_2(8){:}3$	$2^3 \cdot 3^3 \cdot 7$	[28]	$1a + 70a + 84a + 105b + 280a + 420a$

Table 5: The partial fusion maps into $Sp(6,2)$

$U_4(2){:}2$-class	2a	2b	2c	2d	3a	3b	3c	5a								
$\rightarrow\ G$	2A	2B	2C	2D	3B	3A	3C	5A								
h								3								
S_8-class	2a	2b	2c	2d	3a	3b	5a	7a								
$\rightarrow\ Sp(6,2)$	2A	2B	2C	2D	3A	3C	5A	7A								
h							1	1								
$2^5{:}S_6$-class	2a	2b	2c	2d	2e	2f	2g	2h	2i	2j	3a	3b	5a			
$\rightarrow\ Sp(6,2)$	2A	2C	2B	2C	2A	2B	2D	2D	2C	2D	3A	3C	5A			
h													3			
$U_3(3){:}2$-class	2a	2b	3a	3b	7a											
$\rightarrow\ Sp(6,2)$	2B	2D	3B	3C	7A											
h					1											
$2^6{:}L_3(2)$-class	2a	2b	2c	2d	2e	2f	2g	3a	7a	7b						
$\rightarrow\ Sp(6,2)$	2A	2B	2C	2D	2C	2B	2D	3C	7A	7A						
h									1	1						
$2 \cdot 2^6{:}(S_3 \times S_3)$-class	2a	2b	2c	2d	2e	2f	2g	2h	2i	2j	2k	2l	2m	3a	3b	3c
$\rightarrow\ Sp(6,2)$	2B	2C	2A	2C	2B	2A	2D	2D	2C	2B	2D	2C	2D	3A	3B	3C
h														15	9	3
$S_3 \times S_6$-class	2a	2b	2c	2d	2e	2f	2g	3a	3b	3c	3d	3e	5a			
$\rightarrow\ Sp(6,2)$	2A	2A	2B	2C	2C	2D	2D	3A	3A	3C	3C	3B	5A			
h													1			
$L_2(8){:}3$-class	2a	3a	3b	3c	7a											
$\rightarrow\ Sp(6,2)$	2D	3B	3C	3C	7A											
h					1											

Table 6: Values of h for all the maximal subgroups and non-trivial conjugacy classes of G

	H_1	H_2	H_3	H_4	H_5	H_6	H_7	H_8
$2A$	16	16	31	0	15	75	96	0
$2B$	4	12	15	24	39	43	16	0
$2C$	8	8	15	0	15	27	32	0
$2D$	4	4	7	8	7	19	16	16

Table 6 continued

	H_1	H_2	H_3	H_4	H_5	H_6	H_7	H_8
$3A$	10	6	15	0	0	15	21	0
$3B$	1	0	0	3	0	9	12	24
$3C$	1	3	3	6	9	3	3	12
$4A$	4	0	3	12	3	7	0	0
$4B$	2	6	7	0	3	7	4	0
$4C$	6	2	7	0	3	7	4	0
$4D$	0	4	3	4	11	7	0	0
$4E$	2	2	3	0	3	3	4	0
$5A$	3	1	3	0	0	0	1	0
$6A$	4	4	7	0	0	3	9	0
$6B$	4	0	3	0	0	7	1	0
$6C$	1	0	0	3	0	1	4	0
$6D$	2	2	3	0	0	3	5	0
$6E$	1	1	1	0	3	3	3	0
$6F$	1	3	3	0	3	1	1	0
$6G$	1	1	1	2	1	1	1	4
$7A$	0	1	0	1	2	0	0	1
$8A$	2	0	1	2	1	1	0	0
$8B$	0	2	1	2	1	1	0	0
$9A$	1	0	0	0	0	0	0	3
$10A$	1	1	1	0	0	0	1	0
$12A$	2	0	1	0	0	1	1	0
$12B$	0	2	1	0	0	1	1	0
$12C$	1	0	0	3	0	1	0	0
$15A$	0	1	0	0	0	0	1	0

4 The ranks of the classes of $Sp(6,2)$.

Now we study the ranks of $G = Sp(6,2)$ with respect to the various conjugacy classes of all its nonidentity elements. We start our investigation on the ranks of the non-trivial classes of G by looking at the four classes of involutions $2A$, $2B$, $2C$ and $2D$. It is well known that the rank of any of these involutions classes will be at least 3, since two involutions generate a dihedral group, which is not a simple group.

Lemma 3. $rank(G : 2A) \notin \{3, 4\}$.

Proof. We claim that G is not a $(2A, 2A, 2A, nX)$-generated group for any non-trivial conjugacy class nX of G. To do this, we will make use of Ree's Theorem (Theorem 2). Here we have $n = 28$ and $s = 4$. Thus $n(s-2)+2 = 28 \times 2 + 2 = 58$. If there exists nX of G such that G is $(2A, 2A, 2A, nX)$-generated, then we must have $c_{2A} + c_{2A} + c_{2A} + c_{nX} \leq 58$. From Table 3 we have $c_{2A} = 22$ and it follows that $c_{2A} + c_{2A} + c_{2A} + c_{nX} = 3(22) + c_{nX} = 66 + c_{nX}$, which is always greater than 58. This implies that G is not a $(2A, 2A, 2A, nX)$-generated group and consequently $rank(G : 2A) \neq 3$.
Similar arguments can be applied to show that the group G is not a $(2A, 2A, 2A, 2A, nX)$-generated for any non-trivial conjugacy class nX of G. In this case, we have $n = 28$ and $s = 5$. Thus, $n(s-2)+2 = 28\times 3+2 = 86$. Using Table 3, we can see that $4\times c_{2A}+c_{nX} = 88+c_{nX} > 86$ for any nX. Thus G is not a $(2A, 2A, 2A, 2A, nX)$-generated group and it follows that $rank(G : 2A) \neq 4$. □

Lemma 4. $rank(G : 2A) \neq 5$.

Proof. We claim that G is not a $(2A, 2A, 2A, 2A, 2A, nX)$-generated group for any non-trivial conjugacy class nX of G. To do this, we will make use of Ree's Theorem (Theorem 2) to establish the non-generation for some nX of G. For the remaining classes nX of G that can not be handled using Ree's Theorem, we will perform direct computations in GAP. For the purpose of applying Ree's Theorem, we have $n = 28$ and $s = 6$ and it follows that $n(s-2)+2 = 28 \times 4 + 2 = 114$. If there exists nX of G such that G is $(2A, 2A, 2A, 2A, 2A, nX)$-generated, then we must have $5 \times c_{2A} + c_{nX} \leq 114$; that is $110 + c_{nX} \leq 114$; that is $c_{nX} \leq 4$. From Table 3, we can see the group G will not be $(2A, 2A, 2A, 2A, 2A, nX)$-generated since $110 + c_{nX} > 114$ for any nX of G except for the classes $7A$, $8B$, $9A$, $12C$ and $15A$. Now we have used GAP and we computed $\Delta_G(2A, 2A, 2A, 2A, 2A, nX)$ for $nX \in \{7A, 8B, 9A, 12C, 15A\}$, where the computations rendered that $\Delta_G(2A, 2A, 2A, 2A, 2A, nX) = 0$. This argument, together with the above, shows that G is not a $(2A, 2A, 2A, 2A, 2A, nX)$-generated group for any non-trivial conjugacy class nX of G. Therefore, $rank(G : 2A) \neq 5$. □

Lemma 5. $rank(G : 2A) \neq 6$.

Proof. We claim that G is not a $(2A, 2A, 2A, 2A, 2A, 2A, nX)$-generated group for any non-trivial conjugacy class nX of G. To do this, we will make use of Ree's Theorem (Theorem 2) to establish the non-generation for some nX of G. For the remaining classes nX of G that can not be handled using Ree's Theorem; we will use direct computations in GAP. For the purpose of applying Ree's Theorem, we have $n = 28$ and $s = 7$ and it follows that $n(s-2)+$

$2 = 28 \times 5 + 2 = 142$. If there exists nX of G such that G is $(2A, 2A, 2A, 2A, 2A, 2A, nX)$-generated, then we must have $6 \times c_{2A} + c_{nX} \leq 142$; that is $132 + c_{nX} \leq 142$; that is $c_{nX} \leq 10$. From Table 3 we can see that group G will not be $(2A, 2A, 2A, 2A, 2A, 2A, nX)$-generated since $132 + c_{nX} > 142$ for $nX \in \{1A, 2A, 2B, 2C, 2D, 3A, 4C, 6A, 6D\}$. Next, we show that G is not a $(2A, 2A, 2A, 2A, 2A, 2A, nX)$-generated group for $nX \in \{3B, 3C, 4A, 4B, 4D, 4E, 5A, 6B, 6C, 6E, 6F, 6G, 7A, 8B, 9A, 10A, 12C, 15A\}$. For all the classes $nX \in T$ we give in Table 7 the information about the structure constant $\Delta_G(2A, 2A, 2A, 2A, 2A, 2A, nX) := \Delta(G)$, $\Sigma^*_{H_i}(2a, 2a, 2a, nx) := \Sigma^*(H_i)$ for H_i a subgroup of G, $|H_i|$, and the structures of the groups that these 7−tuples generate.

Table 7: The groups generated by $(a, b, c, d, e, x, y, z) \in (2A)^7$ where $abcdexyz$ is a fixed element in nX

nX	$\Delta(G)$	$\Sigma^*(M)$	Structure description	Order
3B	9720	9720	$S_3 \times S_3 \times S_3$	216
3C	114210	2430	$2 \times S_3 \times S_3$	72
		2430	$S_3 \times S_3$	36
		21870	$S_3 \times S_4$	144
		21870	$S_3 \times S_4$	144
		87480	S_6	720
4A	315840	8640	$2^4{:}S_4$	384
		30720	$2^3{:}S_4$	192
		276480	$2^4{:}S_5$	1920
4B	126400	4160	$2 \times S_4$	48
		5760	$2^2 \times S_4$	96
		34560	$2^4{:}S_4$	384
		81920	S_6	720
4D	16000	5760	$2^5{:}S_4$	768
		10240	$S_4 \times S_4$	576
4E	134336	1920	$2^2 \times S_4$	96
		4160	$2 \times S_4$	48
		5120	$S_3 \times S_4$	144
		5120	$S_3 \times S_4$	144
		11520	$2^4{:}S_4$	384
		24576	$2 \times S_5$	240
		40960	S_6	720
		46080	$2^4{:}S_5$	1920
5A	224375	5625	$2 \times S_5$	240
		15625	S_5	120
		62500	S_6	720
		140625	$2^4{:}S_5$	1920
6B	269730	7290	$2^4{:}S_4$	384
		29160	$2^3{:}S_4$	192
		233280	$2^4{:}S_5$	1920
6C	77760	77760	$O_5(3){:}2$	51840
6E	43740	43740	$2^5{:}S_6$	23040
6F	23328	23328	$2 \times S_6$	1440
6G	7776	7776	$2 \times S_6$	1440
7A	16807	16807	S_7	5040
8A	12288	12288	$2 \times (2^4{:}S_5$	3840
8B	40960	40960	$2^5{:}S_6$	5040
9A	59049	59049	$O_5(3){:}2$	51840
10A	31250	31250	$2^5{:}S_6$	23040
12A	2880	2880	$2 \times S_3 \times S_4$	288
12B	15552	15552	$2 \times (2^4{:}S_5)$	1080

Table 7 continued

nX	$\Delta(G)$	$\Sigma^*(M)$	Structure description	Order
12C	41472	41472	$O_5(3){:}2$	51840
15A	5625	5625	$S_5 \times S_3$	720

From the above, we can see that the group G is not $(2A, 2A, 2A, 2A, 2A, 2A, nX)$-generated for all the conjugacy classes nX of G. Therefore, we conclude that $rank(G : nX) \neq 6$. □

Proposition 1. $rank(G : 2A) = 7$.

Proof. The aim here is to show that the group G is $(2A, 7A, 15A)$-generated. The computations with GAP give $\Delta_G(2A, 7A, 15A) = 15$. From Table 4, we see that the only maximal subgroup of G that may contribute to $\Delta^*_G(2A, 7A, 15A)$ is H_2. However computations renders that $\Sigma_{H_2}(2a, 7a, 15a) = 0$ and thus $\Delta^*_G(2A, 7A, 15A) = \Delta_G(2A, 7A, 15A) = 15$. Hence G is $(2A, 7A, 15A)$-generated. It follows from Corollary 1 that $rank(G : 2A) \leq 7$. In Lemmas 3, 4, and 5 we have seen that $rank(G : 2A) \notin \{3, 4, 5, 6\}$. Hence, $rank(G : 2A) = 7$. □

Proposition 2. $rank(G : 2B) = 4$.

Proof. By Table 2, the group G acts on a 15-dimensional irreducible complex module $\mathbb{V}$, and we have $d_{2B} + d_{2B} + d_{2B} + d_{nX} = 3 \times 4 + d_{nX} < 2 \times 15$ for all the conjugacy classes of nX of G. Thus, by Scott's Theorem [22], the group G will not be a $(2B, 2B, 2B, nX)$-generated. Therefore $rank(G : 2B) \neq 3$. The direct computations with GAP show that the structure constant $\Delta_G(2B, 2B, 2B, 2B, 9A) = 4617$. Only two maximal subgroups of G have each an element of order 9. These are the groups H_1 and H_8. The intersection of these two maximal subgroups do not contain elements of order 9. Only the maximal subgroup H_1 meets the classes $2B$ and $9A$ of G. We obtain that $\Sigma_{H_1}(2B, 2B, 2B, 2B, 9A) = 243$ and $h(9A, H_1) = 1$. It follows that $\Delta^*_G(2B, 2B, 2B, 2B, 9A) = 4617 - 243 = 4374$, proving that the group G is $(2B, 2B, 2B, 2B, 9A)$-generated. Hence the result. □

Proposition 3. $rank(G : 2C) = 4$.

Proof. To show that G can not be generated by only three elements from the class $2C$, we use Scott's Theorem on the 7 dimensional irreducible module. If G is a $(2C, 2C, 2C, nX)$-generated group for any non-trivial class nX of G, then we must have $d_{2C} + d_{2C} + d_{2C} + d_{nX} \geq 2 \times 7$. However, it is clear from Table 1 that $3 \times d_{2C} + d_{nX} < 14$, for each nX of G. Thus G is not a $(2C, 2C, 2C, nX)$-generated group, and it follows that $rank(G : 2C) \neq 3$. The direct computations with GAP give $\Delta_G(2C, 4D, 15A) = 45$. Only two maximal subgroups of G have each an element of order 15, namely, H_2 and H_7. The intersection of these two maximal subgroups is isomorphic to $S_3 \times S_5$. The subgroups H_2, H_7 and $S_3 \times S_5$ having elements of order 7 will not have any contributions because their elements of order 4 do not fuse to the class $4D$ of G. Since there is no contribution from any of the three groups, we then have $\Delta^*_G(2C, 4D, 15A) = \Delta_G(2C, 4D, 15A) = 45 > 0$. This shows that the group G is $(2C, 4D, 15A)$-generated. By the application of Lemma 2, we then obtain that the group G is $(2C, 2C, 2C, 2C, (15A)^4)$-generated. Since there is only one class of order 15 in G, we then have $(15A)^4 = 15A$. Thus G becomes a $(2C, 2C, 2C, 2C, 15A)$-generated group. Hence, $rank(G : 2C) = 4$. □

Proposition 4. $rank(G:2D)=3$.

Proof. Using GAP, by taking

$$\begin{aligned} a &= (3,4)(6,22)(7,21)(8,19)(9,23)(10,17)(11,12)(13,24)(15,20)(16,26)(18,28)(25,27) \in 2D, \\ b &= (1,3,25)(2,8,26)(4,24,10)(5,6,20)(7,14,16)(9,13,18)(12,23,19)(15,17,22)(21,28,27) \in 3C. \end{aligned}$$

Then $\langle a,b\rangle = G$ with

$$ab = (1,3,24,18,27)(2,8,12,11,23,13,10,22,20,17,4,25,21,14,16)(5,6,15)(7,28,9,19,26) \in 15A.$$

Thus the group G is $(2D,3C,15A)$-generated and consequently using Lemma 2, it will follow that G is $(2D,2D,2D,(15A)^3)$-generated. Since there is only one conjugacy class of elements order 5 in G, it follows that $(15A)^3 = 5A$. Thus G is $(2D,2D,2D,5A)$-generated group. We deduce that $rank(G:2D)=3$. □

Proposition 5. $rank(G:3A)=4$.

Proof. To show that G cannot be generated by only two (or three) elements from class $3A$, we use Scott's Theorem on the 7 dimensional irreducible module. If G is $(3A,3A,nX)$-generated group (or $(3A,3A,3A,nX)$-generated group) for any non-trivial class nX of G, then we must have $d_{3A}+d_{3A}+d_{nX} \geq 2\times 7$ (or $d_{3A}+d_{3A}+d_{3A}+d_{nX} \geq 2\times 7$). However, it is clear from Table 1 that $2\times d_{3A}+d_{nX} < 14$ (or $3\times d_{3A}+d_{nX} < 14$), for each nX of G. Thus, G is neither $(3A,3A,nX)$- nor $(3A,3A,3A,nX)$-generated group, and it follows that $rank(G:3A) \notin \{2,3\}$. Direct computations show that $\Delta_G(3A,3A,3A,3A,9A)=229797$ and only two maximal subgroups of G have each an element of order 9, namely, H_1 and H_8. The intersection of these two maximal subgroups is isomorphic to 9:6. The subgroups 9:6 and H_8 will not have any contributions because their elements of 3 do not fuse to the class $3A$ of G. Only H_1 meets the classes $3A$ and $9A$ of G. By GAP we have $\Sigma_{H_1}(3b,3b,3b,3b,9a)=118989$, and by Table 6 we have $h(9A,H_1)=1$. We then obtain that $\Delta^*_G(3A,3A,3A,3A,9A) \geq \Delta_G(3A,3A,3A,3A,9A) - \Sigma_{H_1}(3b,3b,3b,3b,9a) = 229797-118989 = 110808$, proving that G is $(3A,3A,3A,3A,9A)$-generated. Hence the result. □

Remark 1. *An alternative way to show that G is not $(3A,3A,nX)$-generated group for any non-trivial class nX of G, we note that the direct computations yield $\Delta_G(3A,3A,nX)=0$ for all non-trivial classes nX of G except for $nX \in \{2C,3A,3C,4A,5A,6B\} := T$. For $nX \in T$, we have $\Delta_G(3A,3A,2C)=32<1536=|C_G(2C)|$, $\Delta_G(3A,3A,3A)=46<2160=|C_G(3A)|$, $\Delta_G(3A,3A,3C)=2<108=|C_G(3C)|$, $\Delta_G(3A,3A,4A)=16<384=|C_G(4A)|$, $\Delta_G(3A,3A,5A)=5<30=|C_G(5A)|$ and $\Delta_G(3A,3A,6B)=6<144=|C_G(6B)|$. Then using Lemma 1, we deduce that G is not $(3A,3A,nX)$-generated group for $nX \in T$ and thus $rank(G:3A)\neq 2$.*

Proposition 6. $rank(G:3B)=3$.

Proof. To show that G is not $(3B,3B,nX)$-generated group for any non-trivial class nX of G, we note that the direct computations yield $\Delta_G(3B,3B,nX)=0$ for all non-trivial classes nX of G except for $nX \in \{2C,3A,3B,3C,4A,4D,5A,6C,6D,7A,9A,15A\}$. By GAP we obtained that $\Delta_G(3B,3B,7A)=7$. Although the maximal subgroups H_4 and H_8 are the only ones meeting the

$3B$, $7A$ classes of G, the maximal subgroup H_4 will not contribute because its relevant structure constant is zero. We obtained that $\sum_{H_8}(3a, 3a, 7a) = 7$ and we have $h(7A, H_8) = 1$. Thus we obtain $\Delta^*_G(3B, 3B, 7A) = \Delta_G(3B, 3B, 7A) - \sum_{H_8}(3a, 3a, 7a) = 7 - 7 = 0$, proving that G is not $(3B, 3B, 7A)$-generated. Direct computations show that $\Delta_G(3B, 3B, 9A) = 9$. Subgroups meeting the classes $3B$ and $9A$ of G are 9:6, H_1 and H_8. The subgroups 9:6 and H_1 will not have any contributions because their relevant structure constants are all zeros. The direct computations show that $\Sigma_{H_8}(3a, 3a, 9x) = \Delta_{H_8}(3a, 3a, 9a) + \Delta_{H_8}(3a, 3a, 9b) + \Delta_{H_8}(3a, 3a, 9c) = 0+0+9 = 9$. We found that $h(9A, H_8) = 1$. It follows that $\Delta^*_G(3B, 3B, 9A) = \Delta_G(3B, 3B, 9A) - \Sigma_{H_8}(3a, 3a, 9x) = 9 - 9 = 0$, showing the non-generation of G by the triple $(3B, 3B, 9A)$. Let $T := \{2C, 3A, 3B, 3C, 4A, 4D, 5A, 6C, 6D, 15A\}$. For $nX \in T$, we have $\Delta_G(3B, 3B, 2C) = 64 < 1536 = |C_G(2C)|$, $\Delta_G(3B, 3B, 3A) = 40 < 2160 = |C_G(3A)|$, $\Delta_G(3B, 3B, 3B) = 28 < 648 = |C_G(3B)|$, $\Delta_G(3B, 3B, 3C) = 20 < 108 = |C_G(3C)|$, $\Delta_G(3B, 3B, 4A) = 16 < 384 = |C_G(4A)|$, $\Delta_G(3B, 3B, 4D) = 16 < 128 = |C_G(4D)|$, $\Delta_G(3B, 3B, 5A) = 10 < 30 = |C_G(5A)|$, $\Delta_G(3B, 3B, 6C) = 12 < 72 = |C_G(6C)|$, $\Delta_G(3B, 3B, 6D) = 8 < 48 = |C_G(6D)|$ and $\Delta_G(3B, 3B, 15A) = 5 < 15 = |C_G(15A)|$. Then using Lemma 1 we deduce that G is not $(3B, 3B, nX)$-generated group for $nX \in T$ and thus $rank(G : 3B) \neq 2$.

Direct computations show that $\Delta_G(3B, 3C, 15A) = 25$. As in Proposition 3, subgroups having elements of order 15 are $S_3 \times S_5$, H_2 and H_7. The subgroups $S_3 \times S_5$ and H_2 will not have any contributions because their elements of 3 do not fuse to the class $3B$ of G. Only H_7 meets the classes $3B$, $3C$ and $15A$ of G. The direct computations show that $\sum_{H_7}(3e, 3x, 15a) = \Delta_{H_7}(3e, 3c, 15a) + \Delta_{H_7}(3e, 3d, 15a) = 5 + 5 = 10$ and we have $h(15A, H_7) = 1$ (see Table 6). We then obtain that $\Delta^*_G(3B, 3C, 15A) \geq \Delta_G(3B, 3C, 15A) - \sum_{H_7}(3e, 3x, 15a) = 25 - 10 = 15 > 0$, proving that G is $(3B, 3C, 15A)$-generated. By the application of Lemma 2, we then obtain that the group G is $(3B, 3B, 3B, (15A)^3)$-generated. Since the class $15A$ is only one of order 15 in G, we then have $(15A)^3 = 5A$ so that G becomes $(3B, 3B, 3B, 5A)$-generated. Hence $rank(G : 3B) = 3$. □

Proposition 7. $rank(G : 3C) = 2$.

Proof. The proof of Proposition 4 included the information that G is a $(2D, 3C, 15A)$-generated group. The result now follows by applications of Corollary 2. □

Proposition 8. $rank(G : 4A) = 3$.

Proof. To show that only two elements from class $4A$ can not generate G, we use Scott's Theorem on the 7 dimensional irreducible module. If G is $(4A, 4A, nX)$-generated group for any non-trivial class nX of G, then we must have $d_{4A} + d_{4A} + d_{nX} \geq 2\times 7$. However, it is clear from Table 1 that $2\times d_{4A} + d_{nX} = 2(4) + d_{nX} < 14$, for each $nX \in \{2B, 2C, 2D, 3A, 3C, 4A, 4B, 4E, 5A, 6B, 6D\} := T$. Thus G is not $(4A, 4A, nX)$-generated group for each $nX \in T$. Let $T_1 := \{2A, 3B, 4C, 4D, 6A, 6E, 6F, 6G, 8A, 10A, 12A, 12B, 15A\}$. By GAP we have $\Delta_G(4A, 4A, nX) = 0$ for $nX \in T_1$, thus G is not $(4A, 4A, nX)$-generated for all $nX \in T_1$. Also the computations yield that $\Delta_G(4A, 4A, 12C) = 11 < 12 = |C_G(12C)|$, showing that the group G is not $(4A, 4A, 12C)$-generated. By GAP, we have $\Delta_G(4A, 4A, 6C) = 114$, $\sum_{H_1}(4a, 4a, 6d) = 24$, $\sum_{H_4}(4b, 4b, 6b) = 14$, $\sum_{H_6}(4d, 4d, 6a) = 18$, and by Table 6, $h(4A, H_4) = 3$. We then obtain that $\Delta^*_G(4A, 4A, 6C) = \Delta_G(4A, 4A, 6C) - \sum_{H_1}(4a, 4a, 6d) - 3 \cdot \sum_{H_4}(4b, 4b, 6b) - \sum_{H_6}(4d, 4d, 6a) = 114 - 24 - 3 \times 14 -$

$18 = 30 < 72 = |C_G(6C)|$. Again by GAP, we have $\Delta_G(4A, 4A, 7A) = 7$. Subgroups fusing to $4A$ and $7A$ have all their relevant structure constants zero except the maximal subgroup H_4. Since $\sum_{H_4}(4c, 4c, 7a) = 7$ and $h(7A, H_4) = 1$, we then obtain that $\Delta^*_G(4A, 4A, 7A) = \Delta_G(4A, 4A, 7A) - \sum_{H_4}(4c, 4c, 7a) = 7 - 7 = 0$. Similarly we obtain the following results, $\Delta^*_G(4A, 4A, 8B) = \Delta_G(4A, 4A, 8B) - \sum_{H_3}(4f, 4f, 8b) - \sum_{H_5}(4a, 4a, 8b) = 36 - 32 - 8 = 0$, $\Delta^*_G(4A, 4A, 9A) = \Delta_G(4A, 4A, 9A) - \sum_{H_1}(4a, 4a, 9a) = 9 - 9 = 0$ and $\Delta^*_G(4A, 4A, 6C) = \Delta_G(4A, 4A, 6C) - \sum_{H_1}(4c, 4c, 6a) = 27 - 27 = 0$. These show that the group G is not $(4A, 4A, 6C)$-, $(4A, 4A, 7A)$-, $(4A, 4A, 8B)$- and $(4A, 4A, 9A)$-generated. Thus $rank(G : 4A) \neq 2$. Easy computations show that $\Delta_G(4A, 4A, 4A, 9A) = 47385$. The subgroups having elements of order 9 are H_1, H_8 and 9:6. The subgroups H_8 and 9:6 do not have elements of order 4. Only H_1 meets the classes $4A$ and $9A$ of group G. We obtained that $\sum_{H_1}(4a, 4a, 4a, 9a) = 5832$. We then have

$$\Delta^*_G(4A, 4A, 4A, 9A) \geq \Delta_G(4A, 4A, 4A, 9A) - \sum_{H_1}(4a, 4a, 4a, 9a) = 47385 - 5832 = 41553 > 0,$$

proving that G is $(4A, 4A, 4A, 9A)$-generated. Hence the result. □

Proposition 9. $rank(G : 4B) = 3$.

Proof. To show that G cannot be generated by only two elements from class $4B$, we use Scott's Theorem on the 7 dimensional irreducible module. If G is $(4B, 4B, nX)$-generated group for any non-trivial classes nX of G, then we must have $d_{4B} + d_{4B} + d_{nX} \geq 2 \times 7$. However, it is clear from Table 1 that $2 \times d_{4A} + d_{nX} = 2(4) + d_{nX} < 14$, for each $nX \in \{2B, 2C, 2D, 3A, 3C, 4A, 4B, 4E, 5A, 6B, 6D\}$ of G. Thus, G is not $(4A, 4A, nX)$-generated group, for each nX in the previous set. Let $T_1 := \{2A, 3B, 4C\}$. By GAP, we have $\Delta_G(4B, 4B, nX) = 0$, so that the group G is not $(4A, 4A, nX)$-generated for all $nX \in T_1$. We obtained that $\Delta_G(4B, 4B, 4D) = 64 < 192 = |C_G(4D)|$, $\Delta_G(4B, 4B, 6A) = 36 < 144 = |C_G(6A)|$, $\Delta_G(4B, 4B, 6C) = 18 < 72 = |C_G(6C)|$, $\Delta_G(4B, 4B, 6E) = 18 < 36 = |C_G(6E)|$, $\Delta_G(4B, 4B, 8A) = 8 < 16 = |C_G(8A)|$, $\Delta_G(4B, 4B, 12A) = 12 < 24 = |C_G(12A)|$, $\Delta_G(4B, 4B, 12B) = 12 < 24 = |C_G(12B)|$ and $\Delta_G(4B, 4B, 12C) = 3 < 12 = |C_G(12C)|$ so that the group G is not $(4B, 4B, nX)$-generated for each $nX \in \{4D, 6A, 6C, 6E, 8A, 12A, 12B, 12C\}$. It follows that $rank(G : 4B) \neq 2$. Easy computations show that $\Delta_G(3B, 4B, 9A) = 9$. Although the subgroup H_1 meets the classes $3B$, $4B$, and $9A$ of the group G, it will not have any contribution because its relevant structure is zero. Therefore $\Delta^*_G(3B, 4B, 9A) = \Delta_G(3B, 4B, 9A) = 9 > 0$, proving that G is $(3B, 4B, 9A)$-generated. Since .G is $(3B, 4B, 9A)$-generated, it is also $(4B, 3B, 9A)$-generated group, and it follows by applications of Lemma 2 that G is $(4B, 4B, 4B, (9A)^3)$-generated; that G is $(4B, 4B, 4B, 3B)$-generated group. We deduce that $rank(G : 4B) = 3$. □

Proposition 10. $rank(G : 4C) = 3$.

Proof. Let $nX \in \{2A, 2B, 2C, 2D, 3A, 3B, 3C, 4A, 4B, 4C, 4D, 4E, 5A, 6A, 6B, 6D, 6E, 10A\} := T$. If G is a $(4C, 4C, nX)$-generated group for $nX \in T$, then we must have $c_{4C} + c_{4C} + c_{nX} \leq 30$. We use Ree's Theorem by acting G on 28 points. By Table 3 we have $c_{4C} + c_{4C} + c_{nX} = 12 + 12 + c_{nX} > 30$ for all $nX \in T$ and by Ree's Theorem we conclude that G is not $4C, 4C, nX)$-generated group for any $nX \in T$. The computations with GAP render that $\Delta_G(4C, 4C, nX) = 0$ for $nX \in \{6F, 6G, 8A, 12A, 12B, 15A\}$. Therefore G is not a $(4C, 4C, nX)$-generated group for

nX in the previous set. We now check the tuple $(4C, 4C, 6C)$ to see whether it generates G or not. The GAP computations give $\Delta_G(4C, 4C, 6C) = 162$. We also obtained that $\sum_{H_1}(4c, 4c, 6d) = 162$ and $h = 1$. Subgroups fusing to $4C$ and $6C$ have all their relevant structure constant zero except the maximal subgroup H_1. Thus $\Delta^*_G(4C, 4C, 6C) = \Delta_G(4C, 4C, 6C) - \sum_{H_1}(4c, 4c, 6d) = 162 - 162 = 0$, showing that G is not a $(4C, 4C, 6C)$-generated group. Similarly, we obtain the following results:

$$\begin{aligned}
\Delta^*_G(4C, 4C, 7A) &= \Delta_G(4C, 4C, 7A) - \sum_{H_2}(4a, 4a, 7a) = 7 - 7 = 0,\\
\Delta^*_G(4C, 4C, 8B) &= \Delta_G(4C, 4C, 8B) - \sum_{H_3}(4f, 4f, 8b) - \sum_{H_5}(4a, 4a, 8b) = 40 - 32 - 8 = 0,\\
\Delta^*_G(4C, 4C, 9A) &= \Delta_G(4C, 4C, 9A) - \sum_{H_1}(4c, 4c, 9a) = 81 - 81 = 0,\\
\Delta^*_G(4C, 4C, 12C) &= \Delta_G(4C, 4C, 12C) - \sum_{H_1}(4c, 4c, 12a) = 27 - 27 = 0.
\end{aligned}$$

This show that the group G is not $(4C, 4C, nX)$-generated for each $nX \in \{6C, 7A, 8B, 9A, 12C\}$. We deduce that $rank(G : 4C) \neq 2$. Now, easy computations show that $\Delta_G(3B, 4C, 9A) = 27$. Although the subgroup H_1 meets the classes $3B$, $4C$ and $9A$ of the group G, it will not have any contribution because its relevant structure is zero. No other subgroup of G meets the classes $3B$, $4C$, and $9A$, and thus no contribution from any maximal subgroup of Gin the computations of $\Delta^*_G(3B, 4C, 9A)$. We then obtain that

$$\Delta^*_G(3B, 4C, 9A) = \Delta_G(3B, 4C, 9A) = 27 > 0,$$

proving that G is $(3B, 4C, 9A)$-generated. Since G is $(3B, 4C, 9A)$-generated, it is also $(4C, 3B, 9A)$-generated group. By applications of Lemma 2, G will be $(4C, 4C, 4C, (9A)^3)$-generated group, that is G is a $(4C, 4C, 4C, 3B)$-generated group. Hence $rank(G : 4C) = 3$. □

Proposition 11. $rank(G : 4D) = 2$.

Proof. We have seen in the proof of Proposition 3 that the group G is a $(2C, 4D, 15A)$-generated. Applications of Corollary 2 shows that $rank(G : 4D) = 2$. □

Proposition 12. $rank(G : 6A) = 3$.

Proof. Let $nX \in \{2A, 2B, 2C, 2D, 3A, 3B, 3C, 4A, 4B, 4C, 4D, 4E, 5A, 6A, 6B, 6D, 6E, 10A\}$. If G is a $(6A, 6A, nX)$-generated group, then we must have $c_{6A} + c_{6A} + c_{nX} \leq 30$. We use Ree's Theorem by acting G on 28 points. Since by Table 3 we have $c_{6A} + c_{6A} + c_{nX} = 12 + 12 + c_{nX} > 30$, it follows by Ree's Theorem that G is not $(6A, 6A, nX)$-generated group for any nX in the previous set. By GAP, we have $\Delta_G(6A, 6A, 12A) = 8 < 24 = |C_G(12A)|$, so that the group G is not $(6A, 6A, 12A)$-generated. By GAP we have $\Delta_G(6A, 6A, 6C) = 144$. Subgroups fusing to $6A$ and $6C$ have all their relevant structure constant zero except the maximal subgroup H_1. Since $\sum_{H_1}(6b, 6b, 6d) = 144$, and the value of h is 1 (see Table 6), we then obtain that

$$\Delta^*_G(6A, 6A, 6C) = \Delta_G(6A, 6A, 6C) - \sum_{H_1}(4a, 4a, 7a) = 144 - 144 = 0.$$

Similarly, we obtain the following results:
$\Delta^*_G(6A,6A,6F) = \Delta_G(6A,6A,6F) - \sum_{H_1}(6b,6b,6e) = 72-72 = 0$, $\Delta^*_G(6A,6A,6G) = \Delta_G(6A,6A,6G) - \sum_{H_2}(6d,6d,6a) = 24-24 = 0$, $\Delta^*_G(6A,6A,7A) = \Delta_G(6A,6A,7A) - \sum_{H_2}(6d,6d,7a) = 63-63 = 0$, $\Delta^*_G(6A,6A,8A) = \Delta_G(6A,6A,8A) - \sum_{H_1}(6a,6a,8a) = 32-32 = 0$, $\Delta^*_G(6A,6A,8B) = \Delta_G(6A,6A,8B) - \sum_{H_2}(6d,6d,8a) = 64-64 = 0$, $\Delta^*_G(6A,6A,9A) = \Delta_G(6A,6A,9A) - \sum_{H_1}(6b,6b,9a) = 81-81 = 0$, $\Delta^*_G(6A,6A,12B) = \Delta_G(6A,6A,12B) - \sum_{H_2}(6b,6b,6a) = 32-32 = 0$, $\Delta^*_G(6A,6A,12C) = \Delta_G(6A,6A,12C) - \sum_{H_1}(6b,6b,12a) = 48-48 = 0$, $\Delta^*_G(6A,6A,15A) = \Delta_G(6A,6A,15A) - \sum_{H_2}(6d,6d,15a) = 25-25 = 0$. From the above we deduce that G is not $(6A,6A,nX)$-generated group for any $nX \in \{6C,6F,6G,7A,8A,8B,9A,12B,12C,15A\}$. This, together with the information that G is not $(6A,6A,nX)$-generated group for all $nX \in \{2A,2B,2C,2D,3A,3B,3C,4A,4B,4C,4D,4E,5A,6A,6B,6D,6E,10A\}$, we deduce that $rank(G:6A) \neq 2$. Easy computations show that $\Delta_G(3B,6A,9A) = 27$. Although the subgroup H_1 meets the classes $3B$, $6A$, and $9A$ of the group G, it will not have any contribution because its relevant structure is zero. We then obtain that $\Delta^*_G(6A,3B,9A) = \Delta^*_G(3B,6A,9A) = 27 > 0$, proving that G is $(6A,3B,9A)$-generated; that is G is $(6A,3B,9A)$-generated group. By Lemma 2 we obtain that G is a $(6A,6A,6A,(9A)^3)$-generated group; that is G is a $(6A,6A,6A,3A)$-generated group. Hence the result. □

Proposition 13. *$rank(G:6B) = 2$.*

Proof. The structure constant gives us $\Delta_G(6B,6B,8B) = 40$ and there are six maximal subgroups of G have each an element of order 8, namely, H_1,H_2,H_3,H_4,H_5, and H_6. Let T be the set of all maximal subgroups of G having elements of order 8. The intersection of conjugacy classes from any 6, 5, 4 or 3 maximal subgroups of T does not contain elements of order 8. The groups formed when taking the intersection of any two maximal subgroups of T will be isomorphic to $PSL_3(2){:}2 := M_1$, $((((2^2\times 2^4){:}2){:}2){:}3){:}2 := M_2$ (3-copies), $2^4{:}S_5 := M_3$, $(3^2{:}3){:}QD_{16} := M_4$, $(((2^3{:}2^2){:}3){:}2){:} := M_5$ (4-copies) or $(S_4\times S_4){:}2 := M_6$. Out of all subgroups having elements of order 8, only M_2, M_3, and M_6 meet the classes $6B$ and $8B$ of G. None of them will have any contributions because their relevant structure constants are all zeros. We then obtain that $\Delta^*_G(6B,6B,8B) \geq \Delta_G(6B,6B,8B) = 40$, proving that G is $(6B,6B,8B)$-generated. The result follows. □

Proposition 14. *Let $nX \in T_1 := \{4E,5A,6C,6D,6E,6F,6G,7A,8A,8B,9A,10A,12A,12B,12C,15A\}$. Then $rank(G:nX) = 2$*

Proof. The maximal subgroups H_2 and H_7 are the only ones containing elements of order 15. The group $S_3\times S_5$ has elements of order 15, and it arises from taking the intersection of these two maximal subgroups of G. We use Table 8 to establish the results of this proposition. In this Table we give the required information needed to calculate $\Theta_G(nX,nX,15A)$ where $nX \in T_1$. The value of h for these contributing subgroups is 1 (see Table 6). We give some information about $\Delta_G(nX,nX,15A)$, h, $\sum_{H_2}(nx,nx,15a)$, $\sum_{H_7}(nx,nx,15a)$ and $\sum_{S_3\times S_5}(nx,nx,15a)$. The last column

$$\Theta_G \quad = \quad \Theta_G(nX,nX,15A) = \Delta_G(nX,nX,15A) - h\cdot\sum_{H_2}(nx,nx,15b)$$

$$- \quad h \cdot \sum_{H_7}(nx, nx, 15b) + h \cdot \sum_{S_3 \times S_5}(nx, nx, 15a)$$

establishes each generation of G by its triples $(nX, nX, 15A)$ because $\Delta^*_G(nX, nX, 15A) \geq \Theta_G(nX, nX, 15A)$ as it appears in Equation (1). Looking at Table 8, we see that $\Delta^*_G(nX, nX, 15A) > 0$. It follows that G is $(nX, nX, 15A)$-generated, where $nX \in T_1$. This proves that $rank(G : nX) = 2$ for all $nX \in T_1$. □

Table 8: Some information on the $nX \in T_1$

nX	$\Delta_G(nX, nX, 15A)$	h	$h \cdot \sum_{H_2}(nx, nx, 15a)$	$h \cdot \sum_{H_7}(nx, nx, 15a)$	$h \cdot \sum_{S_3 \times S_5}(nx, nx, 15a)$	Θ_G
$4E$	1290	1	270	60	45	1005
$5A$	645	1	45	0	0	600
$6C$	280	1	-	40	-	240
$6D$	845	1	155	125	20	585
$6E$	1260	1	510	0	-	750
$6F$	1180	1	35	55	5	1095
$6G$	8580	1	510	120	-	7950
$7A$	28605	1	1620	-	-	26985
$8A$	5100	1	-	-	-	5100
$8B$	5100	1	1140	-	-	3960
$9A$	15645	1	-	-	-	15645
$10A$	15864	1	789	159	24	14940
$12A$	1490	1	-	20	-	1470
$12B$	2450	1	540	20	15	1905
$12C$	10920	1	-	-	-	10920
$15A$	5933	1	308	53	8	5580

We conclude this paper by collecting all the results for the ranks of the non-trivial conjugacy classes of $Sp(6, 2)$ in the following theorem.

Theorem 6. *Let nX be a non-trivial conjugacy class of $G = Sp(6, 2)$. Then*

$$rank(G : nX) = \begin{cases} 7 & \text{if } nX = 2A, \\ 4 & \text{if } nX \in \{2B, 2C, 3A\}, \\ 3 & \text{if } nX \in \{2D, 3B, 4A, 4B, 4C, 6A\}, \\ 2 & \text{if } nX \notin \{1A, 2A, 2B, 2C, 2D, 3A, 3B, 4A, 4B, 6A\}. \end{cases}$$

Proof. Established through Propositions 1 to 14.

□

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