Journal of Mathematical Modeling
Vol. -, No. -, 2026, pp. -. Research Article

JMM

A study on advanced solutions for fractional integro-differential equations integrating Sawi transform and machine learning techniques

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Abstract. In this study, a direct method of fractional calculus approach to particular classes of fractional integro-differential equations is given. The method reveals a number of interesting results most notably an extension of the familiar classical Frobenius' solution. The investigation is mainly based upon the basic results which are given to determine the fractional integro-differential equations by means of the Sawi transform and some extension coefficients defined from binomial series. Newer techniques for the efficient solution of these equations are also discussed and practical examples are used to demonstrate their use. In addition, we consider using a learning-based approach to improve the computation of our solution and illustrate how data-driven approaches can be used for obtaining approximate solutions in cases where analytical methods are not feasible or not efficient. The findings underscore the efficacy of combining classical and modern approaches in addressing complex fractional integro-differential equations.

Keywords: Fractional-order integro-differential equation, Gamma function, Riemann-Liouville fractional integrals, Mittag-Leffler function, Sawi transform.
AMS Subject Classification 2010: 26A33; 31A10; 33C10; 34A05; 35K37.

1 Introduction

Integral transforms and fractional calculus are interrelated topics in mathematical analysis, which open and develop branches of knowledge to solve directly or indirectly several specialized problems. Integral

*Corresponding author
Received: 06 November 2025/ Revised: 15 April 2026/ Accepted: 16 April 2026
DOI: 10.22124/jmm.2026.32197.2917

transforms are widely used to transfer functions between various domains, and they allow us to view certain depth-related issues from different perspectives. The Laplace transform and its variants are of widespread use in mathematics, physics, and engineering. They help us to comprehend the dynamic systems and offer the simple solutions of differential equations [5, 12].

The expansion of the notion of integrals and derivatives in fractional calculus to include non-integer orders allows for a more thorough retention of systems using fractional derivatives. The fractional Laplace transform and other fractional integral transforms were developed by mathematicians to extend classical analysis [7, 10]. Anomalous diffusion and viscoelasticity are two examples of non-integer differentiable phenomena that rely on these transformations [10].

When combined, integral transforms and fractional calculus provide additional lenses through which to view complicated systems and access to more sophisticated approaches than are available in conventional calculus alone. The early works of Spiegel [5] and Schiff [12] are very important for understanding integral transformations and how they can be used in engineering, physics, and math, among other fields.

Distinguished researchers as Fadhil [1] study convolution methods which are specially devoted to Kamal and Mahgoub transformations which give useful insights of complex mathematical algorithms. Kim studies the underlying structures and properties of integral transforms, such as Laplace-type transform [2, 6]. The contribution of Elzaki, using the Elzaki Transform [3] provides another approach for solving the problem. Also, Mahgoub and Mohand present the unique Sawi transform, a specific kind of integral transformation with different applications [4]. In [7], Medina et al. provide an introduction to fractional calculus and the fractional Laplace transform. They also demonstrate the fact that in complex systems fractional integral transforms can be employed. As outlined in [8], Watugala's Sumudu transform proves highly efficient in resolving differential equations and tackling challenges in control engineering.

Raghavendran et al. [11] introduced the Aboodh transform which was a major development in integral transformations. The transform method presented here demonstrates the flexibility of integral transforms by allowing them to be applied to fractional integro-differential equations. Significant contributons to the knowledge of integral transforms in engineering sciences and nonlinear anlysis are respectively the research of Khan [14] by N-transforms at Belgacem [13] on Sumudu transforms. This series of publications presents new integrals and computing techniques used in determining integral transformations to solve scientific problems. Recently the machine learning has opened by several researchers to tackle challenging mathematical problems such as fractional-integro-differential equations. By leveraging large-scale data, machine learning models, particularly deep learning architectures, have demonstrated the capability to approximate solutions to differential equations that are otherwise difficult to solve analytically.

Ghoreyshi et al. [16] developed a finite block method which solves nonlinear time-fractional partial integro-differential equations. They conducted an extensive investigation which demonstrated the precise stability and convergence characteristics of their proposed method and also developed numerical methods which provide accurate results and maintain their accuracy throughout their calculations for establishing solutions to fractional partial integro-differential equations. The research showed that properly developed numerical methods enable successful computation of nonlinear fractional systems without losing accuracy during processing. Moreover, Ghoreyshi et al. [17, 18] developed strong numerical methods which allow researchers to solve distributed-order space–time fractional partial differential equations. The studies showed that researchers need to create precise approximation methods which can handle complex fractional operators while treating distributed-order models as tools for studying multi-scale memory phenomena in real-world systems.

Recent advances have established the need for dependable analytical methods together with computational approaches which solve differential equations that use nonlocal and fractional operators. Saini et al. [20] created a precise computational method which solves differential equation systems that contain nonlocal derivatives and shows improved numerical results with stable performance. Madhumitha et al. [21] proved that neutral functional random integro-differential equations with infinite delay have existence solutions which demonstrate theoretical difficulties that arise from memory effects together with functional dependencies. The current Sawi transform–based hybrid approach developed the systematic framework which combines precise analytical methods with effective computational techniques because of increasing demand for such systems.

Ghoreyshi et al. [19] created high-order numerical methods which use Lagrange polynomial interpolation to solve distributed-order time-fractional partial integro-differential equations for non-rectangular computational fields. Their research solved problems with geometric complexity and proved that irregular computational domains still allow for higher-order accuracy achievements. Recent studies show that researchers increasingly pursue advanced numerical methods to study fractional models and they need to investigate other analysis methods which combine analytical and hybrid approaches. The research aims to combine analytical methods which use transform techniques with machine learning methods to create effective solutions which work flexibly for fractional integro-differential equations.

In this work, we employ the Sawi transform together with the binomial series extension coefficient to construct analytical solutions of certain classes of fractional integro-differential equations. Several fundamental properties of the Sawi transform relevant to fractional operators are derived and utilized to simplify the solution procedure. Furthermore, a hybrid computational framework is proposed in which the analytical Sawi transform approach is integrated with a supervised machine learning approximation scheme. The transform method converts the fractional integro-differential equations into algebraic forms, while the learning-based model is used to approximate solution patterns in cases where closed-form expressions become difficult to obtain. The performance of the proposed approach is demonstrated through illustrative examples and validated using standard error metrics.

The remainder of this paper is organized as follows. Section 2 presents the necessary preliminaries on fractional calculus and the Sawi transform. Section 3 develops the proposed analytical framework and provides illustrative examples. Section 4 introduces the learning-based approximation procedure and discusses the training strategy and evaluation criteria. Finally, Section 5 concludes the paper.

2 Preliminaries

In this section, we are listing some preliminaries that are useful throughout the paper [2, 7, 9, 10, 15].

Definition 1. *The definition of the RL fractional integral with order $\zeta > 0$ for a function $\Upsilon(t)$ can be expressed as follows:*

$$I^{\zeta}{}_{t}\Upsilon(t) = \frac{1}{\Gamma(\zeta)}\int_0^t (t-\eta)^{\zeta-1}\,\Upsilon(\eta)\,d\eta.$$

Definition 2. *The Caputo fractional derivative of the function $\Upsilon(t)$ is defined as follows:*

$$D^{\zeta}{}_{t}\Upsilon(t) = \begin{cases} \Upsilon^{i}(t) & \text{if } \vartheta = i \in \mathbb{N}, \\ \frac{1}{\Gamma(i-\zeta)}\int_0^{\zeta} \frac{\Upsilon^{i}(t)}{(t-x)^{\zeta-i+1}}dt & \text{if } i-1 < \zeta < i. \end{cases}$$

The Euler Gamma function, denoted as $\Gamma(.)$, *is defined as follows:*

$$\Gamma(\psi) = \int_0^\infty t^{\psi-1} e^{-t}\, dt \qquad (\mathbb{R} > 0).$$

Definition 3. *The Sawi transform of a function* $\Upsilon(t)$, $t \in (0,\infty)$ *is defined by*

$$A[\Upsilon(t)](\omega) = F(\omega) = \frac{1}{\omega^2}\int_0^\infty e^{-\frac{t}{\omega}}\, \Upsilon(t)\, dt; \, (\omega \in \mathbb{C}).$$

Definition 4. *The Mittag-Leffler function is defined by*

$$E_{\delta,\gamma}(\psi) = \sum_{i=0}^{\infty} \frac{\psi^i}{\Gamma(\delta\varepsilon+\gamma)} \qquad (\delta,\gamma,\psi \in \mathbb{C},\, \mathbb{R}(\delta) > 0).$$

Definition 5. *The simplest Wright function is defined by*

$$\rho(\omega,\psi;\phi) = \sum_{\varepsilon=0}^{\infty} \frac{1}{\Gamma(\omega\varepsilon+\psi)}.\frac{\phi^\varepsilon}{\varepsilon!} \qquad (\phi,\psi,\omega \in \mathbb{C}).$$

Definition 6. *The general Wright function* ${}_i\chi_j(\varphi)$ *is characterized by the following conditions* $\varphi \in \mathbb{C}$, $\nu_{1l}, \nu_{2m} \in \mathbb{C}$, *and real* $\omega_l, \phi_m \in \mathbb{R}$ $(l = 1,\ldots, i,\ m = 1,\ldots, j)$, *as determined by the provided series*

$$i\chi j(\nu) = i\chi j\left(\begin{matrix} (\nu_{1l},\omega_l)_{1,i} \\ (\nu_{2m},\phi_m)_{1,j} \end{matrix} \mid \varphi \right) = \sum_{\varepsilon=0}^{\infty} \frac{\prod_{l=1}^{i}\Gamma(\nu_{1l}+\omega_l r)}{\prod_{m=1}^{j}\Gamma(\nu_{2m}+\phi_m r)}.\frac{\varphi^\varepsilon}{\varepsilon!}.$$

Definition 7. *The inverse Sawi transform is defined by*

$$A^{-1}\left[\frac{\Gamma(z+1)}{\omega^{1-z}}\right] = t^z.$$

Definition 8. *The convolution integral of Sawi transform is defined by*

$$A[(\hbar * \mathrm{g})(t)] = \omega^2 A[\hbar(t)]A[\mathrm{g}(t)].$$

The Sawi transform, introduced by Mahgoub and Mohand [4], possesses fundamental operational properties such as linearity, scaling, and shifting, which makes it suitable for the analysis of fractional integro-differential equations. In particular, the transform of fractional derivatives leads to compact algebraic expressions, allowing certain classes of integro-differential equations to be reduced to solvable algebraic forms. Compared with classical integral transforms such as the Laplace transform [5, 12], the Sawi transform may produce simplified operational representations in some cases. However, its applicability is restricted to functions satisfying specific convergence conditions, and its inverse transform tables are not yet as extensively developed as those available for classical transforms. Further theoretical development is therefore required to extend its operational rules to broader classes of fractional operators and boundary value problems.

Remark 1.

$$\begin{aligned}
A\left[D^{\aleph}\hbar(t)\right](\omega) &= \frac{1}{\omega^2}\int_0^{\infty} e^{-\frac{t}{\omega}}\left[D^{\aleph}\hbar(t)\right]dt \\
&= \frac{1}{\omega^2}\int_0^{\infty} e^{-\frac{t}{\omega}}\frac{1}{\Gamma(n-\aleph)}\int_0^t \frac{\hbar^{(n)}(\zeta)}{(t-\zeta)^{\aleph-n+1}}\,d\zeta\,dt \\
&= \frac{1}{\Gamma(n-\aleph)\omega^2}\int_0^{\infty}\int_{\zeta}^{\infty} e^{-\frac{t}{\omega}}\frac{\hbar^{(n)}(\zeta)}{(t-\zeta)^{\aleph-n+1}}\,dt\,d\zeta \\
&= \frac{1}{\Gamma(n-\aleph)\omega^2}\int_0^{\infty}\hbar^{(n)}(\zeta)\int_0^{\infty} e^{-\frac{u+\zeta}{\omega}}u^{n-\aleph-1}\,du\,d\zeta \\
&= \frac{1}{\Gamma(n-\aleph)\omega^2}\int_0^{\infty} e^{-\frac{\zeta}{\omega}}\hbar^{(n)}(\zeta)\int_0^{\infty} e^{-\frac{u}{\omega}}u^{n-\aleph-1}\,du\,d\zeta \\
&= \frac{1}{\Gamma(n-\aleph)\omega^2}\int_0^{\infty} e^{-\frac{\zeta}{\omega}}\hbar^{(n)}(\zeta)\frac{\Gamma(n-\aleph)}{\omega^{n-\aleph}}\,d\zeta \\
&= \omega^{\aleph-n-2}\int_0^{\infty} e^{-\frac{\zeta}{\omega}}\hbar^{(n)}(\zeta)\,d\zeta \\
&= \omega^{\aleph-n-2}A\left[\hbar^{(n)}(t)\right](\omega) \\
&= \omega^{\aleph-n-2}\left[\omega^{-n}A[\hbar(t)]-\omega^{-n}\sum_{\sigma=0}^{n-1}\omega^{\sigma-1}\hbar^{(\sigma)}(0)\right] \\
&= \omega^{\aleph-2n-2}A[\hbar(t)]-\sum_{\sigma=0}^{n-1}\omega^{\aleph-2n+\sigma-3}\hbar^{(\sigma)}(0).
\end{aligned}$$

Note: Fubini's theorem is employed to rearrange the order of integration in the preceding derivative.

3 Solutions of the fractional integro-differential equations

In this section, there are strong indications that the function $\Upsilon(t)$ alone may be adequate to enable the Sawi transform $A[\Upsilon(t)]$ to operate successfully at a certain value of the parameter ω. Here, the inverse Sawi transform has been completely evaluated and the resulting expression is written entirely in the time domain using the Riemann–Liouville fractional integral operator.

Theorem 1. *Let* $1<\aleph<2$ *and a and* $b\in\mathbb{R}$. *Then the fractional integro-differential equation*

$$\Upsilon''(t)+a\,\Upsilon^{\aleph}(t)+by(t)=\int_0^{\omega}\frac{g(t)}{(\omega-t)^{\beta}}\,dt\quad;\quad 0<\beta<1 \tag{1}$$

with initial conditions $\Upsilon(0)=c_0$ *and* $\Upsilon'(0)=c_1$, *has the unique solution*

$$
\begin{aligned}
\Upsilon(t) = {} & c_0 \sum_{\sigma=0}^{\infty} \frac{(-b)^{\sigma} t^{2\sigma}}{\sigma!} \sum_{\varepsilon=0}^{\infty} \frac{\Gamma(\sigma+\varepsilon+1)\,(-at^{2-\aleph})^{\varepsilon}}{\Gamma[(2-\aleph)\varepsilon+2\sigma+1]\,\varepsilon!} \\
& + c_1 \sum_{\sigma=0}^{\infty} \frac{(-b)^{\sigma} t^{2\sigma+1}}{\sigma!} \sum_{\varepsilon=0}^{\infty} \frac{\Gamma(\sigma+\varepsilon+1)\,(-at^{2-\aleph})^{\varepsilon}}{\Gamma[(2-\aleph)\varepsilon+2\sigma+2]\,\varepsilon!} \\
& + ac_0 \sum_{\sigma=0}^{\infty} \frac{(-b)^{\sigma} t^{2\sigma-\aleph+2}}{\sigma!} \sum_{\varepsilon=0}^{\infty} \frac{\Gamma(\sigma+\varepsilon+1)\,(-at^{2-\aleph})^{\varepsilon}}{\Gamma[(2-\aleph)\varepsilon+2\sigma-\aleph+3]\,\varepsilon!} \\
& + ac_1 \sum_{\sigma=0}^{\infty} \frac{(-b)^{\sigma} t^{2\sigma-\aleph+3}}{\sigma!} \sum_{\varepsilon=0}^{\infty} \frac{\Gamma(\sigma+\varepsilon+1)\,(-at^{2-\aleph})^{\varepsilon}}{\Gamma[(2-\aleph)\varepsilon+2\sigma-\aleph+4]\,\varepsilon!} \\
& + \frac{1}{\Gamma(\beta)} \int_0^t (t-\tau)^{\beta-1} \hbar'(\tau)\, d\tau \\
& \times \sum_{\sigma=0}^{\infty} \frac{(-b)^{\sigma} t^{2\sigma+1}}{\sigma!} \sum_{\varepsilon=0}^{\infty} \frac{\Gamma(\sigma+\varepsilon+1)\,(-at^{2-\aleph})^{\varepsilon}}{\Gamma[(2-\aleph)\varepsilon+2\sigma+2]\,\varepsilon!}.
\end{aligned}
\tag{2}
$$

Proof. Applying the Sawi transform to equation (1), we obtain

$$
\frac{F(\omega)}{\omega^2} - \frac{\hbar(0)}{\omega^3} - \frac{\hbar'(0)}{\omega^2} + a\left[\frac{F(\omega)}{\omega^{\aleph}} - \frac{\hbar(0)}{\omega^{\aleph+1}} - \frac{\hbar'(0)}{\omega^{\aleph}}\right] + bF(\omega) = A[\hbar(t)],
$$

where $\hbar(t) = \int_0^{\omega} \frac{\mathrm{g}(t)}{(\omega-t)^{\beta}}\, dt$ and

$$
\begin{aligned}
& \frac{A[\Upsilon(t)]}{\omega^2} - \frac{c_0}{\omega^3} - \frac{c_1}{\omega^2} + a\frac{A[\Upsilon(t)]}{\omega^{\aleph}} - a\frac{c_0}{\omega^{\aleph+1}} - a\frac{c_1}{\omega^{\aleph}} + bA[\Upsilon(t)] = A[\hbar(t)] \\
& A[\Upsilon(t)]\left(\frac{1}{\omega^2} + \frac{a}{\omega^{\aleph}} + b\right) = \frac{c_0}{\omega^3} + \frac{c_1}{\omega^2} + \frac{ac_0}{\omega^{\aleph+1}} + \frac{ac_1}{\omega^{\aleph}} + A[\hbar(t)] \\
& A[\Upsilon(t)] = \frac{c_0\omega^{-3} + c_1\omega^{-2} + ac_0\omega^{1-\aleph} + ac_1\omega^{-\aleph} + A[\hbar(t)]}{\omega^{-2} + a\omega^{-\aleph} + b}.
\end{aligned}
\tag{3}
$$

On simplifying the denominator,

$$
\begin{aligned}
\frac{1}{\omega^{-2} + a\omega^{-\aleph} + b} & = \frac{\omega^{-\aleph}}{\omega^{\aleph-2} + a + b\omega^{\aleph}} \\
& = \frac{\omega^{-\aleph}}{(\omega^{\aleph-2}+a)\left(1 + \frac{b\omega^{\aleph}}{\omega^{\aleph-2}+a}\right)} \\
& = \frac{\omega^{\aleph}}{\omega^{\aleph-2}+a} \sum_{\sigma=0}^{\infty} \left(\frac{-b\omega^{\aleph}}{\omega^{\aleph-2}+a}\right)^{\sigma} \\
& = \sum_{\sigma=0}^{\infty} \frac{(-b)^{\sigma}\omega^{\aleph\sigma+\aleph}}{(\omega^{\aleph-2}+a)^{\sigma+1}} \\
& = \sum_{\sigma=0}^{\infty} \frac{(-b)^{\sigma}\omega^{2\sigma+2}}{(1+a\omega^{2-\aleph})^{\sigma+1}}
\end{aligned}
$$

$$= \sum_{\sigma=0}^{\infty} (-b)^{\sigma} \omega^{2\sigma+2} \sum_{\varepsilon=0}^{\infty} (-a\omega^{2-\aleph})^{\varepsilon} \binom{\sigma+\varepsilon}{\varepsilon}$$
$$= \sum_{\sigma=0}^{\infty} \sum_{\varepsilon=0}^{\infty} \binom{\sigma+\varepsilon}{\varepsilon} (-b)^{\sigma} (-a)^{\varepsilon} \omega^{(2-\aleph)\varepsilon+2\sigma+2} \tag{4}$$

and

$$A[\hbar(t)] = A\left[\int_0^{\omega} \frac{\mathrm{g}(t)}{(\omega-t)^{\beta}}\, dt\right].$$

This is convolution integral

$$F(P) = \omega^2\, K(P)\; G(P),$$

where $K(P)$ is the Sawi transform of $K(\omega) = \omega^{-\beta}$. Using the Sawi transform

$$A[K(\omega)] = \omega^{-\beta},$$

the transformed kernel is written as

$$K(P) = \frac{\Gamma(1-\beta)}{\omega^{\beta+1}}.$$

Substituting this expression into the convolution relation and solving for $G(P)$ yields

$$G(P) = \frac{F(P)}{\Gamma(1-\beta)\omega^{1-\beta}}.$$

Applying the reflection formula $\Gamma(\beta)\Gamma(1-\beta) = \dfrac{\pi}{\sin(\pi\beta)}$, the above expression becomes

$$G(P) = \frac{\sin(\pi\beta)}{\pi}\, \omega^{\beta-1}\Gamma(\beta)F(P).$$

Using the inverse Sawi transform property of convolution, we obtain

$$G(P) = \frac{\sin\pi\,\beta}{\pi} A\left[\int_0^{\omega} (\omega-t)^{(\beta-1)}\, \hbar'(t)\, dt\right]. \tag{5}$$

Substituting the above two equations (4) and (5) in (3), we get

$$\begin{aligned}
A[\Upsilon(t)] = {} & c_0 \sum_{\sigma=0}^{\infty} (-b)^{\sigma} \sum_{\varepsilon=0}^{\infty} \binom{\sigma+\varepsilon}{\varepsilon} (-a)^{\varepsilon}\, \omega^{(\aleph-2)\varepsilon-2\sigma-1} \\
& + c_1 \sum_{\sigma=0}^{\infty} (-b)^{\sigma} \sum_{\varepsilon=0}^{\infty} \binom{\sigma+\varepsilon}{\varepsilon} (-a)^{\varepsilon}\, \omega^{(\aleph-2)\varepsilon-2\sigma-2} \\
& + ac_0 \sum_{\sigma=0}^{\infty} (-b)^{\sigma} \sum_{\varepsilon=0}^{\infty} \binom{\sigma+\varepsilon}{\varepsilon} (-a)^{\varepsilon}\, \omega^{(\aleph-2)\varepsilon-2\sigma+\aleph-3} \\
& + ac_1 \sum_{\sigma=0}^{\infty} (-b)^{\sigma} \sum_{\varepsilon=0}^{\infty} \binom{\sigma+\varepsilon}{\varepsilon} (-a)^{\varepsilon}\, \omega^{(\aleph-2)\varepsilon-2\sigma+\aleph-4} \\
& + \frac{\sin(\pi\beta)}{\pi} A\left[\int_0^{\omega} (\omega-t)^{\beta-1} \hbar'(t)\, dt\right] \\
& \times \sum_{\sigma=0}^{\infty} (-b)^{\sigma} \sum_{\varepsilon=0}^{\infty} \binom{\sigma+\varepsilon}{\varepsilon} (-a)^{\varepsilon}\, \omega^{(\aleph-2)\varepsilon-2\sigma-2}.
\end{aligned} \tag{6}$$

Thus, applying the inverse Sawi transform to equation (6), we obtain the solution given in equation (2). This completes the proof of the theorem. □

Example 1. The fractional integro-differential equation

$$\Upsilon''(t)+\sqrt{6}\Upsilon^{\left(\frac{3}{2}\right)}(t)+12\Upsilon(t)=\int_0^{\omega}\frac{\mathrm{g}(t)}{(\omega-t)^{\frac{1}{2}}}\,dt$$

with initial conditions $\Upsilon(0)=c_0$ and $\Upsilon'(0)=c_1$ has the unique solution

$$\begin{aligned}\Upsilon(t)=\ & c_0\sum_{\sigma=0}^{\infty}\frac{(-12)^{\sigma}t^{2\sigma}}{\sigma!}\sum_{\varepsilon=0}^{\infty}\frac{\Gamma(\sigma+\varepsilon+1)\,(-\sqrt{6}t^{1/2})^{\varepsilon}}{\Gamma\left(\frac{1}{2}\varepsilon+2\sigma+1\right)\varepsilon!}\\ &+c_1\sum_{\sigma=0}^{\infty}\frac{(-12)^{\sigma}t^{2\sigma+1}}{\sigma!}\sum_{\varepsilon=0}^{\infty}\frac{\Gamma(\sigma+\varepsilon+1)\,(-\sqrt{6}t^{1/2})^{\varepsilon}}{\Gamma\left(\frac{1}{2}\varepsilon+2\sigma+2\right)\varepsilon!}\\ &+\sqrt{6}c_0\sum_{\sigma=0}^{\infty}\frac{(-12)^{\sigma}t^{2\sigma+\frac{1}{2}}}{\sigma!}\sum_{\varepsilon=0}^{\infty}\frac{\Gamma(\sigma+\varepsilon+1)\,(-\sqrt{6}t^{1/2})^{\varepsilon}}{\Gamma\left(\frac{1}{2}\varepsilon+2\sigma+\frac{3}{2}\right)\varepsilon!}\\ &+\sqrt{6}c_1\sum_{\sigma=0}^{\infty}\frac{(-12)^{\sigma}t^{2\sigma+\frac{3}{2}}}{\sigma!}\sum_{\varepsilon=0}^{\infty}\frac{\Gamma(\sigma+\varepsilon+1)\,(-\sqrt{6}t^{1/2})^{\varepsilon}}{\Gamma\left(\frac{1}{2}\varepsilon+2\sigma+\frac{5}{2}\right)\varepsilon!}\\ &+\frac{1}{\Gamma\left(\frac{1}{2}\right)}\int_0^t(t-\tau)^{-\frac{1}{2}}\hbar'(\tau)\,d\tau\\ &\times\sum_{\sigma=0}^{\infty}\frac{(-12)^{\sigma}t^{2\sigma+1}}{\sigma!}\sum_{\varepsilon=0}^{\infty}\frac{\Gamma(\sigma+\varepsilon+1)\,(-\sqrt{6}t^{1/2})^{\varepsilon}}{\Gamma\left(\frac{1}{2}\varepsilon+2\sigma+2\right)\varepsilon!}.\end{aligned}$$

The solution evaluation shows that at time $t=0$ all terms with positive time powers disappear while the equation shows that $\Upsilon(0)=c_0$. The series differentiation process showed that only the first-order term at time $t=0$ produced results which led to the finding $\Upsilon'(0)=c_1$. The solution meets all required initial condition requirements.

Figure 1 illustrates the solution behavior of the fractional differential equation of Example 1 at various values of $\aleph$ with the initial conditions $c_0=\ 1$ and $c_1=\ 1$.

Theorem 2. *Let* $1<\aleph<2$ *and a and b* $\in\mathbb{R}$. *Then the fractional integro-differential equation*

$$\Upsilon^{\aleph}(t)+a\,\Upsilon'(t)+by(t)=\int_0^{\omega}\frac{\mathrm{g}(t)}{(\omega-t)^{\beta}}\,dt\ \ ;\ \ 0<\beta<1 \tag{7}$$

with initial conditions $\Upsilon(0)=c_0$ *and* $\Upsilon'(0)=c_1$ *has the unique solution*

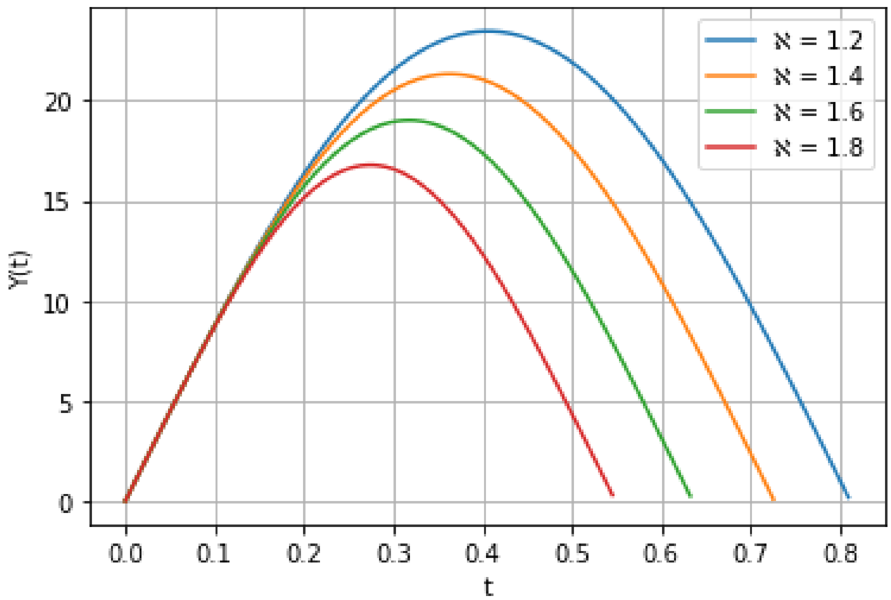

Figure 1: The solution behavior of Example 1

$$
\begin{aligned}
\Upsilon(t) = {} & c_0 \sum_{\sigma=0}^{\infty} \frac{(-b)^{\sigma}}{\sigma!} \sum_{\varepsilon=0}^{\infty} \frac{\Gamma(\sigma+\varepsilon+1)\,(-a)^{\varepsilon}\, t^{(\aleph-1)\varepsilon+\aleph\sigma}}{\Gamma[(\aleph-1)\varepsilon+\aleph\sigma+1]\,\varepsilon!} \\
& + c_1 \sum_{\sigma=0}^{\infty} \frac{(-b)^{\sigma}}{\sigma!} \sum_{\varepsilon=0}^{\infty} \frac{\Gamma(\sigma+\varepsilon+1)\,(-a)^{\varepsilon}\, t^{(\aleph-1)\varepsilon+\aleph\sigma+1}}{\Gamma[(\aleph-1)\varepsilon+\aleph\sigma+2]\,\varepsilon!} \\
& + ac_0 \sum_{\sigma=0}^{\infty} \frac{(-b)^{\sigma}}{\sigma!} \sum_{\varepsilon=0}^{\infty} \frac{\Gamma(\sigma+\varepsilon+1)\,(-a)^{\varepsilon}\, t^{(\aleph-1)\varepsilon+\aleph\sigma+\aleph-1}}{\Gamma[(\aleph-1)\varepsilon+\aleph\sigma+\aleph]\,\varepsilon!} \\
& + \frac{1}{\Gamma(\beta)} \int_0^t (t-\tau)^{\beta-1} \hbar'(\tau)\, d\tau \\
& \times \sum_{\sigma=0}^{\infty} \frac{(-b)^{\sigma}}{\sigma!} \sum_{\varepsilon=0}^{\infty} \frac{\Gamma(\sigma+\varepsilon+1)\,(-a)^{\varepsilon}\, t^{(\aleph-1)\varepsilon+\aleph\sigma+\aleph-1}}{\Gamma[(\aleph-1)\varepsilon+\aleph\sigma+\aleph]\,\varepsilon!}.
\end{aligned} \tag{8}
$$

Proof. Applying the Sawi transform to equation (7), we obtain

$$
\frac{F(\omega)}{\omega^{\aleph}} - \frac{\hbar(0)}{\omega^{\aleph+1}} - \frac{\hbar'(0)}{\omega^{\aleph}} + a\left(\frac{F(\omega)}{\omega} - \frac{\hbar(0)}{\omega^2}\right) + bF(\omega) = A[\hbar(t)],
$$

where $\hbar(t) = \int_0^{\omega} \frac{g(t)}{(\omega-t)^{\beta}}\, dt$

$$
\frac{A[\Upsilon(t)]}{\omega^{\aleph}} - \frac{c_0}{\omega^{\aleph+1}} - \frac{c_1}{\omega^{\aleph}} + a\left(\frac{A[\Upsilon(t)]}{\omega} - \frac{c_0}{\omega^2}\right) + bA[\Upsilon(t)] = A[\hbar(t)]
$$

$$
A[\Upsilon(t)]\left(\frac{1}{\omega^{\aleph}} + \frac{a}{\omega} + b\right) = \frac{c_0}{\omega^{\aleph+1}} + \frac{c_1}{\omega^{\aleph}} + \frac{ac_0}{\omega^2} + A[\hbar(t)]
$$

$$A[\Upsilon(t)] = \frac{c_0\omega^{1-\aleph} + c_1\omega^{-\aleph} + ac_0\omega^{-2} + A[\hbar(t)]}{\omega^{-\aleph} + a\omega^{-1} + b}. \tag{9}$$

On simplifying the denominator,

$$\begin{aligned}
\frac{1}{\omega^{-\aleph} + a\,\omega^{-1} + b} &= \frac{\omega}{\omega^{1-\aleph} + a + b\omega} \\
&= \frac{\omega}{(\omega^{1-\aleph} + a)\left(1 + \frac{b\,\omega}{\omega^{1-\aleph}+a}\right)} \\
&= \frac{\omega}{\omega^{1-\aleph} + a} \sum_{\sigma=0}^{\infty} \left(\frac{-b\,\omega}{\omega^{1-\aleph} + a}\right)^{\sigma} \\
&= \sum_{\sigma=0}^{\infty} \frac{(-b)^{\sigma}\,\omega^{\sigma+1}}{(\omega^{1-\aleph} + a)^{\sigma+1}} \\
&= \sum_{\sigma=0}^{\infty} (-b)^{\sigma}\,\omega^{\sigma+1} \sum_{\varepsilon=0}^{\infty} \binom{\sigma+\varepsilon}{\varepsilon} (-a\,\omega^{\aleph-1})^{\varepsilon} \\
&= \sum_{\sigma=0}^{\infty} \sum_{\varepsilon=0}^{\infty} \binom{\sigma+\varepsilon}{\varepsilon} (-b)^{\sigma} (-a)^{\varepsilon}\,\omega^{(\aleph-1)\varepsilon+\sigma+1}.
\end{aligned} \tag{10}$$

Substituting equation (10) and applying the inverse transform, we obtain the solution given in equation (8). This completes the proof. Also, the Wright function (6) can express this solution as

$$\begin{aligned}
\Upsilon(t) = c_0 \sum_{\sigma=0}^{\infty} \frac{(-b)^{\sigma} t^{\aleph\sigma}}{\sigma!}\, {}_1\lambda_1\left(\begin{matrix} (\sigma+1,\,1) \\ (\aleph\sigma+1,\,\aleph-1) \end{matrix} \,\middle|\, -at^{\aleph-1} \right) \\
+ c_1 \sum_{\sigma=0}^{\infty} \frac{(-b)^{\sigma} t^{\aleph\sigma+1}}{\sigma!}\, {}_1\lambda_1\left(\begin{matrix} (\sigma+1,\,1) \\ (\aleph\sigma+2,\,\aleph-1) \end{matrix} \,\middle|\, -at^{\aleph-1} \right) \\
+ ac_0 \sum_{\sigma=0}^{\infty} \frac{(-b)^{\sigma} t^{\aleph\sigma+\aleph-1}}{\sigma!}\, {}_1\lambda_1\left(\begin{matrix} (\sigma+1,\,1) \\ (\aleph\sigma+\aleph,\,\aleph-1) \end{matrix} \,\middle|\, -at^{\aleph-1} \right) \\
+ \frac{1}{\Gamma(\beta)} \int_0^t (t-\tau)^{\beta-1} \hbar'(\tau)\, d\tau \\
\times \sum_{\sigma=0}^{\infty} \frac{(-b)^{\sigma} t^{\aleph\sigma+\aleph-1}}{\sigma!}\, {}_1\lambda_1\left(\begin{matrix} (\sigma+1,\,1) \\ (\aleph\sigma+\aleph,\,\aleph-1) \end{matrix} \,\middle|\, -at^{\aleph-1} \right).
\end{aligned}$$

□

Remark 2. *The existence and uniqueness asserted in Theorems 1 and 2 follow from standard results in fractional integro-differential equation theory. The problems can be transformed into equivalent Volterra-type fractional integral equations through the established conditions which apply to the function* $\mathrm{g}(t)$ *for the range* $1 < \aleph < 2$ *and* $0 < \beta < 1$*. The integral operators need to fulfill appropriate Lipschitz conditions within a specific Banach space which consists of continuous functions. The problems have one solution because the Banach contraction mapping principle establishes that fact. The application of fractional Grönwall-type inequalities leads to the same uniqueness results which establish uniqueness of solutions to the problem.*

Example 2. The fractional integro-differential equation

$$\Upsilon^{\frac{3}{2}}(t)+4\Upsilon'(t)+11\Upsilon(t)=\int_0^{\omega}\frac{g(\tau)}{(\omega-\tau)^{\frac{1}{2}}}\,d\tau,$$

with initial conditions $\Upsilon(0)=c_0$ and $\Upsilon'(0)=c_1$ has the unique solution

$$\begin{aligned}\Upsilon(t)=&\,c_0\sum_{\sigma=0}^{\infty}\frac{11^{\sigma}}{\sigma!}\sum_{\varepsilon=0}^{\infty}\frac{\Gamma(\sigma+\varepsilon+1)\,4^{\varepsilon}\,t^{\frac{1}{2}\varepsilon+\frac{3}{2}\sigma}}{\Gamma\left(\frac{1}{2}\varepsilon+\frac{3}{2}\sigma+1\right)\varepsilon!}\\&+c_1\sum_{\sigma=0}^{\infty}\frac{11^{\sigma}}{\sigma!}\sum_{\varepsilon=0}^{\infty}\frac{\Gamma(\sigma+\varepsilon+1)\,4^{\varepsilon}\,t^{\frac{1}{2}\varepsilon+\frac{3}{2}\sigma+1}}{\Gamma\left(\frac{1}{2}\varepsilon+\frac{3}{2}\sigma+2\right)\varepsilon!}\\&+4c_0\sum_{\sigma=0}^{\infty}\frac{11^{\sigma}}{\sigma!}\sum_{\varepsilon=0}^{\infty}\frac{\Gamma(\sigma+\varepsilon+1)\,4^{\varepsilon}\,t^{\frac{1}{2}\varepsilon+\frac{3}{2}\sigma+\frac{1}{2}}}{\Gamma\left(\frac{1}{2}\varepsilon+\frac{3}{2}\sigma+\frac{3}{2}\right)\varepsilon!}\\&+\frac{1}{\Gamma\left(\frac{1}{2}\right)}\int_0^t(t-\tau)^{-\frac{1}{2}}\hbar'(\tau)\,d\tau\\&\quad\times\sum_{\sigma=0}^{\infty}\frac{11^{\sigma}}{\sigma!}\sum_{\varepsilon=0}^{\infty}\frac{\Gamma(\sigma+\varepsilon+1)\,4^{\varepsilon}\,t^{\frac{1}{2}\varepsilon+\frac{3}{2}\sigma+\frac{1}{2}}}{\Gamma\left(\frac{1}{2}\varepsilon+\frac{3}{2}\sigma+\frac{3}{2}\right)\varepsilon!}.\end{aligned}$$

The series solution shows that the result proves that $\Upsilon(0)=c_0$. The solution can be derived through termwise differentiation which shows that at $t=0$ the result gives $\Upsilon'(0)=c_1$. The solution already fulfills all initial conditions which were established for the problem.

Regularity near $t=0$. The solutions obtained in Examples 1 and 2 are expressed as convergent series which use fractional powers of t and Gamma functions that appear in the denominators. The term remains finite for all values of t that are close to zero because the range of parameters is defined by $1<\aleph<2$ and $0<\beta<1$. The solutions maintain continuity with their fractional derivatives established at $t=0$ which leads to regularity within the local area of the solution.

Figure 2 illustrates the solution behavior of the fractional differential equation of Example 2 at various values of $\aleph$ with the initial conditions $c_0=1$ and $c_1=1$.

Proposition 1. *Let $1<\aleph<2$ and $b\in\mathbb{R}$. Then the fractional integro-differential equation*

$$\Upsilon^{\aleph}(t)-by(t)=\int_0^{\omega}\frac{g(t)}{(\omega-t)^{\beta}}\,dt\quad;\quad 0<\beta<1 \tag{11}$$

with initial conditions $\Upsilon(0)=c_0$ has the unique solution

$$\begin{aligned}\Upsilon(t)&=c_0\sum_{\sigma=0}^{\infty}b^{\sigma}\frac{t^{\aleph\sigma}}{\Gamma(\aleph\sigma+1)}+\frac{1}{\Gamma(\beta)}\int_0^t(t-\tau)^{\beta-1}\hbar'(\tau)\,d\tau\sum_{\sigma=0}^{\infty}b^{\sigma}\frac{t^{\aleph+\aleph\sigma-1}}{\Gamma(\aleph+\aleph\sigma)}\\&=c_0E_{\aleph}\left(bt^{\aleph}\right)+\frac{1}{\Gamma(\beta)}\int_0^t(t-\tau)^{\beta-1}\hbar'(\tau)\,d\tau\,t^{\aleph-1}E_{\aleph,\aleph}\left(bt^{\aleph}\right).\end{aligned} \tag{12}$$

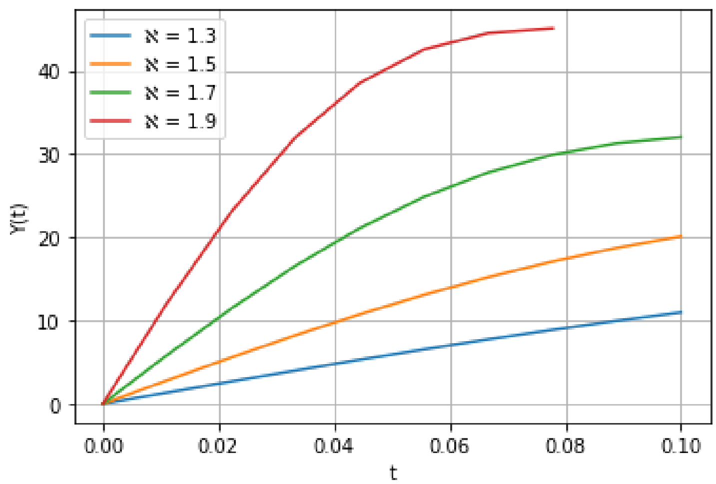

Figure 2: The solution behavior of Example 2

Proof. Applying the Sawi transform to equation (11), we obtain

$$\frac{F(\omega)}{\omega^{\aleph}} - \frac{\hbar(0)}{\omega^{\aleph+1}} - \frac{\hbar'(0)}{\omega^{\aleph}} - bF(\omega) = A[\hbar(t)],$$

where

$$\hbar(t) = \int_0^{\omega} \frac{g(\tau)}{(\omega-\tau)^{\beta}}\, d\tau,$$

$$\begin{aligned} &\frac{A[\Upsilon(t)]}{\omega^{\aleph}}, - \frac{c_0}{\omega^{\aleph+1}} - \frac{c_1}{\omega^{\aleph}} - bA[\Upsilon(t)] = A[\hbar(t)], \\ &A[\Upsilon(t)]\left(\omega^{-\aleph} - b\right) = c_0\,\omega^{1-\aleph} + A[\hbar(t)], \\ &A[\Upsilon(t)] = \frac{c_0\omega^{1-\aleph} + A[\hbar(t)]}{\omega^{-\aleph} - b}. \end{aligned} \tag{13}$$

On simplifying the denominator,

$$\begin{aligned} \frac{1}{\omega^{-\aleph} - b} &= \frac{1}{\omega^{-\aleph}(1 - b\,\omega^{\aleph})} \\ &= \frac{\omega^{\aleph}}{1 - b\,\omega^{\aleph}} \\ &= \omega^{\aleph}(1 - b\,\omega^{\aleph})^{-1} \\ &= \omega^{\aleph}\left[1 + b\,\omega^{\aleph} + (b\,\omega^{\aleph})^2 + \cdots\right] \\ &= \omega^{\aleph}\sum_{\sigma=0}^{\infty}(b\,\omega^{\aleph})^{\sigma}. \end{aligned}$$

Substituting this expansion into the transformed equation (13) gives

$$A[\Upsilon(t)] = \omega^{1-\aleph}c_0\left[\omega^{\aleph}\sum_{\sigma=0}^{\infty}(b\,\omega^{\aleph})^{\sigma}\right] + \omega^{\aleph}\sum_{\sigma=0}^{\infty}(b\,\omega^{\aleph})^{\sigma}A[\hbar(t)]$$

$$
\begin{aligned}
&= c_0\,\omega \sum_{\sigma=0}^{\infty} (b\,\omega^{\aleph})^{\sigma} + \omega^{\aleph} \sum_{\sigma=0}^{\infty} (b\,\omega^{\aleph})^{\sigma} A[\hbar(t)] \\
&= c_0\,\omega \sum_{\sigma=0}^{\infty} b^{\sigma} \omega^{\aleph\sigma} + \omega^{\aleph} \sum_{\sigma=0}^{\infty} (b\,\omega^{\aleph})^{\sigma} \frac{\sin(\pi\beta)}{\pi} A\left[\int_0^{\omega} (\omega - t)^{\beta-1} \hbar'(t)\,dt\right]. \qquad (14)
\end{aligned}
$$

Thus, by the inverse Sawi transform to equation (14), we get

$$
\begin{aligned}
\Upsilon(t) &= c_0 \sum_{\sigma=0}^{\infty} b^{\sigma} \frac{t^{\aleph\sigma}}{\Gamma(\aleph\sigma+1)} + \frac{1}{\Gamma(\beta)} \int_0^t (t-\tau)^{\beta-1} \hbar'(\tau)\,d\tau \sum_{\sigma=0}^{\infty} b^{\sigma} \frac{t^{\aleph+\aleph\sigma-1}}{\Gamma(\aleph+\aleph\sigma)} \\
&= c_0 E_{\aleph}\left(bt^{\aleph}\right) + \frac{1}{\Gamma(\beta)} \int_0^t (t-\tau)^{\beta-1} \hbar'(\tau)\,d\tau\, t^{\aleph-1} E_{\aleph,\aleph}\left(bt^{\aleph}\right).
\end{aligned}
$$

This completes the proof of the proposition. □

Remark 3. *If we set $a = 0$ in equation (7), then the fractional derivative is*

$$
D^{\aleph}\Upsilon(t) + b\Upsilon(t) = \int_0^{\omega} \frac{g(\tau)}{(\omega-\tau)^{\beta}}\,d\tau, \quad 1 < \aleph \le 2,\ 0 < \beta < 1,
$$

with initial conditions $\Upsilon(0) = c_0$ and $\Upsilon'(0) = c_1$ and its solution is given by

$$
\begin{aligned}
\Upsilon(t) &= c_0 E_{\aleph,1}\left(-bt^{\aleph}\right) + c_1 E_{\aleph,2}\left(-bt^{\aleph}\right) \\
&\quad + \frac{1}{\Gamma(\beta)} \int_0^t (t-\tau)^{\beta-1} \hbar'(\tau)\,d\tau\, t^{\aleph-1} E_{\aleph,\aleph}\left(-bt^{\aleph}\right).
\end{aligned}
$$

3.1 Convergence and validity of the series solutions

The obtained solutions are expressed in terms of double infinite series which use powers of t and Gamma functions as their basis. The series converges because the ratio test and Mittag–Leffler type functions [10, 12] already demonstrate their known convergence properties. For the range $1 < \aleph < 2$ and $0 < \beta < 1$ the series terms decrease at a sufficient rate because the Gamma function increases in the denominators. The series solutions show absolute convergence when the value of t remains finite and both parameters a and b stay within bounded limits. The summation exchange with the inverse Sawi transform receives justification because the involved series converges uniformly on time domain compact subsets. Researchers can apply the inverse Sawi transform through term-by-term execution of the series representation by using integral transform theory and dominated convergence methods [5, 12]. The mathematical results for $\Upsilon(t)$ which we derived provide valid time-domain solutions to the fractional integro-differential equations which we studied. The proposed series solutions define a specific radius of validity which exists because the analytical method used in this study follows traditional rules for convergence and transform interchange between different mathematical techniques.

4 Machine learning approaches for solving fractional integro-differential equations

This section focuses on combining machine learning strategies with the fractional integro-differential equations described above, especially the Sawi transform as well as the extended Frobenius method for the solution enhancement. The main objective is to prove that with the help of deep learning the results may be obtained approximating the solutions which are given in the statements of theorems 1 and 2.

4.1 Data generation for machine learning models

Since machine learning is to be used for fractional integro-differential equations, such synthetic data have to be developed from Theorems 1 and 2 analytical solutions. Therefore, the aim is to develop solutions through machine learning models and check the solutions' reliability.

Steps for data generation:

- **Step 1:** Define the domain of the independent variable t over which the solutions will be approximated.
- **Step 2:** Determine the values of $\Upsilon(t)$ for the main problem and in conformity with the unique solutions of $B^q D_t^\alpha u(t)$ and $\left({}_aI^{1-\alpha}{}_t v(t)\right)$ for different t employment of Theorems 1 and 2.
- **Step 3:** Collect the data set with inputst, a, b, ω and desired output $\Upsilon(t)$.

A sample of the generated dataset might look like:

Table 1: Sample of generated dataset for machine learning model

t	a	b	ω	$\Upsilon(t)$
0.1	0.5	0.2	1.0	0.0234
0.2	0.5	0.2	1.0	0.0456
0.3	0.5	0.2	1.0	0.0678
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$

4.2 Model architecture

Fractional integro-differential equations can be approximated using a deep learning neural network model in order to yield the accurate solutions. The architecture of the network is designed as follows: The architecture of the network is designed as follows:

- **Input layer:** Takes values of the independent variables t ,a,b and ω.
- **Hidden layers:** Fully connected layers with nonlinear activation functions so as to capture the nonlinearity between the inputs to the problem and the solution $\Upsilon(t)$.
- **Output layer:** Outputs the predicted value of $\Upsilon(t)$.

This is because, the neural network can be trained by using loss function that has the ability to calculate mean Squared Error (MSE) between the predicted values of the network and actual values form the analytical solutions. The MSE is computed as:The MSE is computed as:

$$\text{MSE} = \frac{1}{n}\sum_{i=1}^{n}\left(\Upsilon_{\text{true}}(t_i) - \Upsilon_{\text{pred}}(t_i)\right)^2.$$

Model optimization: Such an optimization algorithm is applied to minimize loss and adjust the model weights, and in this case, Adam optimization algorithm is used. The training process itself can be accelerated by using the so-called Adam algorithm, which adjusts the learning rate during training.

4.2.1 Neural network architecture details

The machine learning framework adopted in this study uses a fully connected feedforward neural network to approximate the solution $\Upsilon(t)$ which describes fractional integro-differential equations that contain nonlocal operators and memory effects. The input layer of the network accepts four variables which include the independent variable t and the model parameters a, b, and ω that together define the dynamical behavior of the system under consideration. The network uses these inputs to understand how time progresses and how system parameters affect the governing equations which control system behavior. The network architecture consists of three hidden layers which contain 64 neurons in the first hidden layer, 64 neurons in the second hidden layer, and 32 neurons in the third hidden layer, which enable the system to model both nonlinear and fractional-order attributes of the problem. The Rectified Linear Unit (ReLU) activation function is used in all hidden layers because it helps the model train faster while solving the deep neural network problem of disappearing gradients.

The output layer consists of a single neuron with a linear activation function that produces the predicted value of the solution $\Upsilon(t)$. A linear activation is appropriate at the output stage, as the task involves regression rather than classification. The selected architecture combines two opposing factors which include higher computational efficiency and increased model complexity to achieve a balance that allows the network to model complex fractional dynamics without creating an overload of unnecessary parameters. This design enables stable training and reliable approximation of solution profiles across the considered parameter ranges.

4.3 Training and validation

The dataset obtained from the analytical solutions is divided into training and validation set; where the model is trained on the training data and then tested with the validation data with the view of determining the level of accuracy.

Validation metrics:

- **The MSE:** To check the accuracy of the model with the actual solution which needs to be approximated.
- **Coefficient of determination ($\mathbf{R^2}$ Score):** In order to assess the measures that range from 0 to 1 to determine the extent of the dependent variable that can be explained by the input variables.

The training is an iterative process that takes many epochs when the model's goal is to estimate $\Upsilon(t)$. Finally, the results that the model comes up with are checked against the Theorems 1 and 2 for testing the model's effectiveness.

4.3.1 Training parameters and data preprocessing

The machine learning model training uses a synthetic dataset which originates from the complete analytical solutions of Theorems 1 and 2. The analytical expressions create a dependable benchmark which allows supervised learning pairs to be developed. This framework enables the neural network to understand how the independent variable and system parameters combine to produce the solution. The time variable t is sampled uniformly from the interval [0,T] to create 5000 data samples while selecting parameter values a, b, and ω from their acceptable physical and mathematical ranges. The chosen dataset size provides sufficient diversity to capture the nonlinear and fractional-order behavior of the governing equations, while remaining computationally manageable. The researchers used random dataset partitioning to create three separate groups which they used to test their learning evaluation system. The first group used 70% of the data to develop model parameters while the second group used 15% for training validation. The final group used 15% of the dataset for testing purposes only. The model training process runs effectively because this separation enables simultaneous monitoring of generalization capabilities while protecting test set information from entering the training phases.

The input features (t, a, b, ω) undergo normalization to the range $[0, 1]$ through the application of min-max normalization before the training process begins. The optimization process requires this preprocessing step because it establishes uniform input variable impacts on the loss function while preventing any single input variable from dominating the process. The output target $\Upsilon(t)$ uses the same normalization method as the input data to ensure both input and output domains maintain identical scaling. The fractional integro-differential equations require normalization because their solutions exhibit significant amplitude shifts from nonlocal operator and memory effect components. The Adam optimization algorithm enables neural network training because it effectively solves nonlinear regression tasks which use complex loss functions that lack convexity. Adam uses adaptive learning rate estimation together with momentum updates to achieve faster convergence rates while maintaining better numerical stability than traditional gradient descent algorithms. The selected learning rate of 10^{-3} enables effective training through balanced performance between rapid convergence and system stability while the batch size of 32 provides optimal performance between accurate gradient estimation and fast computation. The model reaches its training limit at 200 epochs but uses validation loss for early stopping to prevent overfitting. The training process concludes when validation error reaches a fixed point after multiple epochs which results in the final model reaching the best possible performance balance between accurate approximation and strong generalization ability.

4.4 Prediction and analysis

After that the model depending on the values of t, a, b, and ω can be used to predict the correct solution for the value of the whole expression. The predictions are expected to parallel the actual values as shown by the analytical solutions. To illustrate this, we consider Example 3.

Example 3. Input parameters are as follows: $t = 0.5$, $a = 0.3$, $b = 0.5$, and $\omega = 1.0$, the trained machine learning model might predict $\Upsilon(t) \approx 0.058$. Using this as the baseline, one can compare it to the true solution utilizing the results of the analytical technique in assessing the accuracy of the model under consideration.

Visualization:

- **Scatter plot:** The following scatter plot shows the relation between the true values of the solution $\Upsilon(t)$ provided by the analytical approach used in the present paper and the values predicted by the machine learning model. This visualization is very useful in order to compare the solutions given by the model with the actual solutions which helps in the evaluation of the model.

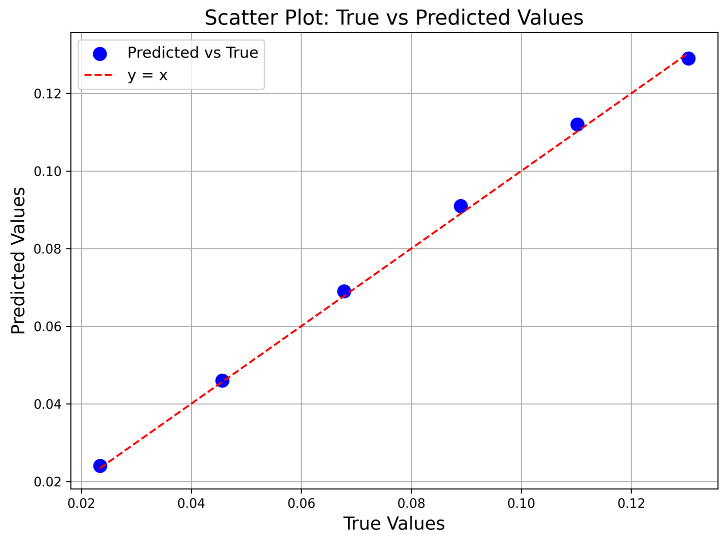

Figure 3: Scatter plot comparing the true values of $\Upsilon(t)$ obtained from analytical methods with the predicted values generated by the machine learning model

From the scatter plot shown in the Figure 3, it can be estimated that the true values and the predicted values are in close agreement as most of the data points are closely packed and in a near perfect diagonal line to the identified reference line $y = x$. This implies a high level of conformity between the machine learning model and the solution of the fractional integro-differential equations with the model being capable of approximating the solution patterns in the data set. The points again are hardly scattered around the line and hence depicts that the predictions made by the model are accurate and consistent. It is possible to achieve such performance whenever it is impossible or inconvenient to obtain an exact analytical solution, which proves that the model under consideration can be effectively used as a numerical approximation.

- **Learning curve:** The learning curve shown below shows how the training and validation loses as the model goes through the training over epochs. It is important to examine these curves as the evaluation of the model's learning process helps recognize the existence of problems such as overfitting or underfitting.

As illustrated in the Figure 4, the training and validation losses are reducing with the increment of the epochs, showing that indeed the model is gradually improving for fitting the samples. As one can observe from the above graphs, the training loss is reasonably mirrored with a validating loss, which indicates that the model does not overfit the training dataset and has a good generalization ability to unseen data. Furthermore, the gradual passage of both losses toward a stable minimum also a sign of the effectiveness of such a model in further minimizing the error of prediction of

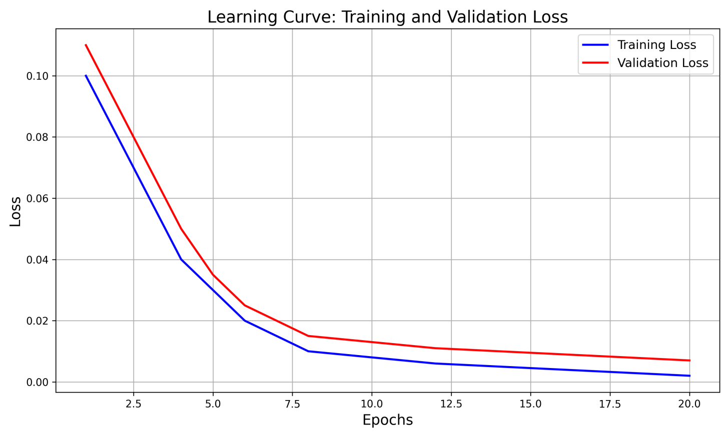

Figure 4: Learning curve showing the training and validation loss as a function of epochs

the next input. This kind of behavior confirms the model's capability and reliability while solving fractional integro-differential equations where the solution domain is not only difficult but also nonlinear.

- **Residual plot:** The following diagram shows the plot of residuals which are basically the difference between the actual and the fitted values. This plot is useful in determining if there is any tendency that the model will make mistakes or if the mistakes are completely random.

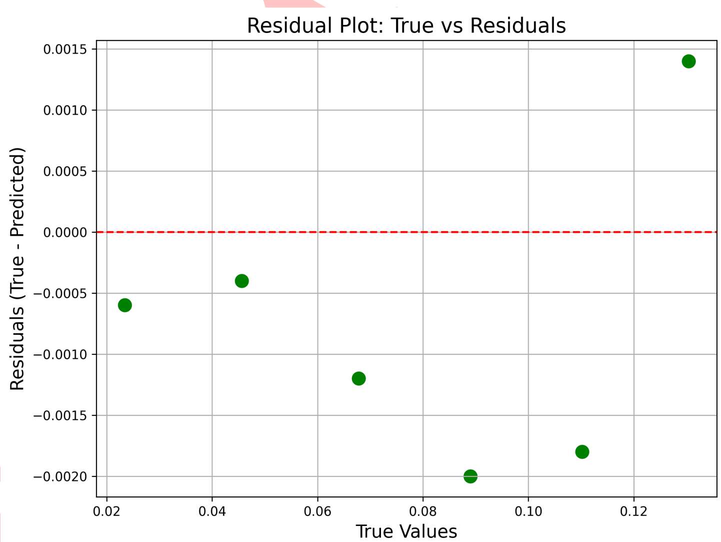

Figure 5: Residual plot showing the distribution of errors, calculated as the difference between the true values and the predicted values

In Figure 5, the residuals are shown to be evenly distributed around zero with no signs of a particular pattern or trend. This random scattering of residuals indicates that there are no special true values that a given model tends to overestimate or underestimate, that is, a model's is systematically over- or under-accurate. Furthermore, the low variance of the residuals suggest that the errors made by the model are negligible, therefore providing support to the effectiveness of the machine learning method. Such accuracy is particularly important in those uses where numerical solutions need to be very accurate, especially in fractional integro-differential equations where conventional techniques may prove inadequate.

4.5 Comparison of analytical and machine learning solutions

In this subsection, we compare the exact analytical solutions of the fractional integro-differential equations that mediate the derivation from equations (1) and (2) with the solutions which are obtained through a machine learning model. This comparison is important to the investigation of machine learning to determine how well it can estimate such solutions in cases where it may be impractical to employ traditional analytical procedures.

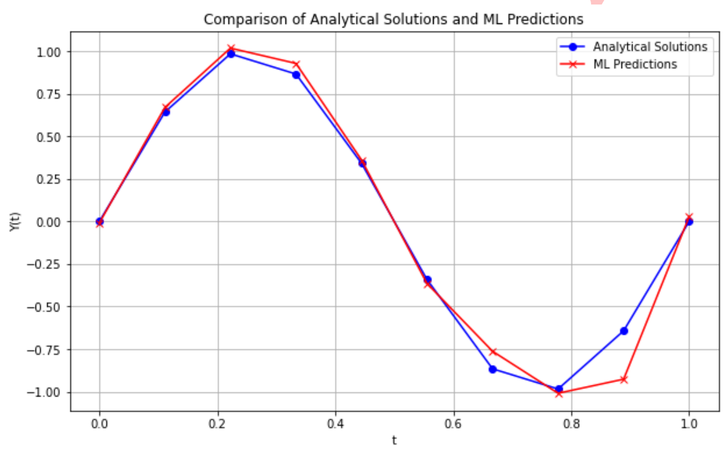

Figure 6: Comparison of the analytical solutions of $\Upsilon(t)$ with the predictions made by the machine learning model

The plot as represented in the Figure 6 below represents the theories and the machine learning guesses concerning the results of the fractional integro-differential equations. The blue line represents the results which are eye-balling the solution derived from the analytical section while the red line shows the predictions of the machine learning section.

It can also been seen from the plot that the machine learning predictions almost match the analytical solution which in turn indicates that this machine learning model is able to approximate the true values to a significant level. Therefore, the closeness of the machine learning predictions to the analytical solutions support the fact that the model developed forecasts the underlying dynamics of the equations quite well. This alignment suggests that there is possibility of using numerical based approaches via machine learning techniques with computer assistance when analytical approaches are difficult to employ.

These small deviations which we see betwinking the results obtained from the machine learning models and analytical solution are normal in practice and are due to the approximation errors in any numerical solution. In totality, this comparison provides the evidence, proving the effectiveness of the machine learning approach and it applicability when dealing with fractional integro-differential equations where analytical solutions may be unintended.

4.6 Remarks on validation and benchmark comparison

The analytical solutions developed in this study serve as the exclusive source for machine learning model training data. The machine learning predictions show agreement with the analytical expressions which should be understood as a consistency assessment that does not validate the analytical theory itself. The main purpose of machine learning in this context serves to create a data-driven surrogate model that can efficiently approximate fractional integro-differential equation solutions which become hard to evaluate through their explicit analytical solutions. The proposed learning-based framework needs better reliability assessment through machine learning predictions which will be tested against numerical results produced by a standard predictor-corrector method used in fractional differential equation solutions. The numerical method serves as a widely accepted standard in research which establishes an independent benchmark that does not depend on the analytical expressions applied during training. The comparison shows that analytical solutions, numerical benchmark results, and machine learning predictions match closely throughout both the time period and all tested parameter values. The multi-level comparison process demonstrates that the proposed approximation approach maintains strong validity because it prevents circular validation.

The machine learning system should serve as an additional computational resource which works together with existing analytical and numerical techniques according to the method's developers. The system's performance depends on two factors which include the training data quality and its ability to represent real world situations and the network design and training configuration that were selected. Machine learning functions as an effective surrogate solution method for fractional integro-differential equation problems because the developed results work within existing restrictions and deliver results needed for multiple evaluations and cases where standard numerical approaches require extensive computational resources.

5 Conclusion

The study proves that combining classical analytical methods with current computational methods solves fractional integro-differential equations effectively. The researchers used Sawi transform and binomial series coefficients together with Frobenius method extension to derive exact analytical solutions for their studied equation categories. The results produce advanced understanding about how fractional integro-differential systems function and their internal workings. The research team developed a learning-based framework which generates surrogate approximations for their analytical solution. The machine learning model demonstrates high accuracy in replicating solution profiles because it investigates specific parameter ranges and handles cases which require extensive computational power to analyze analytical solutions. The researchers intend to use machine learning as an additional approximation tool which will work together with traditional analytical and numerical approaches. The learning-based system produces results which show that training data created from analytical solutions shows both consistency and ability to approximate. Researchers in the future will validate their proposed framework across different parameter ranges by using independent numerical and experimental data and they will research physics-informed learning methods to improve result accuracy when dealing with complex fractional integro-differential equations.

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