

Generalized annihilator in pseudo BCI -algebras

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Abstract. In this paper, for any subsets D, C of a pseudo BCI -algebra A the notion of generalized annihilator of D with respect to C and $(*, \diamond)$ (resp: to C and $(\diamond, *)$), denoted by $(C : D)^{(*, \diamond)}$ (resp: $(C : D)^{(\diamond, *)}$) is introduced and its related properties are investigated. Also, a necessary and sufficient condition for BCI -algebra to be p-semisimple or pseudo BCK -algebra are given. Moreover, it is shown that the equation $(C : D)^{(*, \diamond)} = (C : D)^{(\diamond, *)}$ hold for every p-semisimple BCI -algebra. Finally, it is proved that the set of all involutory ideal, denoted by $S_C^{(*, \diamond)}(A)$ forms a distributive lattice.

Keywords: BCI -algebra, Pseudo BCI -algebra, Annihilator, Pseudo BCI -ideal, Distributive lattice.

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1 Introduction

BCI -algebras represent a significant class of logical algebras introduced by Y. Imai and K. Iseki in 1966 [12, 13]. These concepts have roots in two distinct origins: one derived from set theory and the other from classical and non-classical propositional calculi.

In 2008, W. A. Dudek and Y. B. Jun extended the idea of BCI -algebras to introduce pseudo BCI -algebras [4].

Y. B. Jun et al. introduced the idea of pseudo-ideals and pseudo-homomorphism in a pseudo BCI -algebra, and then they examined several properties [14].

M. Aslam and A. B. Thaheem introduced the concepts of annihilators and involutory ideals in commutative BCK -algebras and studied their properties [1].

In [2], A. Banderi etc. introduced the idea of annihilator in pseudo BCI -algebras and explored various associated properties. They illustrated that the idea of annihilator of D and denoted it by $D^{(*, \diamond)}$, and investigated some related properties. It was also established that for any subset D of a pseudo BCI -algebra A , $D^{(*, \diamond)}$ is closed pseudo BCI -ideal of A .

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Building upon the work in [2], this paper extends the idea of annihilator to pseudo *BCI*-algebra. For any two non-empty subsets D, C of a pseudo *BCI*-algebra A , we introduce the concept of generalized annihilator of D with respect to C and $(*, \diamond)$ (resp: to C and $(\diamond, *)$), denoted by $(C : D)^{(*, \diamond)}$ (resp: $(C : D)^{(\diamond, *)}$), and investigate some related properties. We proved that if A is a p-semisimple *BCI*-algebra then $(C : D)^{(*, \diamond)} = (C : D)^{(\diamond, *)}$, for any two non-empty subsets C, D of A . Also, using the mentioned concept we give some conditions for a pseudo *BCI*-algebra to be 1-pseudo *BCK*-algebra; 2-p-semisimple *BCI*-algebra. Finally, applying the concept of generalized annihilator, we define the involutory ideal of pseudo *BCI*-algebra and prove that the set of all involutory ideals with respect to a normal ideal forms a distributive lattice.

2 Preliminaries

In this section, we present some fundamental aspects essential for this paper. For additional details, the reader is referred to [4, 7, 14].

Definition 1 ([4]). *In a pseudo *BCI*-algebra, the structure $A = (A, \preceq, *, \diamond, 0)$ consists of \preceq as a binary relation on set A , $*$ and \diamond as binary operations on A , and 0 as an element of A satisfying the following axioms: for all $r, s, t \in A$,*

- (a₁) $(r * s) \diamond (r * t) \preceq t * s$, $(r \diamond s) * (r \diamond t) \preceq t \diamond s$,
- (a₂) $r * (r \diamond s) \preceq s$, $r \diamond (r * s) \preceq s$,
- (a₃) $r \preceq r$,
- (a₄) $r \preceq s$, $s \preceq r \implies r = s$,
- (a₅) $r \preceq s \iff r * s = 0 \iff r \diamond s = 0$.

In the context of pseudo *BCI*-algebra, a structure $A = (A, \preceq, *, \diamond, 0)$ that satisfies $0 * r = 0 = 0 \diamond r$ for all $r \in A$ is known as a pseudo *BCK*-algebra.

Proposition 1 ([4]). *Any pseudo *BCI*-algebra A satisfies the following conditions: for any $r, s, t \in A$,*

- (p₁) $r \preceq 0 \implies r = 0$,
- (p₂) $r \preceq s \implies r * t \preceq s * t$, $r \diamond t \preceq s \diamond t$,
- (p₃) $r \preceq s \implies t * s \preceq t * r$, $t \diamond s \preceq t \diamond r$,
- (p₄) $r \preceq s$, $s \preceq t \implies r \preceq t$,
- (p₅) $(r * s) \diamond t = (r \diamond t) * s$,
- (p₆) $r * s \preceq t \iff r * t \preceq s$, $r \diamond s \preceq t \iff r \diamond t \preceq s$,
- (p₇) $(r * s) * (t * s) \preceq r * t$, $(r \diamond s) \diamond (t \diamond s) \preceq r \diamond t$,
- (p₈) $r * (r \diamond (r * s)) = r * s$ and $r \diamond (r * (r \diamond s)) = r \diamond s$,
- (p₉) $r * 0 = r = r \diamond 0$,
- (p₁₀) $r * r = 0 = r \diamond r$,
- (p₁₁) $0 * (r \diamond s) \preceq s \diamond r$, $0 \diamond (r * s) \preceq s * r$,
- (p₁₂) $0 * r = 0 \diamond r$,
- (p₁₃) $0 * (r * s) = (0 * r) \diamond (0 * s)$, $0 \diamond (r \diamond s) = (0 \diamond r) * (0 \diamond s)$.

The relation order \preceq in any pseudo *BCI*-algebra (and pseudo *BCK*-algebra) A , defined by

$$(\forall r, s \in A) r \preceq s \iff r * s = 0 \iff r \diamond s = 0,$$

is a partial order.

Let $(A, \preceq, *, \diamond, 0)$ be a pseudo BCI-algebra and S a non empty subset of A . Then S is called a *subalgebra* of A if $r * s \in S$ and $r \diamond s \in S$ for any $r, s \in S$. It can be checked that $K(A) := \{r \in A \mid 0 * r = 0 = 0 \diamond r\}$ is a subalgebra of A (called the *maximal* pseudo BCK-algebra of A). This establishes that $(K(A), \preceq, *, \diamond, 0)$ forms a pseudo BCK-algebra.

In a pseudo BCI-algebra A , an element c is considered *minimal* if the following condition holds:

$$(\forall r \in A) r \preceq c \implies r = c.$$

The set of all minimal elements of A will be denoted by $M(A)$. Clearly, $0 \in M(A)$. In [11], it has showed that $c \in M(A)$ if and only if $c = 0 * (0 \diamond c)$.

Hence $M(A) = \{c \in A \mid c = 0 * (0 \diamond c)\}$. It can be checked that $K(A) \cap M(A) = \{0\}$. A pseudo BCI-algebra A is considered *p-semisimple* if every element in A is minimal, that is $M(A) = A$.

Proposition 2 ([7]). *Considering a pseudo BCI-algebra A and elements $r, s \in A$, the following conditions are equivalent:*

- (i) A is a *p-semisimple*,
- (ii) $r * (r \diamond s) = s = r \diamond (r * s)$,
- (iii) $0 * (0 \diamond r) = r = 0 \diamond (0 * r)$.

Definition 2 ([15]). *In pseudo BCI-algebra A , a subset J of A is defined a pseudo BCI-ideal when:*

- (J1) $0 \in J$,
- (J2) $(\forall s \in J)(*(s, J) := \{r \in A \mid r * s \in J\} \subseteq J$ and $\diamond(s, J) := \{r \in A \mid r \diamond s \in J\} \subseteq J)$.

Theorem 1 ([5, 7]). *A subset J of A is a pseudo BCI-ideal of A if and only if $0 \in J$ and if $r, s \diamond r \in J$ imply $s \in J$ for all $r, s \in A$.*

Theorem 2 ([5, 7]). *Suppose J is a pseudo BCI-ideal of a pseudo BCI-algebra A . Then the following hold:*

- (i) *If $s \preceq r \in J$, then $s \in J$,*
- (ii) *A pseudo BCI-ideal J of a pseudo BCI-algebra A is closed if and only if for any $r \in J$, $0 * r = 0 \diamond r \in J$.*

Definition 3 ([8]). *pseudo BCI-ideal J of a pseudo BCI-algebra A is called compatible if for all $r, s \in A$,*

$$r * t \in J \iff r \diamond t \in J.$$

A partially ordered set $(L; \leq)$ is called a *lattice* if any two elements x, y in L have both the greatest lower bound, denoted by $x \wedge y$ and the least upper bound, denoted by $x \vee y$. (see [15]).

A distributive Lattice is a lattice L satisfying the following law (called the distributive identity)

$$x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z) \text{ or } x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z)$$

for all $x, y, z \in L$ (see [3], [10]).

3 Generalized annihilator in pseudo BCI-algebras

We start by defining the concept of generalized annihilator of D respect to C in which C, D are non-empty subsets of a pseudo BCI-algebra A , and then investigate some related properties. Throughout this paper, let A will denote a pseudo BCI-algebra unless specified otherwise.

Definition 4. Consider C, D as non-empty subsets of A . The generalized annihilator of D respect to C and $(*, \diamond)$, denoted by $(C : D)^{(*, \diamond)}$, is defined as

$$(C : D)^{(*, \diamond)} := \{r \in A \mid d * (d \diamond r) \in C, \forall d \in D\}.$$

Similarly, $(C : D)^{(\diamond, *)}$ is defined. It is evident that, $0 \in (C : D)^{(*, \diamond)}$ and $0 \in (C : D)^{(\diamond, *)}$.

In the example below, we demonstrate that $(C : D)^{(*, \diamond)}$ and $(C : D)^{(\diamond, *)}$ may not be equal.

Example 1 ([9]). Consider $A = (\{0, g, h, j, k\}, *, \diamond, 0)$ as a pseudo BCI-algebra, with the binary operations $*$ and \diamond defined by the following Cayley's tables:

$*$	0	g	h	j	k	\diamond	0	g	h	j	k
0	0	0	0	j	j	0	0	0	0	j	j
g	g	0	0	j	j	g	g	0	0	j	j
h	h	h	0	j	j	h	h	h	0	j	j
j	j	j	j	0	0	j	j	j	j	0	0
k	k	k	j	g	0	k	k	j	j	h	0

By choosing, $D := \{0, k\}$, $C := \{0, h\}$, it can be verified that $(C : D)^{(*, \diamond)} = \{0\} \neq \{0, g, h\} = (C : D)^{(\diamond, *)}$.

In the following theorem, we give a sufficient condition satisfying the equality $(C : D)^{(*, \diamond)} = (C : D)^{(\diamond, *)}$.

Theorem 3. If A is p-semisimple then $(C : D)^{(*, \diamond)} = (C : D)^{(\diamond, *)}$, for any two non-empty subsets C, D of A .

Proof. Let A be p-semisimple and $r \in (C : D)^{(*, \diamond)}$. Then $d * (d \diamond r) \in C$ for all $d \in D$. By Proposition 2, $d * (d \diamond r) = r = d \diamond (d * r) \in C$. Therefore $d \diamond (d * r) \in C$ for all $d \in D$, and so $r \in (C : D)^{(\diamond, *)}$. Hence $(C : D)^{(*, \diamond)} \subseteq (C : D)^{(\diamond, *)}$. Similarly, we have $(C : D)^{(\diamond, *)} \subseteq (C : D)^{(*, \diamond)}$. Therefore $(C : D)^{(*, \diamond)} = (C : D)^{(\diamond, *)}$. \square

It can be easily see that the proof of results about operation $(\diamond, *)$ is similar to the proof of results about operation $(*, \diamond)$. So from now on, we suffices to discuss the results of operation $(*, \diamond)$.

The following lemma is an immediate consequence from the Definition 4.

Lemma 1. If E, F are non-empty subsets of A such that $E \subseteq F$, then $(C : F)^{(*, \diamond)} \subseteq (C : E)^{(*, \diamond)}$ and $(E : C)^{(*, \diamond)} \subseteq (F : C)^{(*, \diamond)}$.

Lemma 2. If C is a pseudo BCI-ideal of A and $\emptyset \neq D \subseteq A$, Then $C \subseteq (C : D)^{(*, \diamond)}$.

Proof. Let $c \in C$. It follows from Definition 1(a_2) that $d * (d \diamond c) \preceq c$ for any $d \in D$. Then by Theorem 2(i), we have $d * (d \diamond c) \in C$. Thus, according to Definition 4, we get $c \in (C : D)^{(*, \diamond)}$ and so the proof is completed. \square

The following example shows that the pseudo BCI-ideal condition in Lemma 2 is necessary.

Example 2. Let $A = (\{0, g, h, j, k\}, *, \diamond, 0)$ be a pseudo BCI-algebra as in Example 2. By taking $D := \{0\}$ and $C := \{k\}$, clearly C is not a pseudo BCI-ideal of A . It can be checked that $(C : D)^{(*, \diamond)} = \emptyset$. Therefore, $C \not\subseteq (C : D)^{(*, \diamond)}$.

Theorem 4. *If C is a pseudo BCI-ideal of A and $D \subseteq A$, then*

$$(C : D)^{(*, \diamond)} \cap D = C \cap D.$$

Proof. According to Lemma 2, we observe that $C \cap D \subseteq (C : D)^{(*, \diamond)} \cap D$. To demonstrate the reverse inclusion, suppose $r \in (C : D)^{(*, \diamond)} \cap D$. Since $r \in (C : D)^{(*, \diamond)}$, it follows that $d * (d \diamond r) \in C$ for all $d \in D$. By substituting $d = r$, we find $r = r * (r \diamond r) \in C$, implying $r \in C$. Thus, $(C : D)^{(*, \diamond)} \cap D \subseteq C \cap D$, and so the proof is completed. \square

Lemma 3. *For any $\emptyset \neq C \subseteq A$. The following hold:*

- (i) $(A : \{0\})^{(*, \diamond)} = A$,
- (ii) $(A : A)^{(*, \diamond)} = A$,
- (iii) $(\{0\} : A)^{(*, \diamond)} = \{0\}$.

Proof. (i), (ii) by using Lemma 2, the proof is straightforward.

(iii) Let $r \in (\{0\} : A)^{(*, \diamond)}$. Then $d * (d \diamond r) = 0$ for all $d \in A$. By putting $d = r$, we have $r = r * (r \diamond r) = 0$. Therefore $(\{0\} : A)^{(*, \diamond)} = \{0\}$. \square

Lemma 4. *Let A be a p -semisimple pseudo BCI-algebra. Then for any $\emptyset \neq C \subseteq A$, the following hold:*

- (1) $(C : C)^{(*, \diamond)} = C$,
- (2) $(\{0\} : C)^{(*, \diamond)} = \{0\}$.

Proof. (1) Let $r \in (C : C)^{(*, \diamond)}$. Then $d * (d \diamond r) \in C$ for all $d \in C$. By Proposition 2, we have $d * (d \diamond r) = r$. Hence $r \in C$. Thus, $(C : C)^{(*, \diamond)} \subseteq C$. To proof the inverse inclusion, assume that $r \in C$. By a similar argument, we have $d * (d \diamond r) = r$, which implies $d * (d \diamond r) \in C$. Thus, $r \in (C : C)^{(*, \diamond)}$. Therefore, $C \subseteq (C : C)^{(*, \diamond)}$ and the proof of (1) is completed.

(2) Let $r \in (\{0\} : C)^{(*, \diamond)}$. Then $d * (d \diamond r) = 0$ for all $d \in C$. By Proposition 2, we have $d * (d \diamond r) = r$. Hence, $r = 0$. Therefore, $(\{0\} : C)^{(*, \diamond)} = \{0\}$. \square

Using the concept of generalized annihilator, we give a characterization of p -semisimple BCI-algebras.

Theorem 5. *A is p -semisimple if and only if $(C : \{0\})^{(*, \diamond)} = C$ for any $\emptyset \neq C \subseteq A$.*

Proof. (\Rightarrow) Let A be p -semisimple, and let $r \in (C : \{0\})^{(*, \diamond)}$. Then $0 * (0 \diamond r) \in C$, and since $0 * (0 \diamond r) = r$, we get $r \in C$. Therefore, $(C : \{0\})^{(*, \diamond)} \subseteq C$. To prove the reverse inclusion, let $r \in C$. By Proposition 2, we have $0 * (0 \diamond r) = r$, so $0 * (0 \diamond r) \in C$, which implies $r \in (C : \{0\})^{(*, \diamond)}$. Thus, $C \subseteq (C : \{0\})^{(*, \diamond)}$, and so the result holds.

(\Leftarrow) Assume that $(C : \{0\})^{(*, \diamond)} = C$ for every non-empty subset C of A . Let $r \in A$. Putting $C = \{r\}$, we have $(\{r\} : \{0\})^{(*, \diamond)} = \{r\}$. Hence $r \in (\{r\} : \{0\})^{(*, \diamond)}$, and so $0 * (0 * r) \in \{r\}$, which implies $0 * (0 * r) = r$. Therefore A is p -semisimple, as required. \square

Proposition 3. Let $\{D_i | i \in I\}$ be a family of non-empty subset of A and $\emptyset \neq C \subseteq A$. Then we have

$$(i) \bigcap_{i \in I} (C : D_i)^{(*, \diamond)} = (C : \bigcup_{i \in I} D_i)^{(*, \diamond)},$$

$$(ii) \bigcup_{i \in I} (C : D_i)^{(*, \diamond)} \subseteq (C : \bigcap_{i \in I} D_i)^{(*, \diamond)}.$$

Proof. (i) Using Lemma 1, we get $(C : \bigcup_{i \in I} D_i)^{(*, \diamond)} \subseteq \bigcap_{i \in I} (C : D_i)^{(*, \diamond)}$. To proof the reverse inclusion, assume that $r \in \bigcap_{i \in I} (C : D_i)^{(*, \diamond)}$. Thus $r \in (C : D_i)^{(*, \diamond)}$ for all $i \in I$, and consequently $d * (d \diamond r) \in C$ for all $d \in D_i$. Hence $d * (d \diamond r) \in C$ for all $d \in \bigcup_{i \in I} D_i$, and so $r \in (C : \bigcup_{i \in I} D_i)^{(*, \diamond)}$.

Therefore $\bigcap_{i \in I} (C : D_i)^{(*, \diamond)} \subseteq (C : \bigcup_{i \in I} D_i)^{(*, \diamond)}$ and so (i) holds.

(ii) By Lemma 1, the proof is straightforward. \square

The reverse inclusion in Proposition 3 (ii) is not true in general as seen in the following example.

Example 3 ([6]). Let $(A = \{0, g, h, 1\}, *, \diamond, 0)$ be a pseudo BCI-algebra, which binary operations $*$ and \diamond are defined by the following Cayley's tables:

$*$	0	g	h	1		\diamond	0	g	h	1
0	0	0	0	0		0	0	0	0	0
g	g	0	0	0	,	g	g	0	0	0
h	h	h	0	0		h	h	h	0	0
1	1	h	h	0		1	1	1	g	0

Taking $D_1 := \{g, 1\}$, $D_2 := \{h, 1\}$ and $C := \{0, 1\}$, it is routine to check that $(C : D_1)^{(*, \diamond)} = \{0\}$, $(C : D_2)^{(*, \diamond)} = \{0, g\}$, $\bigcup_{i=1}^2 (C : D_i)^{(*, \diamond)} = \{0, g\}$ and $(C : \bigcap_{i=1,2} D_i)^{(*, \diamond)} = \{0, g, 1\}$. Therefore, $(C : \bigcap_{i=1,2} D_i)^{(*, \diamond)} \not\subseteq \bigcup_{i=1,2} (C : D_i)^{(*, \diamond)}$.

Theorem 6. Let A be p -semisimple. If C is a closed pseudo BCI-ideal of A , then so is $(C : D)^{(*, \diamond)}$, for any $\emptyset \neq D \subseteq A$.

Proof. Clearly, $0 \in (C : D)^{(*, \diamond)}$. Now assume that $r, s \diamond r \in (C : D)^{(*, \diamond)}$ for some $r, s \in A$. Then for all $d \in D$, we have

$$d * (d \diamond r) \in C \text{ and } d * (d \diamond (s \diamond r)) \in C,$$

By the p-semisimple property, we have

$$d * (d \diamond (s \diamond r)) = s \diamond r.$$

Using Definition 1 (a_1), we obtain

$$(d * (d \diamond s)) \diamond (d * (d \diamond r)) \preceq (d \diamond r) * (d \diamond s) \preceq s \diamond r \in C$$

It follows that $(d * (d \diamond s)) \diamond (d * (d \diamond r)) \in C$. Hence from $d * (d \diamond r) \in C$, we conclude $d * (d \diamond s) \in C$ for any $d \in D$, and so $s \in (C : D)^{(*, \diamond)}$. Therefore, $(C : D)^{(*, \diamond)}$ is a pseudo BCI-ideal of A .

To proof that $(C : D)^{(*, \diamond)}$ is closed, it suffices to show that $0 * r \in (C : D)^{(*, \diamond)}$ for any $r \in (C : D)^{(*, \diamond)}$. For $r \in (C : D)^{(*, \diamond)}$, we have $d * (d \diamond r) \in C$ for all $d \in D$, and so by the p-semisimple property, we get $r \in C$. By closeness of C we get, $0 * r \in C$. On the other hand, we have

$$d * (d \diamond (0 * r)) = 0 * r \in C.$$

Hence, $d * (d \diamond (0 * r)) \in C$ for all $d \in D$. Thus $0 * r \in (C : D)^{(*, \diamond)}$, and so the proof is completed. \square

Proposition 4. For any non-empty subset D of A , $(K(A) : D)^{(*, \diamond)} = K(A)$.

Proof. By Lemma 2, since $K(A)$ is a pseudo BCI-ideal of A , $K(A) \subseteq (K(A) : D)^{(*, \diamond)}$. Now, let $r \in (K(A) : D)^{(*, \diamond)}$. Then, $d * (d \diamond r) \in K(A)$ for all $d \in D$. Hence, $0 * (d * (d \diamond r)) = 0$. On the other hand, by (a_2), $d * (d \diamond r) \preceq r$, thus $0 * r \preceq 0 * (d * (d \diamond r)) = 0$, and so $0 * r = 0$, that is $r \in K(A)$. Therefore, $(K(A) : D)^{(*, \diamond)} = K(A)$. \square

Lemma 5. Let C, D be non-empty subsets of A with $C \subseteq K(A)$. Then $(C : D)^{(*, \diamond)} \subseteq K(A)$.

Proof. Using Lemma 1 and Proposition 4, the proof is straightforward. \square

In the following, we give a characterization of pseudo BCK-algebras.

Theorem 7. Let A be a pseudo BCI-algebra. Then A is a pseudo BCK-algebra if and only if $(C : \{0\})^{(*, \diamond)} = A$, for any $\emptyset \neq C \subseteq A$.

Proof. Let A be a pseudo BCK-algebra. For any $r \in A$, $0 * r = 0$ and so $0 \diamond (0 * r) = 0 \in C$ for any $r \in A$. This implies that $r \in (C : \{0\})^{(*, \diamond)}$. Therefore $A \subseteq (C : \{0\})^{(*, \diamond)}$ and so $(C : \{0\})^{(*, \diamond)} = A$.

Conversely, taking $C = \{0\}$, by assumption, we get $(\{0\}, \{0\})^{(*, \diamond)} = A$. Then $r \in (\{0\}, \{0\})^{(*, \diamond)}$ for any $r \in A$. It follows that, $0 * (0 \diamond r) = 0$ and so $0 * (0 * (0 \diamond r)) = 0$. Thus, by (p_8), we get $0 \diamond r = 0$. This implies $r \in K(A)$ and so $A = K(A)$. Therefore, A is a pseudo BCK-algebra. \square

Definition 5. Let C be a pseudo BCI-ideal of A . Then C is called

- (i) a $(*, \diamond)$ -normal if it satisfies $(\forall r, s \in A) r * (r \diamond s) \in C \Rightarrow s * (s \diamond r) \in C$,
- (ii) a $(\diamond, *)$ -normal if it satisfies $(\forall r, s \in A) r \diamond (r * s) \in C \Rightarrow s \diamond (s * r) \in C$,
- (iii) normal if it is both $(*, \diamond)$ -normal and $(\diamond, *)$ -normal.

In the example below, we demonstrate that the $(*, \diamond)$ -normal and $(\diamond, *)$ -normal may not be equal.

Example 4. Let $(A = \{0, g, h, 1\}, *, \diamond, 0)$ be a pseudo BCI-algebra as in Example 3. Consider a pseudo BCI-ideal $C = \{0, g\}$ of A . It can be checked that C is a $(*, \diamond)$ -normal but is not $(\diamond, *)$ -normal, because $1 \diamond (1 * h) = g \in C$ but $h \diamond (h * 1) = h \notin C$

Lemma 6. *If $(C : D)^{(*, \diamond)} = A$, then $D \subseteq C$ for any $C, D \subseteq A$.*

Proof. Let $r \in D$. Then by assumption, $r \in (C : D)^{(*, \diamond)}$ and so $d * (d \diamond r) \in C$ for every $d \in D$. Thus from $r \in D$, we get $r = r * (r \diamond r) \in C$. Therefore $D \subseteq C$. \square

Proposition 5. *Let C be a $(*, \diamond)$ -normal pseudo BCI-ideal of A and $D \subseteq A$. Then*

$$(C : D)^{(*, \diamond)} = A \text{ if and only if } D \subseteq C.$$

Proof. By (i), we only need to prove the sufficiency. Assume that $D \subseteq C$ and r is an arbitrary element of A . By (a_2) , we have

$$r * (r \diamond d) \preceq d \in D \text{ for any } d \in D.$$

Thus, by assumption, we get $r * (r \diamond d) \in C$ and so by the $(*, \diamond)$ -normality of C , we conclude $d * (d \diamond r) \in C$ for any $d \in D$, which implies $r \in (C : D)^{(*, \diamond)}$. Therefore $(C : D)^{(*, \diamond)} = A$, and so the proof is completed. \square

Proposition 6. *Every $(*, \diamond)$ -normal pseudo BCI-ideal is closed and contains $M(A)$.*

Proof. Let C be a $(*, \diamond)$ -normal of A . For any $r \in C$, we have $(0 * r) * ((0 * r) \diamond 0) = 0 \in C$. Then it follows from C is a $(*, \diamond)$ -normal, that $0 * (0 \diamond (0 * r)) \in C$, and so by (p_8) , we get $0 * r \in C$. Hence C is closed. Now, let $r \in M(A)$. Then by the minimality of r , we conclude $0 * (0 \diamond r) = r$. On the other hand, from $r * (r \diamond 0) = 0 \in C$ and the fact that C is a $(*, \diamond)$ -normal, we get $0 * (0 \diamond r) \in C$. Therefore $r \in C$ and so $M(A) \subseteq C$, which completes the proof. \square

Theorem 8. *Let A be a pseudo BCI-algebra. Then for any $(*, \diamond)$ -normal D and two pseudo BCI-ideals C, E of A , we have*

$$E \cap D \subseteq C \Leftrightarrow E \subseteq (C : D)^{(*, \diamond)}$$

Proof. (\Rightarrow) Let $E \cap D \subseteq C$ and $e \in E$. For any $d \in D$, using (a_2) , we have $e * (e \diamond d) \preceq d \in D$ hence by the $(*, \diamond)$ -normality of D , we get $d * (d \diamond e) \in D$. Obviously, $d * (d \diamond e) \preceq e \in E$. Therefore $d * (d \diamond e) \in E \cap D$, hence by hypothesis, we get $d * (d \diamond e) \in C$, which implies $e \in (C : D)^{(*, \diamond)}$. Therefore $E \subseteq (C : D)^{(*, \diamond)}$.

(\Leftarrow) Let $E \subseteq (C : D)^{(*, \diamond)}$ and r be an arbitrary element of $E \cap D$. Then $r \in (C : D)^{(*, \diamond)}$, and hence we have

$$d * (d \diamond r) \in C \text{ for any } d \in D \tag{1}$$

Since $r \in D$, putting $d = r$ in (1), we get $r * (r \diamond r) \in C$, that is $r \in C$. Therefore $E \cap D \subseteq C$. \square

In the following, we establish some others properties of the generalized annihilator.

Theorem 9. Let A be a pseudo BCI-algebra, C is a $(*, \diamond)$ -normal pseudo BCI-ideal of A and $D \subseteq A$. Then the following hold:

- (i) $D \subseteq (C : (C : D)^{(*, \diamond)})^{(*, \diamond)}$,
- (ii) $(C : D)^{(*, \diamond)} = (C : (C : (C : D)^{(*, \diamond)})^{(*, \diamond)})^{(*, \diamond)}$.

Proof. (i) Clearly, if $d \in D$, then $d * (d \diamond r) \in C$ for all $r \in (C : D)^{(*, \diamond)}$. Thus, since C is a $(*, \diamond)$ -normal, we conclude $r * (r \diamond d) \in C$, which implies $d \in (C : (C : D)^{(*, \diamond)})^{(*, \diamond)}$. Therefore $D \subseteq (C : (C : D)^{(*, \diamond)})^{(*, \diamond)}$.

(ii) By (i), we have $D \subseteq (C : (C : D)^{(*, \diamond)})^{(*, \diamond)}$ and so by Lemma 1, we get $(C : (C : (C : D)^{(*, \diamond)})^{(*, \diamond)})^{(*, \diamond)} \subseteq (C : D)^{(*, \diamond)}$. On the other hand, using (i), we obtain $(C : D)^{(*, \diamond)} \subseteq (C : (C : (C : D)^{(*, \diamond)})^{(*, \diamond)})^{(*, \diamond)}$, and so the proof of (ii) is completed. \square

The reverse inclusion of Theorem 9(i) may not true in general as seen in the following example.

Example 5. Let $A = \{0, g, h, 1\}$ be a pseudo BCI-algebra as in Example 3. By taking $C = \{0, g\}$ and $D = \{h\}$, it is routine to check that C is a $(*, \diamond)$ -normal. Some routine calculations shows that $(C : (C : D)^{(*, \diamond)})^{(*, \diamond)} = \{0, g\}$. Therefore $(C : (C : D)^{(*, \diamond)})^{(*, \diamond)} \not\subseteq D$.

Proposition 7. Let A be a pseudo BCI-algebra. Then for any two subsets D and E and a pseudo BCI-ideal C of A , $((C : D)^{(*, \diamond)}; E)^{(*, \diamond)} \subseteq (C : (D \cap E))^{(*, \diamond)}$.

Proof. Let $r \in ((C : D)^{(*, \diamond)}; E)^{(*, \diamond)}$. Then $e * (e \diamond r) \in (C : D)^{(*, \diamond)}$ for every $e \in E$, and hence $d * (d \diamond (e * (e \diamond r))) \in C$ for every $d \in D$ and $e \in E$. Consequently, it follows that $v * (v \diamond (v * (v \diamond r))) \in C$ for every $v \in D \cap E$, and so by (p8), we get $v * (v \diamond r) \in C$ for every $v \in D \cap E$. This implies that $r \in (C : (D \cap E))^{(*, \diamond)}$. Therefore the proof is completed. \square

Definition 6. Let A be a pseudo BCI-algebra and C is a pseudo BCI-ideal of A . Then a pseudo BCI-ideal D of A is called an involutory pseudo BCI – ideal with respect to C if $D = (C : (C : D)^{(*, \diamond)})^{(*, \diamond)}$. We denote the set of all involutory pseudo BCI-ideals of A respect to C by $S_C^{(*, \diamond)}(A)$.

Example 6. Let $(A = \{0, 1, 2, 3\}, *, \diamond, 0)$ be a pseudo BCI-algebra which binary operations $*$ and \diamond and are defined by the following Caylye's tables:

$*$	0	1	2	3		\diamond	0	1	2	3
0	0	0	0	0		0	0	0	0	0
1	1	0	1	1	,	1	1	0	1	1
2	2	2	0	2		2	2	1	0	1
3	3	3	3	0		3	3	1	1	0

(i) By taking $D := \{0\}$ and $C := \{0, 1\}$, we can check that C, D are pseudo BCI-ideal of A and $(C : (C : D)^{(*, \diamond)})^{(*, \diamond)} = \{0\}$ and so $D = (C : (C : D)^{(*, \diamond)})^{(*, \diamond)}$. Therefore D is an involutory ideal with respect to C .

(ii) Now, taking $C := \{0, 2\}$ and $D := \{0, 1\}$. D is not involutory, Because $(C : (C : D)^{(*, \diamond)})^{(*, \diamond)} = \{0\}$ and so $D \neq (C : (C : D)^{(*, \diamond)})^{(*, \diamond)}$.

Proposition 8. For any $(*, \diamond)$ -normal pseudo BCI-ideal C of A and $D \subseteq A$, the following hold:

- (i) $A = (C : (C : A)^{(*, \diamond)})^{(*, \diamond)}$, that is $A \in S_C^{(*, \diamond)}(A)$,
- (ii) $(C : D)^{(*, \diamond)} \in S_C^{(*, \diamond)}(A)$.

Proof. (i) By Theorem 9 (i), the result is obvious.

(ii) It is a direct consequence from Theorem 9 (ii). \square

Theorem 10. Let C be a normal pseudo BCI-ideal of A . Then the following hold: for any

$D, E \in S_C^{(*, \diamond)}(A)$,

(i) $\inf\{D, E\} = D \cap E$;

(ii) $\sup\{D, E\} = (C : (C : D \cup E)^{(*, \diamond)})^{(*, \diamond)}$;

(iii) $(S_C^{(*, \diamond)}(A), \wedge, \vee)$ is a distributive lattice, where $D \wedge E = D \cap E$ and $D \vee E = (C : (C : D \cup E)^{(*, \diamond)})^{(*, \diamond)}$.

Proof. (i) Let $D, E \in S_C^{(*, \diamond)}(A)$. Then by Theorem 9 (i), we have $D \cap E \subseteq (C : (C : D \cap E)^{(*, \diamond)})^{(*, \diamond)}$. To proof the reverse inclusion, by Lemma 1, it follows from $D \cap E \subseteq D, E$ that $(C : (C : D \cap E)^{(*, \diamond)})^{(*, \diamond)} \subseteq (C : (C : D)^{(*, \diamond)})^{(*, \diamond)} \cap (C : (C : E)^{(*, \diamond)})^{(*, \diamond)}$. But $D, E \in S_C^{(*, \diamond)}(A)$. Thus $(C : (C : D)^{(*, \diamond)})^{(*, \diamond)} \cap (C : (C : E)^{(*, \diamond)})^{(*, \diamond)} = D \cap E$, and so $D \cap E = (C : (C : D \cap E)^{(*, \diamond)})^{(*, \diamond)}$. Therefore, it follows that $D \cap E \in S_C^{(*, \diamond)}(A)$. Moreover, clearly $D \cap E$ is the biggest involutory ideal contained in D, E . This implies $\inf\{D, E\} = D \cap E$.

(ii) By Theorem 9 (i), $D, E \subseteq D \cup E \subseteq (C : (C : D \cup E)^{(*, \diamond)})^{(*, \diamond)}$. Thus $(C : (C : D \cup E)^{(*, \diamond)})^{(*, \diamond)}$ is an upper bound of D, E . We now assume that $D, E \subseteq F$ for some $F \in S_C^{(*, \diamond)}(A)$. Then by Lemma 1, $(C : F)^{(*, \diamond)} \subseteq (C : D)^{(*, \diamond)}, (C : E)^{(*, \diamond)}$ and so $(C : F)^{(*, \diamond)} \subseteq (C : D)^{(*, \diamond)} \cap (C : E)^{(*, \diamond)}$. From Proposition 3 (i) and Lemma 1, it follows that

$$\begin{aligned} (C : (C : D \cup E)^{(*, \diamond)})^{(*, \diamond)} &= (C : ((C : D)^{(*, \diamond)} \cap (C : E)^{(*, \diamond)}))^{(*, \diamond)} \\ &\subseteq ((C : (C : F)^{(*, \diamond)})^{(*, \diamond)})^{(*, \diamond)} \\ &= F \end{aligned}$$

Hence $(C : (C : D \cup E)^{(*, \diamond)})^{(*, \diamond)}$ is the least involutory ideal containing D, E . Therefore, $\sup\{D, E\} = (C : (C : D \cup E)^{(*, \diamond)})^{(*, \diamond)}$.

(iii) By (i) and (ii) $(S_C^{(*, \diamond)}(A), \wedge, \vee)$ is a lattice. To prove the distributivity of $S_C^{(*, \diamond)}(A)$, Let $E, F, G \in S_C^{(*, \diamond)}(A)$. It is well known that in any lattice, $(E \wedge F) \vee (E \wedge G) \subseteq E \wedge (F \vee G)$. Thus, it suffices to prove that $E \wedge (F \vee G) \subseteq (E \wedge F) \vee (E \wedge G)$. For brevity, we put $R := (E \wedge F) \vee (E \wedge G)$. Since $R \in S_C^{(*, \diamond)}(A)$, it follows from Definition 6 that $R = (C : (C : R)^{(*, \diamond)})^{(*, \diamond)}$. But $E \cap F \subseteq R$. Hence, we have

$$E \cap F \subseteq (C : (C : R)^{(*, \diamond)})^{(*, \diamond)}. \quad (2)$$

Thus, if we intersect the two sides of (2) with $(C : R)^{(*, \diamond)}$, we have

$$(E \cap F) \cap (C : R)^{(*, \diamond)} \subseteq (C : R)^{(*, \diamond)} \cap (C : (C : R)^{(*, \diamond)})^{(*, \diamond)}. \quad (3)$$

On the other hand, by Theorem 4, we have

$$(C : R)^{(*, \diamond)} \cap (C : (C : R)^{(*, \diamond)})^{(*, \diamond)} = C \cap (C : R)^{(*, \diamond)} \quad (4)$$

It follows from (3) and (4) that $F \cap (E \cap (C : R)^{(*, \diamond)}) \subseteq C$. But by Theorem 8, we get $E \cap (C : R)^{(*, \diamond)} \subseteq (C : F)^{(*, \diamond)}$. By the similar argument for G , we have $E \cap (C : R)^{(*, \diamond)} \subseteq (C : G)^{(*, \diamond)}$. Thus $E \cap (C : R)^{(*, \diamond)} \subseteq (C : F)^{(*, \diamond)} \cap (C : G)^{(*, \diamond)}$, and hence from $(C : F)^{(*, \diamond)} \cap (C : G)^{(*, \diamond)} \in S_C^{(*, \diamond)}(A)$, we get

$$E \cap (C : R)^{(*, \diamond)} \subseteq (C : (C : ((C : F)^{(*, \diamond)} \cap (C : G)^{(*, \diamond)}))^{(*, \diamond)})^{(*, \diamond)} \quad (5)$$

For brevity, we put $P := (C : ((C : F)^{(*, \diamond)} \cap (C : G)^{(*, \diamond)}))^{(*, \diamond)}$. Thus, if we intersect the two sides of (5) with P and using Theorem 4, we have

$$(E \cap (C : R)^{(*, \diamond)}) \cap P \subseteq (C : P)^{(*, \diamond)} \cap P = C \cap P, \quad (6)$$

and hence $(C : R)^{(*, \diamond)} \cap (E \cap P) \subseteq C$. But by Theorem 8, we get $E \cap P \subseteq R$. Also, by Proposition 3(i), we get $P = (C : (C : F \cup G)^{(*, \diamond)})^{(*, \diamond)}$, and so $E \cap P = E \wedge (F \vee G)$. Therefore $E \wedge (F \vee G) \subseteq R = (E \wedge F) \vee (E \wedge G)$, and so the proof is completed. \square

Conclusion

In this work, we introduced the notion of generalized annihilator of D with respect to C for subsets D, C of a pseudo BCI-algebra A , and explored its fundamental properties. We demonstrated that the generalized annihilator of a closed pseudo BCI-ideal is a closed pseudo BCI-ideal too. Also, we gave a necessary and sufficient condition for BCI-algebra to be p-semisimple or pseudo BCK-algebra. Additionally, we defined the concept of a normal ideal and examined the relationship between pseudo BCI-algebras and normal ideals. Spatially, we proved that every normal pseudo BCI-ideal is closed and contains the set of all minimal elements. Finally, we introduced the notion of an involutory pseudo BCI-ideal and proved that the collection of all involutory ideals forms a distributive lattice.

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