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Affine-type pairs on hypergroups and their ergodicity

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Abstract. In this paper, we introduce affine-type pairs on locally compact hypergroups and study their ergodicity and weakly ergodicity. Among other results, we give some applicable sufficient conditions for that an affine-type pair to be strongly totally dissipative. Inspiring the main results and concepts of this paper, the ergodicity on hypergroups can be studied extensively.

Keywords: groups, hypergroups, ergodic functions, weakly ergodicity, affine-type pairs, automorphisms, strongly totally dissipative functions, wandering sets.
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1 Introduction and preliminaries

Locally compact hypergroups are an important and applicable generalization of locally compact groups introduced by [4, 9, 16, 18, 19], in which, although there is a convolution product between its Dirac measures and turns its measure space into a convolution algebra, there is not necessarily an action between its elements. Therefore, hypergroups have so many complicated structures and their analysis is of great importance. On the other hand, the ergodic theory on locally compact groups is a rich and active area of research during the last decades. It extends the classical studies of dynamical systems in the context of locally compact groups, with so many applications for number theory, harmonic analysis, and geometry [1, 11–13, 20]. In particular, various studies have been carried out for several ergodic and chaotic versions of functions. See also the monographs [6] and [17] for more details. Among them, in several papers, the ergodic property of topological-algebraic automorphisms on locally compact groups have been investigated [7, 14, 25].

In this paper, we intend to obtain results on automorphisms and affine functions on locally compact hypergroups in this regard. For some recent articles on hypergroups, see [3, 21, 22]. For this, we first give a very effective definition of affine and affine-type functions on hypergroups,

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inspired by the famous results of Wendel, in which algebraic-topological automorphisms on locally compact groups have been characterized; see [26] and also [10, 23, 24, 28–31]. At the end of this article, we completely review a very important class of hypergroups introduced by Dunkl and Ramirez [5]. For the convenience of readers, we first recall the definition of a locally compact hypergroup and some related notations.

A main reference for the concept of hypergroup is the book [2], although so far one can find the basics regarding this structure in the paper [9], where R.I. Jewett has given the name *convo* to the hypergroups.

Next, ϵ_x is a Dirac measure, and $\mathrm{supp}(\lambda)$ is the support of a measure λ.

A locally compact hypergroup (or simply hypergroup) is a locally compact Hausdorff topological space $\mathcal{H}$ equipped with a product $*$ on $\mathcal{M}(\mathcal{H})$, is the space of all complex Radon measures on $\mathcal{H}$, which is called a *convolution* and makes it a Banach algebra, and also with an involutive homeomorphism $x \mapsto \check{x}$ from $\mathcal{H}$ onto $\mathcal{H}$ such that for every $a, b \in \mathcal{H}$ the following condition hold:

1. $\epsilon_a * \epsilon_b$ is a probability measure and $\mathrm{supp}(\epsilon_a * \epsilon_b)$ is compact.
2. $(x, y) \mapsto \epsilon_x * \epsilon_y$ from $\mathcal{H} \times \mathcal{H}$ into $\mathcal{M}^+(\mathcal{H})$ is continuous.
3. $(x, y) \mapsto \mathrm{supp}(\epsilon_x * \epsilon_y)$ from $\mathcal{H} \times \mathcal{H}$ to the family of non-empty compact subsets of $\mathcal{H}$ is continuous.
4. $(\epsilon_a * \epsilon_b)\check{} = \epsilon_{\check{b}} * \epsilon_{\check{a}}$.
5. There exists an (identity) element $e \in \mathcal{H}$ that for each $x \in \mathcal{H}$, $\epsilon_x * \epsilon_e = \epsilon_e * \epsilon_x = \epsilon_x$. Also, $e \in \mathrm{supp}(\epsilon_a * \epsilon_b)$ if and only if $b = \check{a}$.

Throughout this paper, $\mathcal{H}$ is assumed to be a hypergroup with a left Haar measure μ. This means that μ is a non-zero non-negative regular measure such that for every $x \in \mathcal{H}$, $\epsilon_x * \mu = \mu$ [9, 19]. Next, the collection of all Borel subsets of $\mathcal{H}$ is denoted by $\mathcal{B}_{\mathcal{H}}$. The Lebesgue spaces on $\mathcal{H}$ is considered regarding the left Haar measure μ.

For every measurable function $f : \mathcal{H} \to \mathbb{C}$ and $a, b \in \mathcal{H}$, we define

$$f(a * b) := \int_{\mathcal{H}} f \, d(\epsilon_a * \epsilon_b),$$

while this integral exists.

At the end of this section, we recall the definition of ergodicity of functions. The concept of weakly ergodicity was introduced in [15].

For every $A, B \in \mathcal{B}_{\mathcal{H}}$ we denote $A \overset{\mu}{=} B$ if $\mu((A \setminus B) \cup (B \setminus A)) = 0$.

Definition 1. *Assume that $\mathcal{H}$ is a locally compact hypergroup with a left Haar measure μ. A measurable function $\varphi : \mathcal{H} \to \mathcal{H}$ is called*

1. ergodic *whenever for every $A \in \mathcal{B}_{\mathcal{H}}$ with $\varphi(A) \overset{\mu}{=} A$ we have $\mu(A) = 0$ or $\mu(\mathcal{H} \setminus A) = 0$,*
2. weakly ergodic *whenever for each $A \in \mathcal{B}_{\mathcal{H}}$ with $\varphi(A) \overset{\mu}{=} A$ we have $\mu(A) = 0$ or ∞.*

2 Affine-type pairs

In this section, we introduce the new concepts affine-type and affine pairs on hypergroups. These notions would be more general than automorphisms. Recall that if G is a group, a bijection $f : G \to G$ is called affine whenever there exists an element $b \in G$ and an automorphism $h : G \to G$ such that $f = b\,h$. Clearly, this concept is based on the action of group G, while in general, there is no action between elements of a hypergroup. Then, inspiring the interesting and famous theorem by Wendel [26], we present similar and other related concepts via isometric isomorphisms on L^1.

Definition 2. *Let $\alpha : \mathcal{H} \to \mathcal{H}$ be a homeomorphism, and $b \in \mathcal{H}$. Then, the pair (α, b) is called* affine-type *whenever there exists an isometric linear mapping $T : L^1(\mathcal{H}) \to L^1(\mathcal{H})$ and some $c > 0$ such that*

$$T(f)(b * \alpha(x)) = cf(x) \tag{1}$$

for all $x \in \mathcal{H}$ and $f \in L^1(\mathcal{H})$. If in addition, T is a convolution isomorphism, then the pair (α, b) is said to be affine.

Note that the equality (1) means

$$\int_{\mathcal{H}} T(f)(t)\, d(\epsilon_b * \epsilon_{\alpha(x)})(t) = cf(x).$$

Remark 1. *Assume that (α, b) is an affine-type pair corresponding an isometric convolution isomorphism $T : L^1(\mathcal{H}) \to L^1(\mathcal{H})$ and some $c > 0$. Then, for every $A \in \mathcal{B}_{\mathcal{H}}$ with $\mu(A) < \infty$ we have*

$$\begin{aligned} c\mu(A) &= \|c\chi_A\|_1 \\ &= \|cT^{-1}\chi_A\|_1 \\ &= \|\chi_A(b * \alpha(\cdot))\|_1 \\ &= \int_{\mathcal{H}} |\chi_A(b * \alpha(x))|\; d\mu(x) \\ &= \int_{\mathcal{H}} \chi_A(b * x)\, d\mu(\alpha^{-1}(x)) \\ &= \int_{\mathcal{H}} \chi_A(x)\, d(\epsilon_b * \mu(\alpha^{-1}))(x) \\ &= (\epsilon_b * \mu(\alpha^{-1}))(A), \end{aligned}$$

so,

$$\epsilon_b * \mu(\alpha^{-1}) = c\mu. \tag{2}$$

Definition 3. *In the special case, a homeomorphism $\alpha : \mathcal{H} \to \mathcal{H}$ is called an* automorphism-type *if there is some $c > 0$ such that the mapping $T : L^1(\mathcal{H}) \to L^1(\mathcal{H})$ defined by*

$$T(f)(x) = cf(\alpha^{-1}(x)) \tag{3}$$

for all $x \in \mathcal{H}$ and $f \in L^1(\mathcal{H})$, is an isometric linear function. If T is a convolution isomorphism too, then it is called an automorphism.

Example 1. Let $\mathcal{H} := \{e, a, b\}$ equipped with the discrete topology, e as the identity element, the identity mapping as the involution, and the convolution as below:

$$\epsilon_a * \epsilon_a := \frac{1}{2}\epsilon_e + \frac{1}{2}\epsilon_b, \qquad \epsilon_b * \epsilon_b := \frac{1}{2}\epsilon_e + \frac{1}{2}\epsilon_a,$$

and

$$\epsilon_a * \epsilon_b = \epsilon_b * \epsilon_a := \frac{1}{2}\epsilon_a + \frac{1}{2}\epsilon_b.$$

This hypergroup is called *Golden hypergroup*; see [27] for more details. Note that the measure $\mu := \epsilon_e + 2\epsilon_a + 2\epsilon_b$ is a Haar measure on $\mathcal{H}$. Define the function $\alpha : \mathcal{H} \to \mathcal{H}$ by

$$\alpha(e) := e, \quad \alpha(a) := b, \quad \alpha(b) := a.$$

We will show that α is an automorphism-type function with $c = 1$. For this note that α trivially is a homeomorphism, and $\alpha^{-1} = \alpha$. Define $T : L^1(\mathcal{H}) \to L^1(\mathcal{H})$ by

$$T(f)(x) = f(\alpha(x)), \qquad (x = e, a, b). \tag{4}$$

Easily, T is linear and bijective. For every $f \in L^1(\mathcal{H})$,

$$\|f\|_1 = |f(e)| + 2|f(a)| + 2|f(b)| = \|f(\alpha)\|_1.$$

In addition, for any $f, g \in L^1(\mathcal{H})$,

$$T(f * g)(e) = \int_{\mathcal{H}} f(t)g(t)\, d\mu(t) = f(e)g(e) + 2f(a)g(a) + 2f(b)g(b),$$

and

$$(T(f) * T(g))(e) = \int_{\mathcal{H}} T(f)(t)T(g)(t)\, d\mu(t) = f(e)g(e) + 2f(b)g(b) + 2f(a)g(a).$$

Also, by some calculation based on the structure of $\mathcal{H}$ we have

$$(T(f) * T(g))(a) = f(b)g(e) + (f(e) + f(a))g(b) + (f(b) + f(a))g(a) = T(f * g)(a).$$

Similarly, $(T(f) * T(g))(b) = T(f * g)(b)$. Therefore, T is an isomorphic isomorphism, and hence α is an automorphism on $\mathcal{H}$.

3 Ergodicity of affine-type pairs

In this section, we give some facts regarding ergodicity of affine-type pairs. The obtained results would be novel too for the special case automorphisms on hypergroups. First, we recall the following concept from [8].

Definition 4. *Assume that $A \in \mathcal{B}_{\mathcal{H}}$ and $x \in A$. Then, x is called (φ, E)-recurrent whenever there exists some $k \in \mathbb{N}$ with $x \in \varphi^{-n}(A)$.*

Definition 5. *A set $A \in \mathcal{B}_{\mathcal{H}}$ is called* wandering *for φ if $\mu(A) > 0$ and*

$$\varphi^j(A) \cap \varphi^k(A) = \varnothing$$

for all distinct $j, k \in \mathbb{N}_0$, where $\varphi^0 := \mathrm{Id}_{\mathcal{H}}$.

The collection of all wandering sets for φ is denoted by $W(\varphi)$. φ is called dissipative *if $W(\varphi) \neq \varnothing$.*

Definition 6. *φ is called* totally dissipative *whenever for a.e. $x \in \mathcal{H}$ there is some $A \in W(\varphi)$ such that $\bigcup_{j=-\infty}^{\infty} \varphi^j(A)$ is an open neighborhood of x in $\mathcal{H}$.*

Definition 7. *φ is called* strongly totally dissipative *whenever there is some $A \in W(\varphi)$ such that $V := \bigcup_{j=-\infty}^{\infty} \varphi^j(A)$ is an open subset of $\mathcal{H}$ with $\mu(\mathcal{H} \setminus V) = 0$.*

Theorem 1. *Assume that (b, α) is an affine-type pair for a hypergroup $\mathcal{H}$ which for some $B \in \mathcal{B}_{\mathcal{H}}$,*

$$\mu(B) \neq \mu(\alpha(B)). \tag{5}$$

Then, φ is weakly ergodic.

Proof. First, note that by Definition 2, there exist some isometric isomorphism $T : L^1(\mathcal{H}) \to L^1(\mathcal{H})$ and $c > 0$ such that

$$T(f)(b * \alpha(x)) = cf(x)$$

for all $x \in \mathcal{H}$ and $f \in L^1(\mathcal{H})$, and so

$$T(f)(b * x) = cf(\alpha^{-1}(x)).$$

Fix a compact subset A of $\mathcal{H}$. Then, setting $f := \chi_A$ in the above relation we have

$$T(\chi_A)(b * x) = c\chi_A(\alpha^{-1}(x)) = c\,\chi_{\alpha(A)}(x)$$

for all $x \in \mathcal{H}$. Hence, thanks to left invariance of the measure μ and also since T is an isometry, we have

$$\begin{aligned} c\mu(\alpha(A)) &= \|c\chi_{\alpha(A)}\|_1 \\ &= \|T(\chi_A)(b * \cdot)\|_1 \\ &= \|T(\chi_A)\|_1 \\ &= \|\chi_A\|_1 \\ &= \mu(A). \end{aligned}$$

Therefore, by the regularity of measures, we have $c\mu(\alpha(E)) = \mu(E)$ for all Borel subset E of $\mathcal{H}$. So, by (5),

$$c\mu(B) \neq c\mu(\alpha(B)) = \mu(B),$$

hence $c \neq 1$. But for every $D \in \mathcal{B}_{\mathcal{H}}$ with $\alpha(D) \overset{\mu}{=} D$ we have

$$c\mu(D) = c\mu(\alpha(D)) = \mu(D),$$

since $c \neq 1$ we conclude that $\mu(D) = 0$ or ∞. This implies that α is weakly ergodic. $\square$

Example 2. Assume that $\mathcal{H} := \{e, a, b\}$ and the function $\alpha : \mathcal{H} \to \mathcal{H}$ are as in Example 1. Note that since $\mu(\{a\}) = \mu(\{b\}) = 2$, by definition of α one can see that α is μ-invariant. Also, since $\alpha(\{a, b\}) = \{a, b\}$ and $\mu(\{a, b\}) = 4$, the function α is not ergodic and weakly ergodic on $\mathcal{H}$.

Theorem 2. *Assume that (b, α) is an affine-type pair for a hypergroup $\mathcal{H}$ which for some $B \in \mathcal{B}_{\mathcal{H}}$,*

$$\mu(B) \neq \mu(\alpha(B)). \tag{6}$$

Then, α is totally dissipative.

Proof. As we saw in the proof of Theorem 1, there is some $1 \neq c > 0$ such that

$$c\,\mu(\alpha(A)) = \mu(A) \tag{7}$$

for all $A \in \mathcal{B}_{\mathcal{H}}$, and hence $\mu(\alpha^{-1}(A)) = c\mu(A)$. Hence, in the sequel of this proof without loss of the generality, one can let $c > 1$. Easily, by (7) for every $n \in \mathbb{N}$ we have $c^n\,\mu(\alpha^n(A)) = \mu(A)$. Consider an arbitrary open set $U \subseteq \mathcal{H}$ with $0 < \mu(U) < \infty$. We obtain

$$0 < \mu(U) \leq \mu\left(\bigcup_{j=0}^{\infty} \alpha^j(U)\right) \leq \sum_{j=0}^{\infty} \mu\left(\alpha^j(U)\right) = \mu(U) \sum_{j=0}^{\infty} c^{-j} < \infty.$$

We have

$$\alpha\left(\bigcup_{j=0}^{\infty} \alpha^j(U)\right) = \bigcup_{j=1}^{\infty} \alpha^j(U) \subseteq \bigcup_{j=0}^{\infty} \alpha^j(U),$$

so,

$$\begin{aligned}
\mu\left(U \setminus \bigcup_{j=1}^{\infty} \alpha^j(U)\right) &= \mu\left(\bigcup_{j=0}^{\infty} \alpha^j(U) \setminus \bigcup_{j=1}^{\infty} \alpha^j(U)\right) \\
&= \mu\left(\bigcup_{j=0}^{\infty} \alpha^j(U)\right) - \mu\left(\bigcup_{j=1}^{\infty} \alpha^j(U)\right) \\
&= \mu\left(\bigcup_{j=0}^{\infty} \alpha^j(U)\right) - \mu\left(\alpha\left(\bigcup_{j=0}^{\infty} \alpha^j(U)\right)\right) \\
&= \mu\left(\bigcup_{j=0}^{\infty} \alpha^j(U)\right) - c^{-1}\,\mu\left(\bigcup_{j=0}^{\infty} \alpha^j(U)\right) \\
&= (1 - c^{-1})\,\mu\left(\bigcup_{j=0}^{\infty} \alpha^j(U)\right) > 0.
\end{aligned}$$

For every distinct m, n in $\mathbb{N} \cup \{0\}$ we have

$$\alpha^m\left(U \setminus \bigcup_{j=1}^{\infty} \alpha^j(U)\right) \bigcap \alpha^n\left(U \setminus \bigcup_{j=1}^{\infty} \alpha^j(U)\right) = \varnothing.$$

Hence, the set $U \setminus \bigcup_{j=1}^{\infty} \alpha^j(U)$ is wandering. Now, just note that

$$\bigcup_{m=-\infty}^{\infty} \alpha^m \left(U \setminus \bigcup_{j=1}^{\infty} \alpha^j(U) \right)$$

is open. This completes the proof. □

Example 3. The automorphism-type function α in Example 1 is non-dissipative.

Recall that if $(\Omega, \mathcal{A}, \lambda)$ is a measure space, a measurable set $A \in \mathcal{A}$ is called an *atom* for λ whenever $\lambda(A) > 0$, and for each $B \in \mathcal{A}$ with $B \subset A$ we have $\lambda(B) = 0$ or $\lambda(B) = \lambda(A)$. The measure λ is called *non-atomic* if there is no atom for it.

Theorem 3. *Let the left Haar measure μ of a hypergroup $\mathcal{H}$ is non-atomic. Assume that (b, α) is an affine-type pair for $\mathcal{H}$ which for some $B \in \mathcal{B}_{\mathcal{H}}$,*

$$\mu(B) \neq \mu(\alpha(B)). \tag{8}$$

Then, α is non-ergodic.

Proof. As it was mentioned in the above results, by the assumptions, there is some constant $c \neq 1$ such that $c\,\mu(\alpha(F)) = \mu(F)$ for all Borel sets $F \subseteq \mathcal{H}$. Without loss of the generality, let $c > 1$. Pick some non-empty open set $V \subseteq \mathcal{H}$ which its closure is compact. Then, $0 < \mu(V) < \infty$, and setting $D := \bigcup_{n=0}^{\infty} \alpha^n(V)$ we have $0 < \mu(D) < \infty$. As in the proof of Theorem 2, the terms of sequence $\{\alpha^n(V \setminus D)\}_{n=-\infty}^{\infty}$ are disjoint pairwise. Since $\mu(V \setminus D) > 0$ and μ is non-atomic, there are $A, B \in \mathcal{B}_{\mathcal{H}}$ such that $A \cap B = \varnothing$, $A \cup B = V \setminus D$ and $\mu(A), \mu(B) > 0$. Now, set

$$E := \bigcup_{n=-\infty}^{\infty} \alpha^n(A).$$

Easily one can obtain that $\alpha^j(A) \cap \alpha^k(B) = \varnothing$ for all $j, k \in \mathbb{Z}$. Also, $\alpha(E) = E$, $\mu(E) \geq \mu(A) > 0$, and

$$\infty = \mu \left(\bigcup_{n=-\infty}^{\infty} \alpha^n(B) \right) \leq \mu(\mathcal{H} \setminus E).$$

This implies that α is non-ergodic. □

Definition 8. *Let (b, α) be an affine-type pair for a hypergroup $\mathcal{H}$ and $c \in \mathbb{C}$. Then, $\xi : \mathcal{H} \to \mathbb{C}$ is called* c-invariant under α *whenever for each $x \in \mathcal{H}$,*

$$\xi(\alpha(x)) = c\,\xi(x).$$

For every $h : \mathcal{H} \to \mathbb{C}$ we denote

$$\sigma_h := \{x \in \mathcal{H} :\ h(x) = 0\}.$$

Theorem 4. *Let (b, α) be an affine-type pair for a hypergroup $\mathcal{H}$. Assume that V is an open subset of $\mathcal{H}$ with $\mu(\mathcal{H} \setminus V) = 0$ and $\alpha(V) = V$. Assume that there is a measurable real-valued function ξ on $\mathcal{H}$ such that the set*

$$\{\xi(\alpha(x)) - \xi(x) : x \in V\}$$

is a singleton $\{k\}$ with $k \neq 0$. Then, α is strongly totally dissipative.

Proof. Under the hypothesis, denote

$$A := \{t \in V : 0 < \xi(t) \leq k\}.$$

We intend to show that $A \in W(\alpha)$. First, note that A is measurable because ξ is a measurable function. By the hypothesis, for every $x \in V$ and $n \in \mathbb{Z}$ we have

$$\xi(\alpha^n(x)) = \xi(x) + nk.$$

Then, thanks to the definition of A, for every $n \in \mathbb{Z}$ we have $\xi(\alpha^n(A)) \subseteq (nk, (n+1)k]$. This implies that for every distinct $i, j \in \mathbb{N} \cup \{0\}$ we have $\alpha^i(A) \cap \alpha^j(A) = \varnothing$. For every $x \in V$ there is some $n \in \mathbb{Z}$ such that $\xi(x) \in (nk, (n+1)k]$, so $x \in \alpha^{-n}(A)$. Hence,

$$V = \bigcup_{n=-\infty}^{\infty} \alpha^n(A).$$

Finally, since (b, α) is affine-type, there is some $c > 0$ such that $\mu(A) = c^n \mu(\alpha^n(A))$ for all $n \in \mathbb{Z}$. In contrast, if $\mu(A) = 0$, then for every $n \in \mathbb{Z}$, $\mu(\alpha^n(A)) = 0$, so $\mu(V) = 0$, a contradiction because the support of a left Haar measure equals the whole $\mathcal{H}$. Hence, $\mu(A) > 0$, and therefroe $A \in W(\alpha)$. Now, the openness of V and the hypothesis $\mu(\mathcal{H} \setminus V) = 0$ imply that α is a strongly totally dissipative. □

Corollary 1. *Let (b, α) be an affine-type pair for a hypergroup $\mathcal{H}$, $c \in \mathbb{T}$, ν be a continuous c-invariant function under α such that the set σ_ν is null, and for some continuous function $\gamma : \mathcal{H} \to \mathbb{C}$,*

$$\gamma(\alpha(x)) = c\gamma(x) + \nu(x).$$

Then, α is totally dissipative.

Proof. It would be enough to apply Theorem 4 by setting $V := \mathcal{H} \setminus \sigma_\nu$, $k := 1$, and $\xi(x) :=$ the real part of $\frac{c\gamma(x)}{\nu(x)}$ for all $x \in V$ and $\xi(x) = 0$ for all $x \in \mathcal{H} \setminus V$.

□

Theorem 5. *Let (b, α) be an affine-type pair for a hypergroup $\mathcal{H}$, $\omega \in \mathbb{C} \setminus \mathbb{T}$, and ξ be a continuous c-invariant function under α such that the set σ_ξ is null. Then, α is strongly totally dissipative.*

Proof. Without loss of the generality, let $s := |\omega| > 1$. In this case, put

$$A := \{x \in \mathcal{H} : 1 < |\xi(x)| \leq s\}.$$

Then, for every $j \in \mathbb{Z}$, $\alpha^j(A) \subseteq (s^j, s^{j+1}]$, and so $A \in W(\alpha)$. Also, $\bigcup_{j=-\infty}^{\infty} \alpha^j(A) \overset{\mu}{=} \mathcal{H}$ because

$$\mathcal{H} \setminus \bigcup_{j=-\infty}^{\infty} \alpha^j(A) = \sigma_\xi.$$

This completes the proof. Note that the fact $\mu(A) > 0$ holds because there is some $c > 0$ such that $c\,\mu(\alpha(D)) = \mu(D)$ for all $D \in \mathcal{B}_{\mathcal{H}}$. □

4 Example

C.F. Dunkl and C.E. Ramirez in [5] introduced an important class of strong compact countable hypergroups H_a, where $0 < a \leq \frac{1}{2}$. In this section, we intend to verify regarding affine-type functions on the dual of this hypergroup. First, we mention that this dual is as the set $\mathbb{N}_0 = \{0, 1, 2, \ldots\}$ equipped with the identity mapping as involution, 0 as the identity element, and the following convolution: for each distinct $m, n \in \mathbb{N}$ with $m > n$, we have $\epsilon_m * \epsilon_n := \epsilon_m$, and

$$\epsilon_n * \epsilon_n := \frac{a^n}{1-a}\epsilon_0 + \sum_{k=1}^{n-1} a^{n-k}\epsilon_k + \frac{1-2a}{1-a}\epsilon_n.$$

The following measure would be a Haar measure for this Hermitian discrete hypergroup:

$$\mu(\{k\}) := \begin{cases} 1, & \text{if } k = 0, \\ \frac{1-a}{a^k}, & \text{if } k \geq 1. \end{cases}$$

Note that if $a \neq \frac{1}{2}$, then $\frac{1-a}{a^k} \neq 1$ for all $k \geq 1$. Assume that $\sigma : \mathbb{N}_0 \to \mathbb{N}_0$ is a bijection, $c > 0$, and $j \in \mathbb{N}_0$. Let (j, σ) be a c-affine-type pair for $\mathbb{N}_0$. Then, let $\alpha : \mathcal{H} \to \mathcal{H}$ be a homeomorphism, and $b \in \mathcal{H}$. So, for some isometric convolution isomorphism $T : L^1(\mathbb{N}_0) \to L^1(\mathbb{N}_0)$,

$$T(f)(j * \sigma(x)) = cf(x), \qquad (x \in \mathbb{N}_0, f \in L^1(\mathbb{N}_0)). \tag{9}$$

Thanks to the proof of Theorem 1, for every $A \subseteq \mathbb{N}_0$,

$$c\mu(\sigma(A)) = \mu(A).$$

In particular, we have $\mu(\{k\}) = c\,\mu(\{\sigma(k)\}$ for all $k \in \mathbb{N}_0$. Hence,

$$\mu(\{\sigma(0)\}) = c^{-1}, \qquad \mu(\{\sigma(k)\}) = c^{-1} \cdot \frac{1-a}{a^k}$$

for all $k = 1, 2, \ldots$. By definition of μ there are four cases:

1. Case 1: $c = 1$ and $a \neq \frac{1}{2}$. In this case, we have $\sigma(0) = 0$ and so since σ is 1-1, $\sigma(k) \geq 1$ for all $k \geq 1$. Hence, $\frac{1-a}{a^{\sigma(k)}} = \frac{1-a}{a^k}$ for all $k \geq 1$, which implies that $\sigma(k) = k$. This means that σ in the identity mapping.

2. Case 2: $c = 1$ and $a = \frac{1}{2}$. In this case, we have $\sigma(0) = 0$ or 1. If $\sigma(0) = 0$, then similar to the case 1, we conclude that σ is the identity mapping. Let $\sigma(0) = 1$. Then, by the above relations we have $\sigma(1) = 0$, and $\sigma(k) = k$ for all $k \geq 2$.

3. Case 3: $c \neq 1$ and $a \neq \frac{1}{2}$. In this case we have $c = \frac{a^l}{1-a}$ for some fixed $l \in \mathbb{N}$. Thus, $\sigma(0) = l$. Since σ is onto, there is some $t \in \mathbb{N}$ such that $\sigma(t) = 0$. This implies that

$$1 = c^{-1} \cdot \frac{1-a}{a^t} = \frac{1-a}{a^l} \cdot \frac{1-a}{a^t} = \frac{(1-a)^2}{a^{l+t}},$$

so, $a^{l+t} = (1-a)^2$. This holds just if $l = t = 1$, and in this case we have $a = \frac{1}{2}$, a contradiction.

4. Case 4: $c \neq 1$ and $a = \frac{1}{2}$. In this case we have $c = 2^{1-l}$ for some fixed $l \in \{2, 3, \ldots\}$. Thus, $\sigma(0) = l$. Since σ is onto, there is some $t \in \mathbb{N}$ such that $\sigma(t) = 0$. Similar as above we conclude that $l = t = 1$, a contradiction.

In the sequel, we verify regarding the conditions on j in the above possible Cases 1, 2. In both cases we have $c = 1$. So, one can write the relation (9) as

$$\sum_{n=0}^{\infty} T(f)_n \, (\epsilon_j * \epsilon_k)(\{n\}) = f_{\sigma^{-1}(k)}, \qquad (k \in \mathbb{N}_0, \ f := (f_n)_n \in L^1(\mathbb{N}_0)). \tag{10}$$

If $j \in \mathbb{N}_0 \setminus \{0, 1\}$, then for every $k = 0, , 1, \ldots, j-1$,

$$\sum_{n=0}^{\infty} T(f)_n \, (\epsilon_j * \epsilon_k)(\{n\}) = \sum_{n=0}^{\infty} T(f)_n \, \epsilon_j(\{n\}) = T(f)_j,$$

and so

$$T(f)_j = f_{\sigma^{-1}(k)}, \qquad (k = 0, 1, 2, \ldots, j-1).$$

By this relation we obtain that $f_{\sigma^{-1}(0)} = f_{\sigma^{-1}(1)} = \cdots = f_{\sigma^{-1}(j-1)}$, for all $f \in L^1(\mathbb{N}_0)$, which trivially is impossible. Therefore, (j, σ) is not a 1-affine-type function for all $j \in \mathbb{N}_0 \setminus \{0, 1\}$. In the sequel, we consider all the cases for $c = 1$:

1. If $j = 0$ and $a \neq \frac{1}{2}$, then σ is the identity mapping, and this is the trivial case.

2. If $j = 0$, $a = \frac{1}{2}$ and $\sigma(0) = 0$, then again σ is the identity mapping, and this is the trivial case.

3. If $j = 0$, $a = \frac{1}{2}$ and $\sigma(0) \neq 1$, then $\sigma(0) = 1$, $\sigma(1) = 0$, and $\sigma(k) = k$ for all $k \geq 2$. Then, by (10),

$$T(f)_k = f_{\sigma^{-1}(k)}, \qquad (k \in \mathbb{N}_0, \ f := (f_n)_n \in L^1(\mathbb{N}_0)).$$

So, for every $f \in L^1(\mathbb{N}_0)$ we have $T(f)_0 = f_1$, $T(f)_1 = f_0$, and $T(f)_k = f_k$ for all $k = 2, 3, \ldots$. We have $(T(f) * T(g))_k = T(f * g)_k$ for all $k \in \mathbb{N}_0$ and $f, g \in L^1(\mathbb{N}_0)$. But,

$$(T(f) * T(g))_1 = (T(g))_0 (T(f))_1 + (T(g))_1 (T(f))_0 + \sum_{i=2}^{\infty} 2^{i-1} (T(f))_i (T(g))_i$$

$$= g_1 f_0 + g_0 f_1 + \sum_{i=2}^{\infty} 2^{i-1} f_i g_i,$$

and

$$(T(f*g))_1 = (f*g)_0 = f_0 g_0 + f_1 g_1 + \sum_{i=2}^{\infty} 2^{i-1} f_i g_i$$

for all $f, g \in L^1(\mathbb{N}_0)$, which is impossible.

4. If $j = 1$ and $a \neq \frac{1}{2}$, then as we mentioned above, σ is the identity mapping. We have

$$\sum_{n=0}^{\infty} T(f)_n \,(\epsilon_1 * \epsilon_k)(\{n\}) = f_k, \qquad (k \in \mathbb{N}_0,\ f := (f_n)_n \in L^1(\mathbb{N}_0)).$$

This relation implies that $(T(f))_1 = f_0$ and $(T(f))_n = f_n$ for all $n = 2, 3, \ldots$. Also,

$$(T(f))_0 = \frac{1-a}{a}\, f_1.$$

Similar to the item (3) we attend a contradiction, because

$$(T(f)*T(g))_1 = (T(g))_0 (T(f))_1 + (T(g))_1 (T(f))_0 + \sum_{i=2}^{\infty} 2^{i-1} (T(f))_i (T(g))_i$$
$$= \frac{1-a}{a}\, g_1 f_0 + \frac{1-a}{a}\, g_0 f_1 + \sum_{i=2}^{\infty} \frac{1-a}{a^i}\, f_i g_i,$$

and

$$(T(f*g))_1 = (f*g)_0 = f_0 g_0 + \sum_{i=1}^{\infty} \frac{1-a}{a^i}\, f_i g_i$$

for all $f, g \in L^1(\mathbb{N}_0)$, which is impossible.

5. If $j = 1$, $a = \frac{1}{2}$ and $\sigma(0) = 0$, then similarly, we obtain a contradiction.

6. If $j = 1$, $a = \frac{1}{2}$ and $\sigma(0) \neq 1$, then similarly, we obtain a contradiction.

Therefore, for the hypergroup $\widehat{H_a}$, just $(0, id_{\mathbb{N}_0})$ is the affine function with $c = 1$. In other words, the only automorphism for this hypergroup is the identity function while it has some non-trivial automorphism-type function in the case $a = \frac{1}{2}$.

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