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2-nil primary ideals of commutative semirings

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Abstract. In this paper, we extend the concept of primary ideals in commutative rings by introducing a new class, namely 2-nil primary ideals, in a commutative semiring S with $1 \neq 0$. We further investigate the structure of ideals by considering related classes such as 2-nil ideals, 2-absorbing ideals, and quasi-primary ideals. A comprehensive framework has been developed to emphasize the significance of 2-nil primary ideals in semiring theory, focusing on their properties and interrelations. We provide arguments and illustrative examples that demonstrate how 2-nil primary ideals are connected to several well-established classes of ideals, including prime ideals, primary ideals, n-ideals, 2-absorbing ideals, and quasi-primary ideals. Finally, Theorem 6 together with Corollaries 1 and 3 illustrates the applications of these ideals and highlights their significance within the theory of semirings.

Keywords: n-ideal, 2-nil ideal, primary ideal, 2-absorbing primary ideal.
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1 Introduction

The algebraic structure of a semiring plays a significant role not only in diverse areas of pure mathematics but also across many domains of applied science. Since its introduction, semiring theory has been extensively studied and developed by many researchers. The term "semiring" was introduced in 1934 by Vandiver [13], inspired by early examples of semirings found in Dedekind's 1894 work on the algebra of ideals in commutative rings [4]. Although numerous mathematicians contributed significantly to the growth of semiring theory during the 1940s, 1950s, and early 1960s, their efforts did not ultimately elevate semiring theory to a central or widely recognized domain within mathematics. It was only in the late 1960s, when real

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applications of semirings were discovered, that the subject began to be recognized as an essential area of research.

The development of concrete applications brought renewed prominence to semirings, which have since evolved into a central topic of study in a range of mathematical and applied settings. A classical example of a semiring that is not a ring is the set of all non-negative integers under standard addition and multiplication. Another example arises from matrices: for a fixed integer n, the set of all $n \times n$ matrices $(a_{ij})_{n\times n}$ with usual matrix addition and multiplication forms a semiring. For a detailed treatment of the subject, one may refer to the works of Golan [5] and Weinert [6].

A semiring S is a set of elements, together with two binary operations, called addition "$+$" and multiplication "$\cdot$", where $S \neq \phi$ satisfies the properties.

(i) $(S, +)$ is a commutative monoid. (ii) $(S, \cdot)$ is a monoid with $1 \neq 0$. (iii) Multiplication distributes over addition from either side, that is, for all r, s, and t in S, $r\cdot(s+t) = (r\cdot s)+(r\cdot t)$ and $(r + s)\cdot t = (r\cdot t) + (s\cdot t)$. (iv) The identity element 0 is multiplicatively absorbing, such that $0\cdot s = 0 = s\cdot 0 = 0 \ \forall \ s \in S$.

In this paper, all semirings are assumed to be commutative. A semiring S is said to be commutative if $rs = sr$ for all $r, s \in S$. A commutative division semiring is called a semifield. A non-empty subset I of S is called a left (respectively right) ideal of S if I is a subsemigroup of $(S, +)$ and $rs \in I$ (respectively $sr \in I$) for all $r \in S$ and $s \in I$. If I is either a left or a right ideal of S, then I is referred to as an ideal of S. An ideal I of a semiring S is called a proper ideal if $I \neq S$. An ideal I of a semiring S is said to be subtractive if for all $r, s \in S$, $r + s \in I$ and $r \in S$ implies that $s \in I$. Every commutative ring is subtractive. Recall from Golan [5] that prime and primary ideals play a fundamental role in the study of commutative semirings. A proper ideal I of a semiring S is called prime if for $r, s \in S, rs \in I$ implies that either $r \in I$ or $s \in I$. Let S be a commutative semiring. For an ideal I of S, the radical of I is defined as $\sqrt{I} = \{a \in S : a^m \in I \text{ for some } m \in \mathbb{N}\}$. Also, $\sqrt{I}$ is equal to the intersection of all prime ideals of S containing I. In particular, $\sqrt{I}$ is an ideal of S. An ideal I of a semiring S is called a primary ideal if for any $r, s \in S$, $rs \in I$ implies that either $r \in I$ or $s^n \in I$ for some $n \in \mathbb{N}$ or I is said to be primary if for any $r, s \in S$, $rs \in I$ implies that either $r \in I$ or $s \in \sqrt{I}$. It is clear that if I is a primary ideal of S, then $\sqrt{I}$ is the smallest prime ideal containing I. In this case, if we set $P = \sqrt{I}$ then we say that I is P-primary [8]. For more on primary ideals of semirings, one can refer to [8]. Further, the set of nilpotent elements of S is denoted by $\sqrt{0}$ and nil radical of $\sqrt{0}$ is defined as $\sqrt{0} = \{a \in S : a^m = 0 \text{ for some } m \in \mathbb{N}\}$. Recall from Sarohe [10], that a proper ideal I of a semiring S is said to be a quasi-primary ideal of S if for any $r, s \in S$, $rs \in I$ then either $r \in \sqrt{I}$ or $s \in \sqrt{I}$. A prime ideal P of S is said to be a minimal prime ideal over an ideal I of S if it is minimal among all prime ideals containing I. The set of all minimal prime ideals over an ideal I is denoted by $\text{Min}(S)$. Let I be an ideal of a semiring S, then $(I : x) = \{r \in S : rx \in I\}$.

These classical notions have been extended to more generalized structures, such as 2-absorbing ideals and 2-absorbing primary ideals. The concept of a 2-absorbing ideal and its extension to 2-absorbing primary ideals were introduced in [1,2]. A proper ideal I of a commutative semiring S is called 2-absorbing ideal if for all $r, s, t \in S$, $rst \in I$ implies that either $rs \in I$, or $st \in I$, or $rt \in I$. In a similar manner, the concept of a 2-absorbing primary ideal in a commutative semiring S is defined as, for all $r, s, t \in S$, $rst \in I$ implies that either $rs \in I$ or $st \in \sqrt{I}$ or

$rt \in \sqrt{I}$. In continuation of extending these concepts, the notion of an n-ideal was introduced by Tekir [12]. An ideal I of S is called n-ideal if for $r, s \in S$, $rs \in I$ then either $r \in \sqrt{0}$ or $s \in I$. This concept has been further expanded to $(2, n)$-ideals by Tamekkante [11], that is, for all $r, s, t \in S$, if $rst \in I$ then either $rs \in I$ or $rt \in \sqrt{0}$ or $st \in \sqrt{0}$.

Çelikel [14] introduced an alternative generalization of $(2, n)$-ideals, termed *2-nil ideals*, and investigated their properties within commutative rings. In particular, 2-nil ideals were examined in relation to other classical generalizations of ideals, such as *2-absorbing ideals* and *n-ideals*. Furthermore, Qaralleh [9] introduced the concept of *2-nil primary ideals* in commutative rings, to bridge the conceptual gaps and establish a broader foundation for the development of ideal theory in commutative rings. In ring theory, substantial work has been devoted to the study of *2-nil ideals*, *2-absorbing ideals*, *2-nil primary ideals*, and related generalizations, which serve as refined tools for analyzing nilpotency, primary decomposition, and the behaviour of zero divisors. These notions have proven essential for characterizing the structural features of rings and for extending classical ideal-theoretic frameworks.

However, extending these concepts to the semiring context is not straightforward. The study of semirings has gained increasing prominence due to their central role in applied mathematics, algebraic computation, and various non-classical algebraic frameworks in which additive inverses are absent. The lack of additive inverses introduces significant technical challenges and often necessitates the development of new methods. This motivates a systematic investigation of *2-nil ideals*, *2-absorbing ideals*, and *2-nil primary ideals* in semirings. It is not only to determine the extent to which ring-theoretic results can be generalized, but also to identify novel phenomena that arise uniquely in semiring structures. By examining these generalized notions, the present work seeks to construct a link between classical ring theory and the developing theory of semiring ideals, offering structural insights and potential applications in computational algebra.

In this paper, we extend the developments of [9] and [14] to the setting of commutative semirings by introducing and studying the notions of *2-nil ideals* and *2-nil primary ideals*. In addition, we investigate further classes of ideals that contribute to a more comprehensive understanding of the structure of commutative semirings and their applications. The properties of *2-nil ideals*, *2-nil primary ideals*, and *2-absorbing quasi-primary ideals* in commutative semirings are examined, supported by illustrative examples, and their applications are established.

2 2-nil ideals in semirings

In this section, we compare various generalizations of ideals, namely 2-absorbing ideals, 2-absorbing primary ideals, n-ideals, $(2, n)$-ideals, and 2-nil ideals, with prime and primary ideals. It is shown that both prime ideals and n-ideals are 2-nil ideals, and every 2-nil ideal is a 2-absorbing primary ideal (cf. Proposition 1). In general, the converse implications do not hold (see Examples 1, 2, 3, 4). Additional implications concerning 2-nil ideals are also established (cf. Theorem 2). Furthermore, for the radical I of S, the notions of 2-absorbing ideals, 2-absorbing primary ideals, n-ideals, $(2, n)$-ideals, and 2-nil ideals coincide, as described in Theorem 6.

Definition 1. *A proper ideal I of a semiring S is called* 2*-nil ideal if for all $r, s, t \in S$ such that $rst \in I$, then either $rs \in \sqrt{0}$ or $st \in I$ or $rt \in I$.*

Proposition 1. *Let S be a semiring. Then the prime ideals and n-ideals are 2-nil ideals. Further, 2-nil ideals are 2-absorbing primary ideals of S. Converse is not true.*

The following examples show that the converse implications do not hold.

Example 1. Let S be a semiring and I an ideal of S. Consider $S = \mathbb{Z}_0$ and $I = p\mathbb{Z}_0$, where p is a prime integer and $\mathbb{Z}_0$ is the set of all non negative integers. Then I is a 2-nil ideal but not an n-ideal, since $1 \cdot p \in I$ but neither $1 \in I$ nor $p \in \sqrt{0}$.

Example 2. Let $S = \mathbb{Z}_0$ and $I = 4\mathbb{Z}_0$. Then I is a 2-nil ideal but not a prime ideal, because $2 \cdot 2 \in I$ but $2 \notin I$.

Example 3. Let $S = \mathbb{Z}_0$ and $I = pq\mathbb{Z}_0$, where p, q are prime integers. Then I is a 2-absorbing and 2-absorbing primary ideal but not 2-nil ideal of S. Since, $p \cdot q \cdot 1 \in I$ but $p \cdot q \notin \sqrt{0}$ or $p \cdot 1 \notin I$ or $q \cdot 1 \notin I$.

Now, the following example shows that a 2-nil ideal is not necessarily a $(2, n)$-ideal.

Example 4. Let $S = \mathbb{Z}_0$ be a semiring. Then the ideal $I = 2\mathbb{Z}_0 \subset \mathbb{Z}_0$ is a 2-nil ideal but not a $(2, n)$-ideal of S as $1 \cdot 1 \cdot 2 \in I$ but neither $1 \cdot 1 \in I$ nor $1 \cdot 2 \in \sqrt{0}$.

Theorem 1. *Let S be a semiring and I a proper 2-nil ideal of S. Then*

1. *$\sqrt{I}$ is a 2-nil ideal of S;*
2. *$\sqrt{I} = I \cup \sqrt{0}$;*
3. *either $\sqrt{I} = I$ or $\sqrt{I} = \sqrt{0}$.*

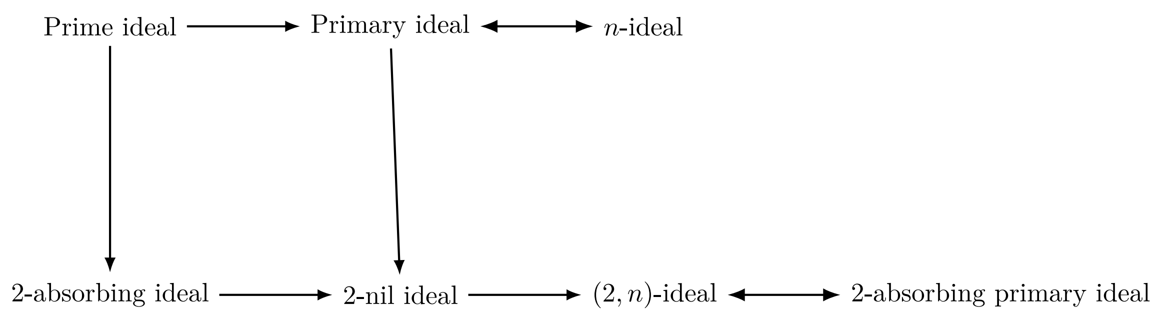
Proof. (1) Let that $rst \in \sqrt{I}$ for some $r, s, t \in S$, and that $rs \notin \sqrt{0}$. Then for some integer $n \geq 1$, we have $(rst)^n = r^n s^n t^n \in I$ and $r^n s^n \notin \sqrt{0}$. This implies that either $s^n t^n \in I$ or $r^n t^n \in I$. Therefore, either $st \in \sqrt{I}$ or $rt \in \sqrt{I}$. Thus, $\sqrt{I}$ is 2-nil ideal of S.
(2) It is evident that $I \cup \sqrt{0} \subseteq \sqrt{I}$. We now prove the reverse inclusion, that is $\sqrt{I} \subseteq I \cup \sqrt{0}$. Let $r \in \sqrt{I}$. If $r \in I$, then clearly $r \in I \cup \sqrt{0}$, and there is nothing to prove. Suppose that $r \notin I$. By definition of the radical, there exists an integer $n \geq 2$ such that $r^n \in I$ but $r^{n-1} \notin I$. Now, $r^n = 1 \cdot r \cdot r^{n-1} \in I$, $1 \cdot r \notin I$ and $1 \cdot r^{n-1} \notin I$. This implies that $r^n = r \cdot r^{n-1} \in \sqrt{0}$, since I is 2-nil ideal. It follows that $r \in \sqrt{0}$. Therefore, $r \in \sqrt{0} \subset I \cup \sqrt{0}$. Hence, $\sqrt{I} \subseteq I \cup \sqrt{0}$.
(3) Assume that $I \subsetneq \sqrt{I}$. Let $s \in \sqrt{I} \setminus I$. If $r \in I$, then $r + s \in \sqrt{I} \setminus I$. It follows from (2) that $r + s, s \in \sqrt{0}$. Hence, $r \in \sqrt{0}$. This shows that $I \subseteq \sqrt{0}$, and by (2), we have $\sqrt{I} = I \cup \sqrt{0} = \sqrt{0}$. □

The converse of the above Theorem 1 does not hold in general.

Example 5. Let p and q be prime integers, and consider the semiring $S = \mathbb{Z}_{p^2 q}$. The zero ideal I satisfies all the conditions of Theorem 1, but I is not a 2-nil ideal, since $p \cdot p \cdot q \in I$, while $p \cdot p \notin \sqrt{0}$ and $p \cdot q \notin I$.

Let $I \subseteq \sqrt{0}$ be an ideal of a commutative semiring S. Then $\sqrt{I} = \sqrt{0}$. By comparing the definitions of 2-absorbing ideals, 2-absorbing primary ideals, n-ideals, $(2, n)$-ideals, and 2-nil ideals with the classical notions of prime and primary ideals in semirings, we obtain a chain of implications among these ideal-theoretic properties. These relationships hold in semirings in much the same way as in commutative rings, thus establishing an analogous hierarchy of generalizations.

Theorem 2. *Let I be an ideal of a semiring S such that $I \subseteq \sqrt{0}$. Then the relationships among the various algebraic properties of the ideal I can be expressed through the logical implications shown in the following diagram.*



Further, it is observed that a primary ideal $I \nsubseteq \sqrt{0}$ need not be a 2-nil ideal, and a 2-nil ideal is not necessarily a 2-absorbing ideal in a Semiring S.

Example 6. Let $S = \mathbb{Z}_0$ be a semiring and $p > 1$ a prime number. Consider $I = p^2\mathbb{Z}_0$ be a proper ideal of S then I is not a 2-nil ideal as $p \cdot p \cdot 1 \in I$ but $p \cdot p \notin \sqrt{0}$ and $p \cdot 1 \notin I$. However, I is a primary ideal of S.

Example 7. Let $S = \mathbb{Z}_{p^3}$ be a semiring and $p > 1$ a prime number. Then the ideal $I = (0) \subset \mathbb{Z}_{p^3}$ is a 2-nil ideal but not a 2-absorbing ideal, since $p \cdot p \cdot p \in I$ but $p \cdot p \notin I$.

Definition 2. *A semiring S is said to be divided if for every prime ideal P of S, we have* $\{p/r \mid p \in P, \ r \in S - P\}$

Theorem 3. *Let $I \subseteq \sqrt{0}$ be an ideal of a divided semiring S. If $\sqrt{0}$ is a prime ideal, then I is a 2-nil ideal if and only if I is a primary ideal.*

Proof. Let us assume that $rs \in I$ and $r \notin \sqrt{0}$ for some $r, s \in S$. Since $\sqrt{0}$ is a prime ideal, so $rs \in \sqrt{0}$ and $r \notin \sqrt{0}$ implies that $s \in \sqrt{0}$. Now, since S is a divided semiring, there exists an element $t \in S$ such that $s = rt$. Therefore, $rs = rrt \in I$. Note that $r^2 \notin \sqrt{0}$ and I is 2-nil ideal, so it follows that $rt = s \in I$. This implies that I is an n-ideal. Hence, by Theorem 2, I is a primary ideal. Converse follows from Theorem 2. □

The following theorem is proved in [2, Theorem 2.3] for rings.

Theorem 4 ([2])**.** *Suppose that I is a 2-absorbing primary ideal of a ring R. Then one of the following statements must hold:*

1. $\sqrt{I} = P$ *is a prime ideal,*

2. *$\sqrt{I} = P_1 \cap P_2$, where P_1 and P_2 are the only distinct prime ideals of S that are minimal over I.*

Now, we will prove the theorem 4 in the case of semirings as follows:

Theorem 5. *Let S be a semiring and I a 2-absorbing primary ideal of S. Then one of the following statements must hold:*

1. *$\sqrt{I} = P$ is a prime ideal,*

2. *$\sqrt{I} = P_1 \cap P_2$, where P_1 and P_2 are the only distinct prime ideals of S that are minimal over I.*

Proof. Suppose that I is a 2-absorbing primary ideal of S. Then by Theorem 14, $\sqrt{I}$ is a 2-absorbing ideal. Also, by Golan [5], $\sqrt{\sqrt{I}} = \sqrt{I}$. Suppose that $\sqrt{I} = \{P_i \mid P_i$ is a prime ideal of S and minimal over I $\}$. We will show that there are at most two distinct minimal prime ideals of S over I. On the contrary, suppose that there are three distinct minimal prime ideals, say P_1, P_2, P_3 of S over I. By minimality of these primes we can choose elements $r \in P_1 \setminus (P_2 \cup P_3)$, $s \in P_2 \setminus (P_1 \cup P_3)$, $t \in P_3 \setminus (P_1 \cup P_2)$. Since $r \in P_1$, $s \in P_2$, $t \in P_3$, we have $rst \in P_1 \cap P_2 \cap P_3 = \sqrt{I} = P$. Since $\sqrt{I}$ is 2-absorbing ideal of S, so $rst \in \sqrt{I}$ implies that either $rs \in \sqrt{I}$ or $rt \in \sqrt{I}$ or $st \in \sqrt{I}$. Now, the following cases arise:

Case 1: If $rs \in \sqrt{I}$, then $rs \in P_3 \subseteq \sqrt{I}$. Since P_3 is prime, so $rs \in P_3$ implies that either $r \in P_3$ or $s \in P_3$, which is a contradiction. So, none of r and s belongs to P_3.

Case 2: If $st \in \sqrt{I}$, then $st \in P_1 \subseteq \sqrt{I}$. Since P_1 is prime, so $rs \in P_1$ implies that either $s \in P_1$ or $t \in P_1$, again a contradiction. So, none of s and t belongs to P_1.

Case 3: If $rt \in \sqrt{I}$, then $rt \in P_2 \subseteq \sqrt{I}$. Since P_2 is prime, so $rt \in P_2$ implies that either $r \in P_2$ or $t \in P_2$, a contradiction. So, none of r and t belongs to P_2.

Therefore, there cannot be three distinct minimal primes over I, so the number of minimal primes over I is either one or two. If there is one prime ideal, say P over I, then $\sqrt{I} = P$. If there are two prime ideals say P_1, P_2, then $\sqrt{I} = P_1 \cap P_2$, and P_1, P_2 are the only primes minimal ideals over I. □

An argument analogous to that of [1, Theorem 2.3] establishes the following result for semirings.

Proposition 2. *Let S be a semiring. If $I \subseteq S$ is a 2-nil ideal of S, then there exist at most two minimal prime ideals over I.*

As an application of Proposition 2 and for 2-nil property of radical ideals, we have the following result.

Theorem 6. *Let S be a commutative semiring. Then the following statements are equivalent:*

1. *The number of minimal prime ideals of S satisfying that $|Min(S)| \leq 2$.*

2. *The nil-radical $\sqrt{0}$ is 2-absorbing ideal of S.*

3. *The nil-radical $\sqrt{0}$ is 2-nil ideal of S.*

4. *$\sqrt{0}$ is 2-absorbing primary ideal of S.*

5. *$\sqrt{0}$ is $(2,n)$-ideal of S.*

Let $S = S_1 \times S_2$, where each S_i for $i = 1, 2$ are commutative semirings. Define operations of addition and multiplication on S by $(r_1, r_2)(s_1, s_2) = (r_1 + s_1, r_2 + s_2)$ and $(r_1, r_2)(s_1, s_2) = (r_1 s_1, r_2 s_2)$ for all $r_1, s_1 \in S_1$ and $r_2, s_2 \in S_2$. Then $S = S_1 \times S_2$ becomes a semiring with additive identity $(0, 0)$ and multiplicative identity $(1, 1)$. Since S_1 and S_2 are commutative semiring so is S.

Example 8. Let S_1 and S_2 be two commutative semirings and I a 2-nil ideal of S_1. Then $I \times S_2$ is not necessarily 2-nil ideal of $S_1 \times S_2$. Let us consider $S_1 = S_2 = \mathbb{Z}_0$ and $I = 6\mathbb{Z}_0$, where $\mathbb{Z}_0$ is the set of all non negative integers. Then I is a 2-nil ideal, but $I \times S_2 = 6\mathbb{Z}_0 \times \mathbb{Z}_0 \subset \mathbb{Z}_0 \times \mathbb{Z}_0$ is not a 2-nil ideal. However, $(2, 1)(3, 1)(1, 1) \in I$ but neither $(2, 1)(3, 1) \in \sqrt{0}_S$ nor $(2, 1)(1, 1) \in 6\mathbb{Z}_0 \times \mathbb{Z}_0$ nor $(3, 1)(1, 1) \in 6\mathbb{Z}_0 \times \mathbb{Z}_0$.

Theorem 7. *Let S_1 and S_2 be two commutative semirings, and I a proper ideal of $S = S_1 \times S_2$. Then the following statements are equivalent:*

1. *$I \times S_2$ forms a 2-nil ideal in a semiring S.*

2. *The ideal I is prime in a semiring S_1.*

3. *$I \times S_2$ is a prime ideal in S.*

Proof. (1) implies (2). Let $r, s \in S_1$ such that $rs \in I$. Then $(r, 1)(s, 1)(1, 1) \in I \times S_2$. Since $(r, 1)(s, 1) \notin \sqrt{0}_S$, so it follows that either $(r, 1)(1, 1) \in I \times S_2$ or $(s, 1)(1, 1) \in I \times S_2$. Hence, we conclude that either $r \in I$ or $s \in I$.

(2) implies (3). Let $(r_1, r_2), (s_1, s_2) \in S$ be such that $(r_1, r_2)(s_1, s_2) \in I \times S_2$. Then $(r_1 s_1, r_2 s_2) \in I \times S_2$. This implies that $r_1 s_1 \in I$. Therefore, either $r_1 \in I$ or $s_1 \in I$, since I is a prime ideal of S_1. If $r_1 \in I$, then $(r_1, r_2) \in I \times S_2$. Similarly, if $s_1 \in I$, then $(s_1, s_2) \in I \times S_2$. Hence, $I \times S_2$ is a prime ideal of S.

(3) implies (1). Let $(r_1, r_2), (s_1, s_2), (t_1, t_2) \in S$ be such that $(r_1, r_2)(s_1, s_2)(t_1, t_2) \in I \times S_2$. Then $(r_1, r_2)(s_1 t_1, s_2 t_2) \in I \times S_2$. Since $I \times S_2$ is a prime ideal of S so either $(r_1, r_2) \in I \times S_2$ or $(s_1 t_1, s_2 t_2) \in I \times S_2$. This implies that $(s_1, s_2)(t_1, t_2) \in I \times S_2$. Hence, $I \times S_2$ is a 2-nil ideal of S. □

Theorem 8. *Let S_1 and S_2 be two commutative semirings. If I_1 and I_2 are proper ideals of S_1 and S_2 respectively and $I = I_1 \times I_2$ be an ideal of $S = S_1 \times S_2$, then the following statements are equivalent:*

1. *I is a 2-nil ideal of S.*

2. *$I_1 = \sqrt{0_{S_1}} \subseteq S_1$ and $I_2 = \sqrt{0_{S_2}} \subseteq S_2$ are prime ideals.*

3. *$I = I_1 \times I_2$ is a 2-absorbing ideal of S and $I \subseteq \sqrt{0_S}$.*

Proof. (1) implies (2). Let $I_1 \neq \sqrt{0_{S_1}}$ such that $r \in I_1 \setminus \sqrt{0_{S_1}}$. Then $(r,1)(1,0)(1,1) \in I$ and $(r,1)(1,0) \notin \sqrt{0_S} = \sqrt{0_{S_1}} \times \sqrt{0_{S_2}}$. Since I is a 2-nil ideal of S, so we have either $(r,1)(1,1) \in I$ or $(1,0)(1,1) \in I$, which is a contradiction. Thus, $I_1 = \sqrt{0_{S_1}}$. If $I_1 = \sqrt{0_{S_1}}$ is not a prime ideal, then there exist $r, s \notin \sqrt{0_{S_1}} = I_1$ such that $rs \in I_1$. Then $(r,1)(s,1)(1,0) \in I$, but neither $(r,1)(s,1) \in \sqrt{0_S}$, nor $(r,1)(1,0) \in I$, nor $(s,1)(1,0) \in I$, which is again a contradiction. Hence, I_1 is a prime ideal of S_1. Similarly, $I_2 = \sqrt{0_{S_2}}$ is a prime ideal of S_2.

(2) implies (3). Suppose that $I_1 = \sqrt{0_{S_1}} \subseteq S_1$ and $I_2 = \sqrt{0_{S_2}} \subseteq S_2$ are prime ideals. Then $I_1 \times S_2$ and $S_1 \times I_2$ are prime ideals of $S = S_1 \times S_2$. Since the intersection of two prime ideals is always 2-absorbing ideal [3, Theorem 2.3], so it follows that $I = (I_1 \times S_2) \cap (S_1 \times I_2)$ is a 2-absorbing ideal of S.

(3) implies (1). Follows from the Theorem 2. □

The following example shows that the second condition of Theorem 8 is mandatory.

Example 9. Let $S_1 = \mathbb{Z}_{16}$ and $S_2 = \mathbb{Z}_{125}$ be two commutative semirings respectively and I an ideal of $S = S_1 \times S_2$. Let $I_1 = (4)$ and $I_2 = (25)$ be proper ideals of S_1 and S_2 respectively. Consider the ideal $I = I_1 \times I_2 = (4) \times (25) \subset S$. Then $(4) \subsetneq \sqrt{0_{S_1}} = (2)$ and $(25) \subsetneq \sqrt{0_{S_2}} = (5)$. However, I is not 2-nil ideal, since $(2,1)(2,1)(1,25) \in I, (2,1)(2,1) \notin \sqrt{0_S} = (2) \times (5)$ and $(2,1)(1,25) \notin I$.

As an application of Theorem 7 and Theorem 8, we obtain the following corollary, which characterizes the 2-nil ideals of a semiring $S = S_1 \times S_2$.

Corollary 1. *Let I_1 be a prime ideal of a commutative semiring S_1 and I_2 a prime ideal of a commutative semiring S_2. Let $S = S_1 \times S_2$ and $I = I_1 \times I_2$. Then I is a 2-nil ideal of S if and only if $I_1 \times S_2 = I = S_1 \times I_2$ and $I = \sqrt{0_{S_1}} \times \sqrt{0_{S_2}}$, where $\sqrt{0_{S_1}}$ and $\sqrt{0_{S_1}}$ are prime ideals.*

Definition 3. *Let S and S' be two semirings. Then a function $f : S \to S'$ is a morphism of semirings if and only if :*

1. *$f(0_S) = 0_{S'}$*
2. *$f(1_S) = 1_{S'}$*
3. *$f(r+s) = f(r) + f(s)$ and $f(rs) = f(r)f(s)$ for all $r, s \in S$.*

Theorem 9. *Let $f : S \to S'$ be a morphism between two commutative semirings.*

1. *If f is a monomorphism and I' is a 2-nil ideal of S', then the preimage $f^{-1}(I')$ is a 2-nil ideal of S.*
2. *If f is an epimorphism and I is a 2-nil ideal of S such that $Ker(f) \subseteq I$, then the image $f(I)$ is a 2-nil ideal of S'.*

Proof. (1) Let $r, s, t \in S$ be such that $rst \in f^{-1}(I')$. Then $f(rst) = f(r)f(s)f(t) \in I'$. Since I' is a 2-nil ideal in S', so it follow from the definition that either $f(r)f(s) = f(rs) \in \sqrt{0_{S'}}$ or $f(s)f(t) = f(st) \in I'$ or $f(r)f(t) = f(rt) \in I'$. Since f is a monomorphism, that is $ker f = 0$, so $f^{-1}(\sqrt{0_{S'}}) \subseteq \sqrt{0_S}$. This implies that either $rs \in \sqrt{0_S}$ or $st \in f^{-1}(I')$ or $rt \in f^{-1}(I')$. Thus,

$f^{-1}(I')$ is a 2-nil ideal of S.
(2) Let $r = f(x)$, $s = f(y)$, $t = f(z) \in S'$ be such that $rst = f(x)f(y)f(z) = f(xyz)$. Since $Ker(f) \subseteq I$, so $xyz \in I$. This implies that $rst \in f(I)$. Also, I is a 2-nil ideal of S, therefore it follows that either $xy \in \sqrt{0_S}$ or $yz \in I$ or $xz \in I$. Hence, $rs \in \sqrt{0_{S'}}$ or $st \in f(I)$ or $rt \in f(I)$, as required. □

3 2-nil primary ideals in semirings

In this section, we introduce the notions of 2-nil primary ideals, 2-absorbing primary ideals, and quasi-primary ideals in a semiring S. A detailed study of 2-nil primary ideals is carried out in commutative rings by Qaralleh [9], including their characterizations and fundamental properties. Therefore, the theoretical developments are supported with illustrative examples, which highlight the practical implications of the results in semirings.

Definition 4. *Let I be a proper ideal of a semiring S. Then I is called 2-nil primary ideal if whenever $r, s, t \in S$ such that $rst \in I$ then either $rs \in \sqrt{0}$ or $rt \in \sqrt{I}$ or $st \in \sqrt{I}$.*

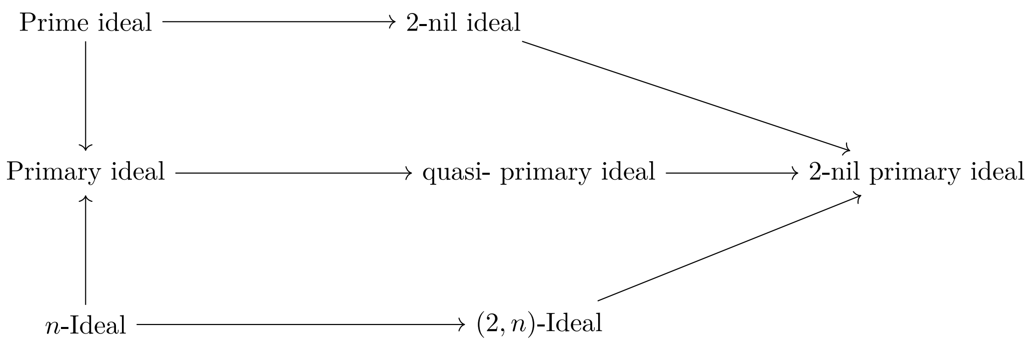
By incorporating the collection of nilpotent elements of S, denoted by $\sqrt{0}$, into the conditions for an ideal to be 2-nil primary, the above definition offers a distinctive perspective. Through the study of their properties and interrelations, this work aims to clarify the significance and relevance of 2-nil primary ideals within the broader framework of semiring theory.

It is evident that every 2-nil ideal is also a 2-nil primary ideal of a semiring S. However, the converse does not generally hold, as demonstrated in the following example.

Example 10. Let $S = \mathbb{Z}_0$ be a semiring and $I = 9\mathbb{Z}_0$ be an ideal of S. Then I is not 2-nil ideal as $3 \cdot 3 \cdot 1 \in I$ but $3 \cdot 3 \notin \sqrt{0}$ and $3 \cdot 1 \notin I$. Let $r, s, t, \in S$ such that $rst = 9k$, where $k \in \mathbb{Z}_0$. At least one of r, s, and t should be a multiple of 3. If $rs \notin \sqrt{0}$, then $rt \in \sqrt{I} = 3\mathbb{Z}_0$ or $st \in \sqrt{I} = 3\mathbb{Z}_0$. Thus, I is a 2-nil primary ideal of S.

The following diagram illustrates the relationships between 2-nil primary ideals of a semiring S and several well-known classes of ideals, including prime ideals, 2-nil ideals, primary ideals, quasi-primary ideals, n-ideals, and $(2, n)$-ideals. In particular, it is evident that there exists a strict relationship between 2-nil primary ideals and 2-nil ideals in a semiring S.

Theorem 10. *Let I be an ideal of a semiring S. Then the relationships among the various algebraic properties of the ideal I can be expressed through the logical implications shown in the following diagram.*



However, a proper ideal I of a semiring S is a 2-nil primary ideal if and only if its radical is a 2 -nil ideal as follows.

Theorem 11. *Let I be a proper ideal of a semiring S. Then I is a 2 -nil primary ideal of S if and only if $\sqrt{I}$ is a 2 -nil ideal.*

Proof. Let $r, s, t \in S$ be such that $rst \in \sqrt{I}$ and $rs \notin \sqrt{0}$. Then for some $n \in \mathbb{N}$, we have $(rst)^n = r^n s^n t^n \in I$. Since I is a 2-nil primary ideal, so it follows that either $r^n t^n \in \sqrt{I}$ or $s^n t^n \in \sqrt{I}$. This implies that either $rt \in \sqrt{I}$ or $st \in \sqrt{I}$. Hence, $\sqrt{I}$ is a 2-nil ideal of S. Conversely, assume that $\sqrt{I}$ is a 2-nil ideal and let $r, s, t \in S$ such that $rst \in I$. Since $I \subseteq \sqrt{I}$, so either $rs \in \sqrt{0}$ or $st \in \sqrt{I}$ or $rt \in \sqrt{I}$. Hence, I is a 2-nil primary ideal of S. □

Now, we establish the notions of 2-nil primary ideals and quasi-primary ideals. The following example demonstrates that, in general, a 2-nil primary ideal is not necessarily a quasi-primary ideal.

Example 11. Let $n \geq 2$, $n \in \mathbb{N}$, and $0 \leq i \leq n$, and define $m = n - i$. Then $B(n, i) = \{0, 1, 2, \ldots, n-1\}$ forms a semiring under the following operations:

$$a +_{B(n,i)} b = \begin{cases} a + b & \text{if } a + b \leq n - 1 \\ l & \text{if } a + b \geq n, \text{where } l \equiv (a + b) \mod m, \end{cases}$$

$$a \cdot_{B(n,i)} b = \begin{cases} ab & \text{if } ab \leq n - 1 \\ l & \text{if } ab \geq n, \text{where } l \equiv (ab) \mod m, i \leq l \leq n - 1. \end{cases}$$

Thus, $B(n, 0)$ is a semiring isomorphic to Z_n. Let $S = B(12, 0)$ be a semiring and $I = 6B(12, 0)$ be an ideal of S. Then I is not quasi- primary ideal, since $2 \cdot 3 \in I$ but neither $2 \in \sqrt{I}$ nor $3 \in \sqrt{I}$. Let $r, s, t, \in S$ be such that $rst \in I$ for some $r, s, t \in S$. Since $2 \mid rst$ and $3 \mid rst$, so clearly, $6 \mid rst$. Hence, either $rs \in \sqrt{0}$, or $rt \in \sqrt{I}$, or $st \in \sqrt{I}$.

Remark 1. *Let $S = \mathbb{Z}_0$, the set of all non-negative integers, be a semiring, and I an ideal of S.*

1. *If I is a 2-nil primary ideal of S then either $I = (0)$ or $I = (p^n)$ for some prime integer p and $n \geq 1$.*

2. *If either $I = (0)$ or $I = (p^n)$ for some prime integer p and $n \geq 1$ then I is a primary ideal of S.*

3. *If I is a primary ideal of S then I is a 2-nil primary ideal of S.*

4. *Let S be a commutative semiring. Then S is a semifield if and only if (0) is the only 2-nil primary ideal of S.*

Remark 2. *The characterization of 2-nil primary ideals in terms of proper ideals I of a semiring S is as follows:*

1. *If I is a 2-nil primary ideal of S then for any $r, s \in S$ and $rs \notin \sqrt{0}$, $(I : rs) \subseteq (\sqrt{I} : r) \cup (\sqrt{I} : s)$.*

2. *If for any $r, s \in S$ and $rs \notin \sqrt{0}$, $(I : rs) \subseteq (\sqrt{I} : r) \cup (\sqrt{I} : s)$, then either $(I : rs) \subseteq (\sqrt{I} : r)$ or $(I : rs) \subseteq (\sqrt{I} : s)$.*

3. *If for any $r, s \in S$, $rs \notin \sqrt{0}$, either $(I : rs) \subseteq (\sqrt{I} : r)$ or $(I : rs) \subseteq (\sqrt{I} : s)$, $rsJ \subseteq I$ for some $r, s \in S$ and J is an ideal of S, then either $rs \in \sqrt{0}$ or $rJ \subseteq \sqrt{I}$ or $sJ \subseteq \sqrt{I}$.*

4. *If $JKL \subseteq I$ for some ideals J, K, L of S such that either $JK \subseteq \sqrt{0}$ or $JL \subseteq \sqrt{I}$ or $KL \subseteq \sqrt{I}$, then I is a 2-nil primary ideal of S.*

Since the statements in Remark 2 are equivalent, we now have the following result.

Theorem 12. *Let S be a semiring and I a proper ideal of S. Suppose that for any $rsJ \subseteq I$ with $r, s \in S$ and J an ideal of S, such that either $rs \in \sqrt{0}$ or $rJ \subseteq \sqrt{I}$ or $sJ \subseteq \sqrt{I}$. Let J, K, L be ideals of S such that $JKL \subseteq I$. Then either $JK \subseteq \sqrt{0}$ or $JL \subseteq \sqrt{I}$ or $KL \subseteq \sqrt{I}$.*

Proof. Assume, to the contrary, that $JKL \subseteq I$ but none of the inclusions $JK \subseteq \sqrt{0}$, $JL \subseteq \sqrt{I}$, or $KL \subseteq \sqrt{I}$ holds. Then there exist $j_1, j_2 \in J$, $k_1, k_2 \in K$ such that $j_1k_1 \notin \sqrt{0}, j_2L \nsubseteq \sqrt{I}, k_2L \nsubseteq \sqrt{I}$. Since $j_2k_2L \subseteq I$ and both $j_2L \nsubseteq \sqrt{I}$ and $k_2L \nsubseteq \sqrt{I}$, the assumption yields $j_2k_2 \in \sqrt{0}$.

Next, from $j_1k_1L \subseteq I$ and $j_1k_1 \notin \sqrt{0}$, we have either $j_1L \subseteq \sqrt{I}$ or $k_1L \subseteq \sqrt{I}$. We now consider all possible cases.

Case 1. $j_1L \subseteq \sqrt{I}$ and $k_1L \nsubseteq \sqrt{I}$. Since $j_2k_1L \subseteq I$ and both $j_2L \nsubseteq \sqrt{I}$ and $k_1L \nsubseteq \sqrt{I}$, we conclude that $j_2k_1 \in \sqrt{0}$. Because $j_1L \subseteq \sqrt{I}$ but $j_2L \nsubseteq \sqrt{I}$, we have $j_1L + j_2L \nsubseteq \sqrt{I}$. Now $(j_1 + j_2)k_1L \subseteq I$ and $k_1L \nsubseteq \sqrt{I}$ imply $(j_1 + j_2)k_1 \in \sqrt{0}$. Since $j_2k_1 \in \sqrt{0}$, we obtain $j_1k_1 \in \sqrt{0}$, a contradiction to our initial choice of j_1, k_1.

Case 2. If $k_1L \subseteq \sqrt{I}$ and $j_1L \nsubseteq \sqrt{I}$, then a similar argument to Case 1 yields a contradiction, so we omit the repetition.

Case 3. $j_1L \subseteq \sqrt{I}$ and $k_1L \subseteq \sqrt{I}$. Since $k_1L \subseteq \sqrt{I}$ and $k_2L \nsubseteq \sqrt{I}$, we have $(k_1+k_2)L \nsubseteq \sqrt{I}$. As $j_2(k_1 + k_2)L \subseteq I$ and $j_2L \nsubseteq \sqrt{I}$, we get $j_2(k_1 + k_2) \in \sqrt{0}$. As $j_2k_2 \in \sqrt{0}$, we have $j_2k_1 \in \sqrt{0}$.

On the other hand, as $j_1L \subseteq \sqrt{I}$ and $j_2L \nsubseteq \sqrt{I}$, we obtain $(j_1 + j_2)L \nsubseteq \sqrt{I}$. Now, $(j_1 + j_2)k_2L \subseteq I$ implies $(j_1 + j_2)k_2 \in \sqrt{0}$. Since $j_2k_2 \in \sqrt{0}$, we deduce $j_1k_2 \in \sqrt{0}$.

Finally, from $(j_1 + j_2)(k_1 + k_2)L \subseteq I$, together with $(j_1 + j_2)L \nsubseteq \sqrt{I}, (k_1 + k_2)L \nsubseteq \sqrt{I}$, we conclude $(j_1 + j_2)(k_1 + k_2) = j_1k_1 + j_1k_2 + j_2k_1 + j_2k_2 \in \sqrt{0}$. Since $j_2k_2,\ j_1k_2,\ j_2k_1 \in \sqrt{0}$, we obtain $j_1k_1 \in \sqrt{0}$, again contradicting the selection of j_1, k_1.

In all cases, we reach a contradiction. Therefore, one of the inclusions $JK \subseteq \sqrt{0}$, $JL \subseteq \sqrt{I}$, or $KL \subseteq \sqrt{I}$ must hold. □

Theorem 13. *Let S be a semiring. If I is a 2-nil primary ideal of S, then $(\sqrt{I} : x)$ is a 2-nil primary ideal of S for all $x \in S \backslash \sqrt{I}$ if and only if $(\sqrt{I} : x) = (\sqrt{I} : x^2)$ for all $x \in S \backslash \sqrt{I}$.*

Proof. Let $r, s, t \in S$ be such that $rst \in (\sqrt{I} : x)$. Then $rstx \in I$. Since I is a 2-nil primary ideal of S, therefore either $rs \in \sqrt{0}$ or $stx \in \sqrt{I}$ or $trx \in \sqrt{I}$, that is $st \in (\sqrt{I} : x)$ or $tr \in (\sqrt{I} : x)$. Hence, $(\sqrt{I} : x)$ is a 2-nil primary ideal of S. Conversely, it is clear that $(\sqrt{I} : x) \subseteq (\sqrt{I} : x^2)$. Let $y \in (\sqrt{I} : x^2)$. Then $x^2 \in \sqrt{0}$ or $xy \in \sqrt{I}$. If $xy \in \sqrt{I}$, then $y \in (\sqrt{I} : x)$ and we are done. If $x^2 \in \sqrt{0} \subseteq \sqrt{I}$, then $x^2 \in \sqrt{I}$. This implies that $x \in \sqrt{I}$, a contradiction. Hence, $(\sqrt{I} : x) = (\sqrt{I} : x^2)$. □

Definition 5 ([10])**.** *Let S be a commutative semiring and I a proper ideal of S. Then I is said to be a 2-absorbing quasi-primary ideal of S if $\sqrt{I}$ is a 2-absorbing ideal of S.*

Example 12. Taking $n = 10$ and $i = 8$ in Example 11, we get $S = B(10, 8) = \{0, 1, 2, \cdots, 9\}$. Let $I = \{0, 4, 8\}$ be an ideal of S. Then it is easy to see that $\sqrt{I} = \{0, 2, 4, 6, 8\}$ is a 2-absorbing ideal of S.

Proposition 3 ([10])**.** *A proper ideal I of a semiring S is a 2-absorbing quasi-primary ideal of S if and only if $r, s, t \in S$ and $rst \in I$ such that either $rs \in \sqrt{I}$ or $st \in \sqrt{I}$ or $rt \in \sqrt{I}$.*

Remark 3. *Let $I \subseteq \sqrt{0}$ be a 2-absorbing quasi-primary ideal of a semiring S. Then I is a 2-nil primary ideal of S. For, let $I \subseteq \sqrt{0}$ be an ideal of a semiring S and $rst \in I$ such that $rs \notin \sqrt{0}$. Then $rs \notin I$ and since I is 2-absorbing primary ideal of S, we have either $rt \in \sqrt{I}$ or $st \in \sqrt{I}$. Hence, I is 2-nil primary ideal of S.*

The following theorems are proved in [7].

Theorem 14 ([7])**.** *Let I be an ideal of a semiring S. If I is a 2-absorbing primary ideal of S, then $\sqrt{I}$ is a 2-absorbing ideal of S.*

Theorem 15 ([7])**.** *Let S be a semiring. Suppose that I_1 is a P_1–primary ideal of S for some prime ideal P_1 of S, and I_2 is a P_2–primary ideal of S for some prime ideal P_2 of S. Then the following statements hold:*

1. *I_1I_2 is a 2-absorbing primary ideal of S.*

2. *$I_1 \cap I_2$ is a 2-absorbing primary ideal of S.*

Theorem 16. *Let S be a commutative semiring and I_1, I_2 be P_1-quasi- primary and P_2-quasi-primary ideals of S respectively for some prime ideals P_1 and P_2 of S. Then I_1I_2 and $I_1 \cap I_2$ are 2-absorbing quasi-primary ideals of S.*

Proof. By Golan [5, Proposition 7.30] and Theorem 5, we have $\sqrt{I_1I_2} = \sqrt{I_1 \cap I_2} = P_1 \cap P_2$. By Theorem 15, I_1I_2 and $I_1 \cap I_2$ is a 2-absorbing primary ideal of S, so by Theorem 14, we conclude that $\sqrt{I_1I_2}$ and $\sqrt{I_1 \cap I_2}$ is a 2-absorbing ideal of S. Hence, I_1I_2 and $I_1 \cap I_2$ are 2-absorbing quasi-primary ideal of S. □

As a consequence of Theorem 16, we obtain the following result.

Corollary 2. *Let S be a commutative semiring and I_1, I_2 be P_1-quasi- primary and P_2-quasi-primary ideals of S respectively for some prime ideals P_1 and P_2 of S. Then $I_1^n I_2^m$ and $I_1^n \cap I_2^m$ are 2-absorbing quasi-primary ideals of S for every positive integers m, n.*

Proposition 4. *Let S_1 and S_2 be two commutative semirings, and I an ideal of $S = S_1 \times S_2$. Then the following conditions are equivalent:*

1. *$I \times S_2$ is a 2-nil primary ideal in the semiring $S_1 \times S_2$.*
2. *I is a quasi- primary ideal of S_1.*
3. *$I \times S_2$ is a quasi-primary ideal of $S_1 \times S_2$.*

Proof. (1) implies (2). Let $r, s \in S$ be such that $rs \in I$. Then $(r,1)(s,1)(1,1) \in I \times S_2$. Since $(r,1)(s,1) \notin \sqrt{0}_S$, so it follows that either $(r,1)(1,1) \in \sqrt{I \times S_2}$ or $(s,1)(1,1) \in \sqrt{I \times S_2} \subseteq \sqrt{I} \times S_2$. Hence, we conclude that either $r \in \sqrt{I}$ or $s \in \sqrt{I}$, as needed.
(2) implies (3). Follows from Proposition 2.9 [10].
(3) implies (1). By Theorem 10, it is straightforward. □

Theorem 17. *Let S_1 and S_2 be two commutative semirings and I_1, I_2 be two proper ideals of S_1 and S_2 respectively. Let $I = I_1 \times I_2$ be an ideal of $S = S_1 \times S_2$. Then the following statements are equivalent:*

1. *I is a 2-nil primary ideal of S.*
2. *$I_1 \subseteq \sqrt{0_{S_1}}$ and $I_2 \subseteq \sqrt{0_{S_2}}$ are quasi-primary ideals of S_1 and S_2 respectively.*
3. *I is a 2-absorbing quasi-primary ideal of S and $I \subseteq \sqrt{0_S}$.*

Proof. (1) implies (2). Let $I_1 \nsubseteq \sqrt{0_{S_1}}$, such that $r \in I_1 \setminus \sqrt{0_{S_1}}$. Then $(r,1)(1,0)(1,1) \in I$ and $(r,1)(1,0) \notin \sqrt{0_{S_1}} \times \sqrt{0_{S_2}}$. Since I is a 2-nil primary ideal of S, so we have either $(r,1)(1,1) \in \sqrt{I}$ or $(1,0)(1,1) \in \sqrt{I}$, which is a contradiction. Thus, $I_1 \subseteq \sqrt{0_{S_1}}$, and similarly, $I_2 \subseteq \sqrt{0_{S_2}}$. Now, if I_1 is not quasi-primary ideal, then there exist $r, s \in S_1 \setminus I_1$ such that $rs \in I_1$, but neither $r \in \sqrt{I_1}$ nor $s \in \sqrt{I_1}$. Hence, $(r,1)(s,1)(1,0) \in I$, but neither $(r,1)(s,1) \in \sqrt{0_S}$, nor $(r,1)(1,0) \in \sqrt{I}$, nor $(s,1)(1,0) \in \sqrt{I}$, which is a contradiction. Thus, I_1 is a quasi-primary ideal in S_1. In the same way, I_2 is a quasi-primary ideal of S_2.
(2) implies (3). Suppose that $I_1 \subseteq \sqrt{0_{S_1}} \subset S_1$ and $I_2 \subseteq \sqrt{0_{S_2}} \subset S_2$ are quasi-primary ideals. Hence, by Proposition 4, $I_1 \times S_2$ and $S_1 \times I_2$ are quasi-primary ideals of S. By using the Theorem 16, we conclude that $I = I_1 \times I_2 = (I_1 \times S_2) \cap (S_1 \times I_2)$ is a 2-absorbing quasi-primary ideal of S.

(3) implies (1). Follows from Remark 3. □

Lemma 1. *Let $S = S_1 \times S_2$, where S_1 and S_2 are commutative semirings with nonzero identity, and let I be a proper ideal of S. Then the following statements are equivalent:*

1. *I is a quasi-primary ideal of S.*
2. *Either $I = I_1 \times S_2$ for some quasi-primary ideal I_1 of S_1, or $I = S_1 \times I_2$ for some quasi-primary ideal I_2 of S_2.*

Proof. (i) implies (ii). Assume that I is a quasi-primary ideal of S. Then $I = I_1 \times I_2$ for some ideals I_1 of S_1 and I_2 of S_2. Since $\sqrt{I} = \sqrt{I_1} \times \sqrt{I_2}$ is a prime ideal of S, it follows that either $\sqrt{I_1}$ is a prime ideal of S_1 and $\sqrt{I_2} = S_2$, or $\sqrt{I_1} = S_1$ and $\sqrt{I_2}$ is a prime ideal of S_2. Hence, either $I = I_1 \times S_2$ for some quasi-primary ideal I_1 of S_1, or $I = S_1 \times I_2$ for some quasi-primary ideal I_2 of S_2.
(ii) implies (i). If $I = I_1 \times S_2$ for some quasi-primary ideal I_1 of S_1, or $I = S_1 \times I_2$ for some quasi-primary ideal I_2 of S_2, then it is clear that I is a quasi-primary ideal of S. □

Theorem 18. *Let $I_1, I_2, \dots, I_n$ be proper ideals of commutative semirings $S_1, S_2, \dots, S_n$ respectively, where $n \geq 3$. Let $I = I_1 \times I_2 \times \cdots \times I_n$ be an ideal of $S = S_1 \times S_2 \times \cdots \times S_n$. Then the following conditions are equivalent:*

(1) I is a 2-nil primary ideal of S.

(2) I_k is a quasi-primary ideal of S_k for some $k \in \{1, 2, \dots, n\}$ and $I_j = S_j$ for all $j \neq k$.

(3) I is a quasi-primary ideal of S.

Proof. (1) implies (2). Assume that $I = I_1 \times I_2 \times \cdots \times I_n$ is a 2-nil primary ideal of S. We show that at most one of the I_k can be proper. Assume, on the contrary, that say I_1 and I_2 are proper ideals of S_1 and S_2 respectively. Consider $x, y, z \in S$ be such that $x = (0, 1, 1, \dots, 1)$, $y = (1, 0, 1, \dots, 1)$, $z = (1, 1, 1, \dots, 1)$. Now, $xyz = (0 \cdot 1 \cdot 1,\ 1 \cdot 0 \cdot 1,\ 1 \cdot 1 \cdot 1, \dots, 1) = (0, 0, 1, \dots, 1) \in I$, because $0 \in I_1$ and $0 \in I_2$. However, $xy = (0 \cdot 1,\ 1 \cdot 0,\ 1 \cdot 1, \dots, 1) = (0, 0, 1, \dots, 1) \notin \sqrt{0}$, the last $n-2$ entries are non-nilpotent units, since the semirings are commutative and 1 is not nilpotent. Since I is 2-nil primary ideal and $xyz \in I$. But $xy \notin \sqrt{0}$, we must have either $xz \in \sqrt{I}$ or $yz \in \sqrt{I}$. Now, $xz = (0, 1, 1, \dots, 1), yz = (1, 0, 1, \dots, 1)$. Thus, $(0, 1, 1, \dots, 1) \in \sqrt{I}$. This implies that $1 \in \sqrt{I_2}$, $I_2 = S_2$, or $(1, 0, 1, \dots, 1) \in \sqrt{I}$, $1 \in \sqrt{I_1}$, $I_1 = S_1$. Either case contradicts the assumption that both I_1 and I_2 are proper. Hence, at most one of $I_1, \dots, I_n$ is proper.

Now, assume that exactly, say I_1 is proper and $I = I_1 \times S_2 \times \cdots \times S_n$. Let $r, s \in S_1$ satisfying $rs \in I_1$. Then $(r, 1, \dots, 1)(s, 1, \dots, 1)(1, 1, \dots, 1) = (rs, 1, \dots, 1) \in I$. If $(r, 1, \dots, 1)(s, 1, \dots, 1) \notin \sqrt{0}$, then by 2-nil primary ideal property, $(r, 1, \dots, 1)(1, 1, \dots, 1) \in \sqrt{I}$ or $(s, 1, \dots, 1)(1, 1, \dots, 1) \in \sqrt{I}$. This implies that $r \in \sqrt{I_1}$ or $s \in \sqrt{I_1}$. Thus, I_1 is 2-nil primary in S_1. Hence, condition (2) holds.

(2) implies (3). Assume I_k is quasi-primary in S_k for some k, and $I_j = S_j$ for all $j \neq k$. We prove by induction on n that such an I is quasi-primary in S. For $n = 2$, the result follows from Lemma 1. Assume the statement holds for $n-1$ semirings. Let $S = K \times S_n$, $K = S_1 \times \cdots \times S_{n-1}$, and $I = L \times I_n$, $L = I_1 \times \cdots \times I_{n-1}$. By Lemma 1, I is quasi-primary in S iff either: $I = L \times S_n$ where L is quasi-primary in K, or $I = K \times I_n$ where I_n is quasi-primary in S_n. Both cases fall under the induction hypothesis. Thus, I is quasi-primary in S.

(3) implies (1). This follows immediately from the definitions; every quasi-primary ideal is, in particular, 2-nil primary.

□

Theorem 19. *Let $f : S \to S'$ be a morphism between two commutative semirings. If f is one-one and $I' \subset S'$ is a 2-nil primary ideal, then the preimage $f^{-1}(I')$ is a 2-nil primary ideal of S.*

Definition 6. *A proper ideal I of a semiring S is said to be a strong ideal if for each $r \in I$ there exists $s \in I$ such that $r + s = 0$.*

Theorem 20. *Let $f : S \to S'$ be a morphism between two commutative semirings. If f is an epimorphism such that $f(0) = 0$ and I is a subtractive and strong 2-nil primary ideal of S such that $kerf \subseteq I \cap \sqrt{0_S}$, then $f(I)$ is 2-nil primary ideal of S'.*

Proof. Let $x, y, z \in S'$ be such that $xyz \in f(I)$. Then there exists an element $m \in I$ such that $xyz = f(m)$. Since f is an epimorphism, therefore there exist $r, s, t \in S$ such that $f(r) = x, f(s) = y$ and $f(t) = z$. Also, since I is a strong ideal of S and $m \in I$, therefore there exists $n \in I$ such that $m + n = 0$. This implies that $f(m + n) = f(0) = 0'$, that is, $f(rst + n) = 0'$. This implies that $rst + n \in kerf \subseteq I$. Since I is subtractive ideal of S, we have $rst \in I$. Again, I is a 2-nil primary ideal of S, therefore either $rs \in \sqrt{0_s}$ or $st \in \sqrt{I}$ or $rt \in \sqrt{I}$. Since $kerf \subseteq I \cap \sqrt{0_S}$, so we have $f(\sqrt{0_S}) \subseteq \sqrt{0_{S'}}$ and $f(\sqrt{I}) \subseteq \sqrt{f(I)}$. Thus, $rs \in f(\sqrt{0_s}) \subseteq \sqrt{f(0_s)}$ or $st \in f(\sqrt{I}) \subseteq \sqrt{f(I)}$ or $rt \in f(\sqrt{I}) \subseteq \sqrt{f(I)}$. Hence, $f(I)$ is a 2-nil primary ideal of S'. □

As an application of Theorem 19 and Theorem 20, we characterize 2-nil primary ideals of a semiring S in terms of the following Corollary.

Corollary 3. *Let S be a semiring and I, J ideals of S such that $J \subseteq I \subset S$. Then*

1. *If I is a 2-nil primary ideal of S, then I/J is a 2-nil primary ideal of S/J.*
2. *If S_1 is a subsemiring of S and I is a 2-nil primary ideal of S, then $I \cap S_1$ is a 2-nil primary ideal of S_1.*
3. *If I/J is a 2-nil primary ideal of S/J and $J \subseteq \sqrt{0_s}$, then I is a 2-nil primary ideal of S.*

4 Conclusions

In this work, we examine the concepts of 2-nil ideals and 2-nil primary ideals in the setting of commutative semirings. We establish several fundamental properties of 2-nil ideals, thereby extending the theory of $(2, n)$-ideals and providing a comprehensive algebraic framework for the systematic study of nilpotent elements and their behaviour in commutative semirings. In addition, we characterize 2-nil primary ideals and investigate their connections with prime, primary, and quasi-primary ideals, emphasizing the structural conditions under which a 2-nil ideal becomes a 2-nil primary ideal. These findings not only enhance the ideal theory of commutative semirings but also open new directions for research, particularly in radical theory, ideal decompositions, and applications of semiring theory to algebraic models in computer science and tropical mathematics.

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