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Rees ring and integral closure of a filtration

M. Yahyapour-Dakhel[†], **F. Dorostkar**[‡*]

[†] [‡] *Department of Pure Mathematics, Faculty of Mathematical Sciences, University of Guilan, Rasht, Iran*
Emails: masoudyahyapour@phd.guilan.ac.ir, dorostkar@guilan.ac.ir

Abstract. In this paper, we will study some properties of the integral closure of a filtration relative to a module in the Rees ring.

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1 Introduction

Throughout this paper, R is a commutative ring with a non-zero identity. Further, M is an R-module and $\mathbb{N}$ denotes the set of positive integers.

The concepts of reduction and integral closure of an ideal introduced by D. G. Northcott and D. Rees in [7]. Let R be a commutative Noetherian ring and I and J be ideals of R. Then I is a *reduction* of J if $I \subseteq J$ and there exists a positive integer $s \in \mathbb{N}$ such that $IJ^s = J^{s+1}$. An element x of R is said to be *integrally dependent on* I if there exist a positive integer $n \in \mathbb{N}$ and elements $a_1, \ldots, a_n \in R$ with $a_i \in I^i$ for $i = 1, 2, \ldots, n$ such that

$$x^n + a_1x^{n-1} + \cdots + a_n = 0.$$

We know, the set of all elements of R which are integrally dependent on I, is an ideal of R. This ideal is called the *integral closure of* I and is denoted by $\overline{I}$. In fact, $\overline{I}$ is the largest ideal of R which has I as a reduction.

In [9], R. Y. Sharp, Y. Tiraş and M. Yassi introduced concepts of reduction and integral closure of an ideal I of a commutative ring R relative to a Noetherian module M. Here, we recall some of these definitions. Let I and J be ideals of R and M be a Noetherian R-module. Then I is said to be a reduction of J relative to M if $I \subseteq J$ and there exists a positive integer

*Corresponding author

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$s \in \mathbb{N}$ such that $IJ^sM = J^{s+1}M$. Also, an element x of R is said to be integrally dependent on I relative to M if there exists a positive integer $n \in \mathbb{N}$ such that

$$x^nM \subseteq \sum_{i=1}^{n} x^{n-i}I^iM.$$

The set of all elements of R which are integrally dependent on I relative to M, is an ideal of R. This ideal is called the integral closure of I relative to M and is denoted by $I^{-(M)}$. It is the largest ideal of R which has I as a reduction relative to M.

A *filtration* $f = \{I_n\}_{n\geq 0}$ on R is a sequence of ideals of R such that $I_0 = R$, $I_{n+1} \subseteq I_n$, and $I_nI_m \subseteq I_{n+m}$, for all non-negative integers m and n. If $f = \{I_n\}_{n\geq 0}$ is a filtration on R and k is a positive integer then $\{I_{nk}\}_{n\geq 0}$ is a filtration on R and it is denoted by $f^{(k)}$.

Let I be an ideal of R. The filtration $f = \{I^n\}_{n\geq 0}$ is called the *I-adic filtration* on R.

Now, let R be a Noetherian ring. A filtration $f = \{I_n\}_{n\geq 0}$ on R is called a *Noetherian filtration* if there exists a positive integer k such that

$$I_n = \sum_{i=1}^{k} I_{n-i}I_i,$$

for all $n \geq 1$. Clearly, if R is Noetherian, then the I-adic filtration is a Noetherian filtration (See [8]).

In [3], H. Dichi defined the integral closure of a filtration. An element $x \in R$ is said to be *integral over a filtration* $f = \{I_n\}_{n\geq 0}$ on R if there exists a positive integer m such that

$$x^m + a_1x^{m-1} + \cdots + a_m = 0,$$

where $a_i \in I_i$ for every $1 \leq i \leq m$. The set of all elements $x \in R$, which are integral over $f = \{I_n\}_{n\geq 0}$, is an ideal. This ideal is called the *integral closure of a filtration* $f = \{I_n\}_{n\geq 0}$ and is denoted by $Clos_R(f)$.

Also, in [4], the integral closure of a filtration relative to a module is introduced. An element $x \in R$ is integral over a filtration $f = \{I_n\}_{n\geq 0}$ relative to an $R-$module M, if there exists a positive integer m such that

$$x^m + a_1x^{m-1} + \cdots + a_m \in (0 :_R M),$$

where $a_i \in I_i$ for every $1 \leq i \leq m$. The set of all elements of R which are integral over a filtration $f = \{I_n\}_{n\geq 0}$ relative to a module M is an ideal. This ideal is called the integral closure of a filtration $f = \{I_n\}_{n\geq 0}$ relative to a module M and is denoted by $Clos_R(f, M)$.

In this paper we will extend some of the results which were proved for integral closure of an ideal in [10], to the integral closure of a filtration.

2 Preliminary definitions

In this section we review some preliminary definitions and results which will be needed in the rest of this paper. We begin with definition of graded rings and graded modules.

A ring R is called a $\mathbb{Z}$-*graded* or a *graded* ring if there exists a family $\{R_n\}_{n\in\mathbb{Z}}$ of additive subgroups of R such that $R = \bigoplus_{n\in\mathbb{Z}} R_n$ and for every $n, m \in \mathbb{Z}$, $st \in R_{n+m}$ for all $s \in R_n$ and $t \in R_m$. Every element of $h(R) = \bigcup_{n\in\mathbb{Z}} R_n$ is called a *homogeneous* element. Further, every non-zero homogeneous element of R_n is called a homogeneous element of degree n. If R is a graded ring then every non-zero element a of R is a unique finite sum of non-zero homogeneous elements, which each of them is called a homogeneous component of a. An ideal I of a graded ring R is called a *graded ideal* if for every $a \in I$, all homogeneous components of a belong to I.

Let $R = \bigoplus_{n\in\mathbb{Z}} R_n$ be a graded ring. An R-module M is said to be a $\mathbb{Z}$-*graded* or a *graded* R-module if there exists a family $\{M_n\}_{n\in\mathbb{Z}}$ of additive subgroups of M such that $M = \bigoplus_{n\in\mathbb{Z}} M_n$ and $R_nM_m \subseteq M_{n+m}$ for every $n, m \in \mathbb{Z}$. Every element of $h(M) = \bigcup_{n\in\mathbb{Z}} M_n$ is called a homogeneous element. Also, every non-zero homogeneous element of M_n is called a homogeneous element of degree n (see [6]).

If M is a graded $R-$module, then every non-zero element $x \in M$ has a unique expression $x = x_{n_1} + x_{n_2} + \cdots + x_{n_k}$ as a finite sum of non-zero homogeneous elements. The elements $x_{n_1}, x_{n_2}, \ldots, x_{n_k}$ are called the homogeneous components of x. A submodule N of a graded $R-$module M is called a *graded submodule* if for every $x \in N$, all homogeneous components of x belong to N.

Now, we remember the definitions of the Rees ring and the Rees module with respect to a filtration.

Let $f = \{I_n\}_{n\geq 0}$ be a filtration on R. Let t be an indeterminate and $u = t^{-1}$. The Rees ring of R with respect to f is denoted by $\mathcal{R} = \mathcal{R}(R, f)$. This is defined by

$$\mathcal{R} = \mathcal{R}(R, f) = \bigoplus_{n\in\mathbb{Z}} I_nt^n = \{\textstyle\sum_{i=-q}^{p} a_it^i \in R[u, t] : a_i \in I_i\}.$$

We know, $\mathcal{R}(R, f) = R[u, I_1t, I_2t^2, \ldots]$ is a (graded) subring of $R[u, t]$ (see [2, 8]).

Also, for an R-module M, the Rees module of M with respect to f is

$$\mathcal{M} = \mathcal{R}(M, f) = \bigoplus_{n\in\mathbb{Z}} I_nMt^n = \{\textstyle\sum_{i=-r}^{s} m_it^i \in M[u, t] : m_i \in I_iM\}.$$

For every $\sum_{i=-q}^{p} c_it^i \in \mathcal{R}(R, f)$ and $\sum_{j=-r}^{s} m_jt^j \in \mathcal{R}(M, f)$, we consider the following scalar multiplication

$$(\textstyle\sum_{i=-q}^{p} c_it^i).(\sum_{j=-r}^{s} m_jt^j) = \sum_{i=-q}^{p}\sum_{j=-r}^{s} c_im_jt^{i+j},$$

where $c_im_j \in I_{i+j}M$, then $\mathcal{R}(M, f)$ is an $\mathcal{R}(R, f)$-module.

Further, if f is the I-adic filtration then we denote the Rees ring with respect to I by $\mathcal{R} = \mathcal{R}(R, I)$ and the Rees module with respect to I by $\mathcal{M} = \mathcal{R}(M, I)$.

Remark 1. *Let R be a Noetherian ring and $f = \{I_n\}_{n\geq 0}$ be a Noetherian filtration. Further let M be a finitely generated R-module. Then $\mathcal{R}(R, f)$ is a Noetherian ring by [8, Remark 2.2, P. 139]. Also $\mathcal{R}(M, f)$ is a finitely generated $\mathcal{R}(R, f)$-module by [2, Theorem 4.4.6].*

3 Main Results

We have seen the definition of the integral closure of a filtration of R relative to an $R-$module M. When M is a finitely generated R-module, we have the following proposition.

Proposition 1. (See [4, Proposition 2.6].) *Let $f = \{I_n\}_{n\geq 0}$ be a filtration on R and let M be a finitely generated R-module. Then x is integral over f relative to M if and only if there exists a positive integer $n \in \mathbb{N}$ such that*

$$x^nM \subseteq \sum_{i=1}^{n} x^{n-i}I_iM.$$

Proposition 2. (See [3, Corollary 2.3 (iii)].) *Let $f = \{I_n\}_{n\geq 0}$ be a filtration on R. Then $\widetilde{f} = \{I_n + (0:_R M)/(0:_R M)\}_{n\geq 0}$ is a filtration of the ring $\widetilde{R} = R/(0:_R M)$. Further we have*

$$Clos_{\widetilde{R}}(\widetilde{f}) = Clos_R(f, M)/(0:_R M).$$

Remark 2. *Let $f = \{I_n\}_{n\geq 0}$ be a filtration on R. Let t be an indeterminate and $u = t^{-1}$. Let $\mathcal{R} = \mathcal{R}(R, f)$ be the Rees ring with respect to a filtration $f = \{I_n\}_{n\geq 0}$. In the rest of this section, the homogeneous ideal $\mathcal{R}u$ is denoted by $\mathcal{I}$ and the $\mathcal{I}$-adic filtration on the Rees ring $\mathcal{R} = \mathcal{R}(R, f)$ is denoted by $\mathfrak{f}_{\mathcal{I}} = \{\mathcal{R}u^n\}_{n\geq 0}$.*

Remark 3. *Let $\mathcal{R} = \mathcal{R}(R, f) = R[u, I_1t, I_2t^2, \ldots]$ be the Rees ring of R with respect to a filtration $f = \{I_n\}_{n\geq 0}$. Let $\mathcal{R} = \bigoplus_{n\in\mathbb{Z}} R_n$ where R_n denotes the subgroup of $\mathcal{R}$ consisting of zero and the homogeneous elements of $\mathcal{R}$ of degree n. By [6, Prop. 28, P. 115], we have*

$$\mathcal{R}u^k = \bigoplus_{i\in\mathbb{Z}}(R_i \cap \mathcal{R}u^k)$$

for every positive integer $k \in \mathbb{N}$. Therefore, as we have seen for an I-adic filtration in [10], we can see

$$R_i \cap \mathcal{R}u^k = \begin{cases} I_{i+k}t^i & \text{if } i > -k \\ Rt^i & \text{if } i \leq -k. \end{cases}$$

We can get a simpler proof for [10, Theorem 2.2] by using the following theorem.

Theorem 1. *Let R be a Noetherian ring and $f = \{I_n\}_{n\geq 0}$ be a Noetherian filtration on R. Let M be a finitely generated R-module. If $\mathcal{R} = \mathcal{R}(R, f)$ is the Rees ring of R with respect to $f = \{I_n\}_{n\geq 0}$ and $\mathcal{M} = \mathcal{R}(M, f)$ is the Rees module of M with respect to f then for every positive integer $k \in \mathbb{N}$, we have*

$$Clos_{\mathcal{R}}(\mathfrak{f}_{\mathcal{I}}^{(k)}, \mathcal{M}) \cap R = Clos_R(f^{(k)}, M).$$

Proof. Let $x \in Clos_R(f^{(k)}, M)$. Then there exists a positive integer $n \in \mathbb{N}$ such that

$$x^nM \subseteq \sum_{i=1}^{n} x^{n-i}I_{ki}M,$$

by Proposition 1. We will show that

$$x^n\mathcal{M} \subseteq \sum_{i=1}^{n} x^{n-i}(\mathcal{R}u^{ki})\mathcal{M}.$$

Let $x^nm_jt^j \in x^n\mathcal{M}$, where $m_j \in I_jM$. Since $I_jMt^j \subseteq \mathcal{M}$ and by Remark 3, $I_{ki}t^0 = I_{0+ki}t^0 = R_0 \cap \mathcal{R}u^{ki} \subseteq \mathcal{R}u^{ki}$, we have

$$\begin{aligned} x^nm_jt^j \in x^nI_jMt^j &\subseteq \sum_{i=1}^{n} x^{n-i}I_{ki}(I_jMt^j) \\ &\subseteq \sum_{i=1}^{n} x^{n-i}(I_{ki}t^0)\mathcal{M} \\ &\subseteq \sum_{i=1}^{n} x^{n-i}(\mathcal{R}u^{ki})\mathcal{M}. \end{aligned}$$

For the converse, let $x \in Clos_{\mathcal{R}}(\mathfrak{f}_{\mathcal{I}}^{(k)}, \mathcal{M}) \cap R$. Then there exists $n \in \mathbb{N}$ such that

$$x^n\mathcal{M} \subseteq \sum_{i=1}^{n} x^{n-i}(\mathcal{R}u^{ki})\mathcal{M}$$

by Proposition 1. Let $y \in x^nM$. Then $y = x^nm$, for some $m \in M$. Hence,

$$x^nm \in x^n\mathcal{M} \subseteq \sum_{i=1}^{n} x^{n-i}(\mathcal{R}u)^{ki}\mathcal{M}.$$

Therefore,

$$x^nm = \sum_{i=1}^{n} x^{n-i}u^{ki}\gamma_i$$

for $\gamma_i \in \mathcal{M}$, and this shows $\gamma_i \in I_{ki}Mt^{ki}$. Now since $u^{ki}\gamma_i \in I_{ki}M$, we have $x^nm \in \sum_{i=1}^{n} x^{n-i}I_{ki}M$ and so $x^nM \subseteq \sum_{i=1}^{n} x^{n-i}I_{ki}M$. Then $x \in Clos_R(f^{(k)}, M)$ by Proposition 1. □

Corollary 1. (See [10, Theorem 2.2].) *Let R be a Noetherian ring and let I be an ideal of R and $\mathcal{R} = \mathcal{R}(R, I)$. Let M be a finitely generated R-module and $\mathcal{M} = \mathcal{R}(M, I)$. Then for every positive integer $k \in \mathbb{N}$, we have*

$$(\mathcal{R}u^k)^{-(\mathcal{M})} \cap R = (I^k)^{-(M)}.$$

Proof. Let $f = \{I^n\}_{n\geq 0}$ be an I-adic filtration. Then by Theorem 1, we have

$$Clos_{\mathcal{R}}(\mathfrak{f}_{\mathcal{I}}^{(k)}, \mathcal{M}) \cap R = Clos_R(f^{(k)}, M).$$

But, we can see that

$$Clos_{\mathcal{R}}(\mathfrak{f}_{\mathcal{I}}^{(k)}, \mathcal{M}) \cap R = (\mathcal{R}u^k)^{-(\mathcal{M})} \cap R$$

and

$$Clos_R(f^{(k)}, M) = (I^k)^{-(M)}.$$

So the proof is completed. □

In the next proposition we will show that, if f is a filtration of homogeneous ideals of R then $Clos_R(f)$ is a homogeneous ideal of R.

Proposition 3. *Let $R = \bigoplus_{n\in\mathbb{Z}} R_n$ be a graded ring and $f = \{I_n\}_{n\geq 0}$ be a filtration of homogeneous ideals of R. Then $Clos_R(f)$ is a homogeneous ideal of R.*

Proof. Let $\mathcal{R} = R[u, I_1t, I_2t^2, \ldots]$ be the Rees ring with respect to f. Let $x \in Clos_R(f)$. By [3, Lemma 2.1 (i)], we know that $x \in Clos_R(f)$ if and only if xt is integral over $\mathcal{R}$. So $x \in Clos_R(f)$ implies that xt is integral over $\mathcal{R}$. But by [1, Proposition 20, P. 321], all homogeneous components of xt are integral over $\mathcal{R}$. Again, by using [3, Lemma 2.1 (i)], we see that all homogeneous components of x belong to $Clos_R(f)$. Then the proof is completed. □

Corollary 2. *Let $R = \bigoplus_{n\in\mathbb{Z}} R_n$ be a graded ring and let $f = \{I_n\}_{n\geq 0}$ be a filtration of homogeneous ideals of R. Let M be a graded R-module. Then $Clos_R(f, M)$ is a homogeneous ideal of R.*

Proof. Let φ be the natural ring homomorphism $R \longrightarrow R/(0 :_R M) = \widetilde{R}$. By Proposition 3, $Clos_{\widetilde{R}}(\widetilde{f}) = Clos_R(f, M)/(0 :_R M)$ is a homogeneous ideal of $\widetilde{R}$. Therefore, $Clos_R(f, M)$ is a homogeneous ideal of R. □

Theorem 2. *Let R be a Noetherian ring and let M be a finitely generated R-module. Let $f = \{I_n\}_{n\geq 0}$ be a Noetherian filtration on R and $\mathcal{R} = \mathcal{R}(R, f)$ and $\mathcal{M} = \mathcal{R}(M, f)$. Then*

$$Clos_{\mathcal{R}}(\mathfrak{f}_{\mathcal{I}}^{(k)}, \mathcal{M})\mathcal{M} \cap M = Clos_R(f^{(k)}, M)M.$$

Proof. We know, the zero component of $Clos_{\mathcal{R}}(\mathfrak{f}_{\mathcal{I}}^{(k)}, \mathcal{M})\mathcal{M}$ is $(R \cap Clos_{\mathcal{R}}(\mathfrak{f}_{\mathcal{I}}^{(k)}, \mathcal{M}))M$. But by Theorem 1, we have

$$\begin{aligned} Clos_R(f^{(k)}, M)M &= (R \cap Clos_{\mathcal{R}}(\mathfrak{f}_{\mathcal{I}}^{(k)}, \mathcal{M}))M \\ &\subseteq Clos_{\mathcal{R}}(\mathfrak{f}_{\mathcal{I}}^{(k)}, \mathcal{M})\mathcal{M} \cap M. \end{aligned}$$

Let $m \in Clos_{\mathcal{R}}(\mathfrak{f}_{\mathcal{I}}^{(k)}, \mathcal{M})\mathcal{M} \cap M$. Since m is a homogeneous element of $Clos_{\mathcal{R}}(\mathfrak{f}_{\mathcal{I}}^{(k)}, \mathcal{M})\mathcal{M}$ of degree zero, it belongs to $Clos_R(f^{(k)}, M)M$ and the proof is completed. □

The following theorem is proved by applying the main idea used in the proof of theorem [10, Proposition 2.7].

Proposition 4. *Let R be a Noetherian ring and let M be a finitely generated R-module. Let $f = \{I_n\}_{n\geq 0}$ be a Noetherian filtration and $\mathcal{R} = \mathcal{R}(R, f)$ and $\mathcal{M} = \mathcal{R}(M, f)$. If*

$$p \in Ass_R(M/Clos_R(f^{(k)}, M)M),$$

then there exists

$$\mathcal{P} \in Ass_{\mathcal{R}}(\mathcal{M}/Clos_{\mathcal{R}}(\mathfrak{f}_{\mathcal{I}}^{(k)}, \mathcal{M})\mathcal{M})$$

such that $\mathcal{P} \cap R = p$.

Proof. Let

$$\mathcal{G} = \mathcal{M}/Clos_{\mathcal{R}}(\mathfrak{f}_{\mathcal{I}}^{(k)}, \mathcal{M})\mathcal{M} = \bigoplus_{n\in\mathbb{Z}} G_n.$$

By Theorem 2, we have

$$G_0 = M/Clos_R(f^{(k)}, M)M.$$

Let $p \in Ass_R(G_0)$. Then there is a non-zero element $g_0 \in G_0$ such that $p = (0 :_R g_0)$. Since $\mathcal{R}$ is a Noetherian ring, the zero submodule of $\mathcal{R}g_0$ has a minimal primary decomposition as $0 = \bigcap_{i=1}^n \mathcal{N}_i$, such that $\mathcal{N}_i$ is $\mathcal{P}_i$-primary homogeneous submodule of $\mathcal{R}g_0$ for every $1 \leq i \leq n$. By [5, Theorem 6.8 (ii)],

$$Ass_{\mathcal{R}}(\mathcal{R}g_0) = \{\mathcal{P}_1, \ldots, \mathcal{P}_n\}.$$

Since

$$\sqrt{(0 :_R g_0)} = R \cap \sqrt{(0 :_{\mathcal{R}} \mathcal{R}g_0)},$$

we have

$$p = R \cap \sqrt{(0 :_{\mathcal{R}} \mathcal{R}g_0)} = \bigcap_{\substack{i=1 \\ g_0 \notin \mathcal{N}_i}}^{n} (R \cap \sqrt{(\mathcal{N}_i :_{\mathcal{R}} \mathcal{R}g_0)}).$$

This implies that $p = R \cap \sqrt{(\mathcal{N}_i :_{\mathcal{R}} \mathcal{R}g_0)}$ for some $1 \leq i \leq n$ with $g_0 \notin \mathcal{N}_i$. Therefore, $p = R \cap \mathcal{P}_i$ and $\mathcal{P}_i \in Ass_{\mathcal{R}}(\mathcal{R}g_0) \subseteq Ass_{\mathcal{R}}(\mathcal{G})$. □

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