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On closedness of subvarieties of bands

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Abstract. In this paper, first we proved that all subvarieties of the variety of left (right) regular bands are closed in the variety of n-nilpotent extension of bands. Secondly, we proved the closedness of rectangular bands in the variety $\mathcal{V} = [ac = ab^nc]$ $(n \in \mathbf{N})$, of semigroups. Further, we have shown that all subvarieties of the variety of left (right) normal bands are closed in the variety $\mathcal{V} = [axy = a^py^qx^r]$ $(p, q, r \in \mathbf{N})$, of semigroups and lastly, we proved that all subvarieties of the variety of left (right) quasinormal bands are closed in the variety $\mathcal{V} = [axy = a^px^qa^ry]$ $(p, q, r \in \mathbf{N})$, of semigroups.

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1 Introduction and preliminaries

Fennemore, in [6] , had described the lattice of all varieties of bands and had given its diagram. After that, Petrich [11, Theorem II.5.1] had classified identities on bands in at most three variables. Though the varieties of semi-lattices and left (right) zero semi-groups are absolutely closed, but left (right) normal bands and rectangular bands are not absolutely closed (see Higgins [7, Chapter 4]). Therefore, it is important to find such sub-varieties of the variety of all semi-groups that are closed in itself or closed in the containing varieties of semi-groups. In [2–4, 12] authors had shown that left (right) regular band, left(right) quasi-normal band, left (right) semi-normal band and the normal band were closed respectively. Afterwards, authors in [1] had shown that the variety of all left (right) regular bands was closed in the variety of all bands respectively.

A semigroup identity $u = v$ is the formal equality of two words u and v formed by the letters over an alphabet set X. For any word u, the content of u (necessarily finite) is the set of all distinct variables appearing in u and is denoted by $C(u)$. The class of semigroups, in which a finite or an infinite collection $u_1 = v_1$, $u_2 = v_2$ $\cdots$ of identities is satisfied, is called the variety

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of semigroups determined by these identical relations. Let U be a subsemigroup of a semigroup S. Isbell [9] defined the dominion of U in S as

$$Dom(U,S) = \{d \in S : \forall \alpha, \beta : S \longrightarrow T, \text{ if } \alpha|_U = \beta|_U \implies d\alpha = d\beta\}.$$

It is well known that $Dom(U,S)$ is a subsemigroup of S containing U. If $Dom(U,S) = U$, then U is called closed in S; and absolutely closed if $Dom(U,S) = U$ in every containing semigroup S. If each member of a variety $\mathcal{V}$ is closed, then the variety $\mathcal{V}$ of semigroups is said to be closed. Let $\mathcal{V}_1$ and $\mathcal{V}_2$ be any varieties of semigroups such that $\mathcal{V}_1 \subseteq \mathcal{V}_2$. Then the variety $\mathcal{V}_1$ is said to be closed in the variety $\mathcal{V}_2$ if whenever a semigroup $U \in \mathcal{V}_1$ is a subsemigroup of a member S of $\mathcal{V}_2$, then U is closed in S. Obviously, if $\mathcal{V}_1$ is closed in $\mathcal{V}_2$, then all subvarieties of $\mathcal{V}_1$ are closed in containing subvarieties of $\mathcal{V}_2$. A variety $\mathcal{V}$ will be called absolutely closed if all its members are absolutely closed. Let S^n, for each $n \geq 1$, denote the set of all products of n elements of any semigroup S. If S^n belongs to a class $\mathcal{C}$ of semigroups for some $n \geq 1$, then we say that S is an n-nilpotent extension of a semigroup in $\mathcal{C}$.

The following theorem provided by Isbell [9], known as Isbell's zigzag theorem, is a most useful characterization of semigroup dominions and is of basic importance to our investigations.

Theorem 1 ([9], Theorem 2.3)**.** *Let U be a subsemigroup of a semigroup S and let $d \in S$. Then $d \in Dom(U,S)$ if and only if $d \in U$ or there exists a series of factorizations of d as follows:*

$$d = a_0t_1 = y_1a_1t_1 = y_1a_2t_2 = y_2a_3t_2 = \cdots = y_ma_{2m-1}t_m = y_ma_{2m} \qquad (*)$$

where $m \geq 1,\ a_i \in U\ (i = 0, 1, \ldots, 2m),\ \ y_i, t_i \in S\ (i = 1, 2, \ldots, m)$, *and*

$$a_0 = y_1a_1, \qquad a_{2m-1}t_m = a_{2m},$$
$$a_{2i-1}t_i = a_{2i}t_{i+1}, \qquad y_ia_{2i} = y_{i+1}a_{2i+1} \qquad (1 \leq i \leq m-1).$$

Such a series of factorizations is called a zigzag in S over U with value d, length m and spine $a_0, a_1, \ldots, a_{2m}$.

The following result is from Khan [10] and is also vital for our examinations.

Theorem 2 ([10], Result 3)**.** *Let U and S be semigroups with U as a subsemigroup of S. Take any $d \in S \backslash U$ such that $d \in Dom(U,S)$. Let (*) be a zigzag of shortest possible length m over U with value d. Then $t_j, y_j \in S \backslash U$ for all $j = 1, 2, \ldots, m$.*

Definition 1. *A semigroup S is called a band if $a^2 = a$ for all $a \in S$.*

Definition 2. *A semigroup S is called a n-nilpotent extension of band if S^n is a band for some $n \in \mathbf{N}$.*

Definition 3. *A semigroup S is called a rectangular band if $a = axa$ for all $a, x \in S$.*

Definition 4. *A semigroup S is called a Left (Right) normal band if $axy = ayx$ [$axy = xay$] for all $a, x, y \in S$.*

Definition 5. *A semigroup S is called a Left (Right) regular band if $ax = axa(xa = axa)$ for all $a, x \in S$.*

Definition 6. *A semigroup S is called a Left (Right) Quasinormal band if $axy = axay(axy = ayxy)$ for all $a, x, y \in S$.*

For a complete description of all varieties of bands, we refer the reader to Petrich [11]. Throughout this text, we will be using the semigroup theoretic notations and conventions established by Clifford and Preston [5], as well as Howie [8], without explicitly stating them.

2 Closedness of left (right) regular bands

In [1], Ahanger, Nabi and Shah had proved that the variety of left (right) regular bands is closed in the variety of bands. In this section, we improve their result and show that the variety of left (right) regular bands is closed in the variety of n- nilpotent extension of bands.

Lemma 1. *Let U be any band and S be an n- nilpotent extension of bands containing U with U as a subband. Let $d \in Dom(U, S)\backslash U$ has zigzag equations of type (*) in S over U of length m. If $n \geq 2$, then*

$$a_0a_2 = a_0a_2y_2a_3a_2a_0a_2.$$

Proof. For $n = 2$, we have

$$\begin{aligned}
a_0a_2 &= y_1a_1(a_2) \text{ (by equation (*))}\\
&= y_1a_1a_2a_2 \text{ (as } U \text{ is a band)}\\
&= y_1a_1(a_2)a_2y_1a_1a_2a_2\\
&\qquad \text{(as } S^2 \text{ is a band and } y_1a_1a_2a_2 \in S^2)\\
&= y_1a_1(a_2a_2a_2y_1)a_1a_2a_2 \text{ (since } U \text{ is a band)}\\
&= y_1a_1a_2a_2a_2(y_1a_2)a_2a_2y_1a_1a_2a_2\\
&\qquad \text{(as } S^2 \text{ is a band and } a_2a_2a_2y_1 \in S^2)\\
&= y_1a_1(a_2a_2a_2)y_2a_3(a_2a_2)y_1a_1(a_2a_2) \text{ (by equation (*))}\\
&= (y_1a_1)a_2y_2a_3a_2(y_1a_1)a_2 \text{ (as } U \text{ is a band)}\\
&= a_0a_2y_2a_3a_2a_0a_2 \text{ (by equation (*))}.
\end{aligned}$$

For $n = 3$, we have

$$\begin{aligned}
a_0a_2 &= y_1a_1(a_2) \text{ (by equation (*))}\\
&= y_1a_1a_2a_2a_2a_2 \text{ (as } U \text{ is a band)}\\
&= y_1a_1a_2(a_2)a_2a_2y_1a_1a_2a_2a_2a_2\\
&\qquad \text{(as } S^3 \text{ is a band and } y_1a_1a_2a_2a_2a_2 \in S^3)\\
&= y_1a_1(a_2a_2a_2a_2a_2y_1)a_1a_2a_2a_2a_2 \text{ (as } U \text{ is a band)}
\end{aligned}$$

$$
\begin{aligned}
&= y_1a_1a_2a_2a_2a_2a_2(y_1a_2)a_2a_2a_2a_2y_1a_1a_2a_2a_2a_2 \\
&\quad \text{(as } S^3 \text{ is a band and } a_2a_2a_2a_2a_2y_1 \in S^3) \\
&= y_1a_1(a_2a_2a_2a_2a_2)y_2a_3(a_2a_2a_2a_2)y_1a_1(a_2a_2a_2a_2) \text{ (by equation (*))} \\
&= (y_1a_1)a_2y_2a_3a_2(y_1a_1)a_2 \text{ (as } U \text{ is a band)} \\
&= a_0a_2y_2a_3a_2a_0a_2 \text{ (by equation (*))}.
\end{aligned}
$$

Now, we have

$$
\begin{aligned}
a_0a_2 &= y_1a_1(a_2) \text{ (by equation (*))} \\
&= y_1a_1a_2a_2^{n-3} \text{ (as } U \text{ is a band)} \\
&= y_1a_1(a_2)a_2^{n-3}y_1a_1a_2a_2^{n-3} \\
&\quad \text{(as } S^n \text{ is a band and } y_1a_1a_2a_2^{n-3} \in S^n) \\
&= y_1a_1(a_2a_2a_2^{n-3}y_1)a_1a_2a_2^{n-3} \text{ (as } U \text{ is a band)} \\
&= y_1a_1(a_2a_2a_2^{n-3})y_1a_1a_2(a_2^{n-3})y_1a_1(a_2a_2^{n-3}) \\
&\quad \text{(as } S^n \text{ is a band and } a_2a_2a_2^{n-3}y_1 \in S^n) \\
&= (y_1a_1)a_2(y_1a_2)a_2a_2(y_1a_1)a_2 \text{ (as } U \text{ is a band)} \\
&= a_0a_2y_2a_3(a_2a_2)a_0a_2 \text{ (by equation (*))} \\
&= a_0a_2y_2a_3a_2a_0a_2 \text{ (as } U \text{ is a band)},
\end{aligned}
$$

as required, □

Lemma 2. *Assume that U is a left regular band and S is an n-nilpotent extension of bands, with U being a subband of S. If there exists an element $d \in Dom(U,S)\backslash U$, such that d has zigzag equation of type (*) in S over U with a length of m. Then*

$$a_0a_2\cdots a_{2i} = a_0a_2\cdots a_{2i}y_{i+1}a_{2i+1}a_{2i}a_0a_2\cdots a_{2i-2} \ (i = 2,3,\ldots,m-1).$$

Proof. We prove it by induction on i. For i=1, we have

$$a_0a_2 = a_0a_2y_2a_3a_2a_0a_2 \text{ (by Lemma 1)}.$$

Thus the result is true for i=1. Assume, next, inductively that the result is true for i=k $(1 \le k < m-1)$. Then, we have

$$a_0a_2\cdots a_{2k} = a_0a_2\cdots a_{2k}y_{k+1}a_{2k+1}a_{2k}a_0a_2\cdots a_{2k-2}. \tag{1}$$

We now show that the result also holds for i=k+1. To show this, let

$$s_i = a_0a_2a_4\cdots a_{2i-2}a_{2i} \ (i \in \{k-1,k,k+1\}). \tag{2}$$

Then, equation (1) becomes

$$s_k = s_ky_{k+1}a_{2k+1}a_{2k}s_{k-1}. \tag{3}$$

With this notation, we need to show that

$$s_{k+1} = s_{k+1}y_{k+2}a_{2k+3}a_{2k+2}s_k.$$

As U is a left regular band, then for i=k and i=$k+1$, we have

$$s_i \quad = s_{i-1}a_{2i} \quad = s_{i-1}a_{2i}s_{i-1} = s_is_{i-1} \tag{4}$$

Now,

$$\begin{aligned}
s_{k+1} &= s_{k+1}s_k \text{ (by equation (4))}\\
&= s_ka_{2k+2}(s_k) \text{ (by equation (2))}\\
&= s_k(a_{2k+2})s_ky_{k+1}a_{2k+1}a_{2k}s_{k-1} \text{ (by equation (3))}\\
&= s_k(a_{2k+2}^{n-2}s_ky_{k+1})a_{2k+1}a_{2k}s_{k-1} \text{ (as } U \text{ is a band)}\\
&= s_ka_{2k+2}^{n-2}s_ky_{k+1}a_{2k+2}^{n-2}s_ky_{k+1}a_{2k+1}a_{2k}s_{k-1}\\
&\qquad \text{(as } S^n \text{ is a band and } a_{2k+2}^{n-2}s_ky_{k+1} \in S^n)\\
&= s_ka_{2k+2}s_k(y_{k+1}a_{2k+2})a_{2k+2}s_ky_{k+1}a_{2k+1}a_{2k}s_{k-1} \text{ (as } U \text{ is a band)}\\
&= (s_ka_{2k+2}s_k)y_{k+2}a_{2k+3}a_{2k+2}s_ky_{k+1}a_{2k+1}a_{2k}s_{k-1} \text{ (by equation (*))}\\
&= s_{k+1}y_{k+2}a_{2k+3}a_{2k+2}(s_ky_{k+1}a_{2k+1}a_{2k}s_{k-1}) \text{ (by equation (2))}\\
&= s_{k+1}y_{k+2}a_{2k+3}a_{2k+2}s_k \text{ (by equation (3))}
\end{aligned}$$

This shows that the result holds for $i = k+1$. Hence, by induction, the lemma follows.

□

Theorem 3. *Left regular band is closed in n-nilpotent extension of bands.*

Proof. Let U be any left regular band and S be any n-nilpotent extension of bands with U as a subband of S . Take any $d \in Dom(U,S)\backslash U$. Suppose that d has a zigzag of type (*) in S over U with value d of shortest possible length m. Now,

$$\begin{aligned}
d &= y_1(a_1)t_1 \text{ (by equation (*))}\\
&= (y_1a_1)(a_1t_1) \text{ (as } U \text{ is a band)}\\
&= (a_0a_2)t_2 \text{ (by equation (*))}\\
&= a_0a_2y_2(a_3(a_2a_0a_2))t_2 \text{ (by Lemma 1)}\\
&= (a_0a_2y_2a_3a_2a_0a_2)a_3t_2 \text{ (as } U \text{ is a left regular band)}\\
&= a_0a_2(a_3t_2) \text{ (by Lemma 1)}\\
&= (a_0a_2a_4)t_3 \text{ (by equation (*))}\\
&= a_0a_2a_4y_3(a_5(a_4a_0a_2))t_3 \text{ (by Lemma 2)}\\
&= (a_0a_2a_4y_3a_5a_4a_0a_2)a_5t_3 \text{ (as } U \text{ is a left regular band)}\\
&= a_0a_2a_4(a_5t_3) \text{ (by Lemma 2)}\\
&= a_0a_2a_4a_6t_4 \text{ (by equation (*))}
\end{aligned}$$

$$\vdots$$
$$= (a_0a_2a_4\cdots a_{2m-2})t_m$$
$$= a_0a_2a_4\cdots a_{2m-2}y_m(a_{2m-1}(a_{2m-2}a_0a_2a_4\cdots a_{2m-4}))t_m \text{ (by Lemma 2)}$$
$$= (a_0a_2a_4\cdots a_{2m-2}y_ma_{2m-1}a_{2m-2}a_0a_2a_4\cdots a_{2m-4})a_{2m-1}t_m$$
$$\text{(as } U \text{ is a left regular band)}$$
$$= a_0a_2a_4\cdots a_{2m-2}(a_{2m-1}t_m) \text{ (by Lemma 2)}$$
$$= a_0a_2a_4\cdots a_{2m-2}a_{2m} \text{ (by equation (*))} \in U$$

Therefore $Dom(U,S) = U$, and, hence, the theorem is proved. □

The following corollary can be deduced from Theorem 3.

Corollary 1. *The variety of all left (right) regular bands is closed in the variety of all n-nilpotent extension of bands.*

3 Closedness of rectangular bands

In general, varieties of bands containing the varieties of rectangular bands are not absolutely closed as Higgins [7, Chapter 4] gave an example to show that the variety of all rectangular bands is not absolutely closed. Therefore, for the varieties of semigroups, it is worthwhile to find largest subvarieties of the variety of all semigroups in which the variety of rectangular bands is closed. We have shown that the variety of rectangular bands is closed in the variety $\mathcal{V} = [ac = ab^nc]$ $(n \in \mathbf{N})$ of semigroups.

Theorem 4. *Rectangular band is closed in a semigroup satisfying an identity $ac = ab^nc$ $(n \in \mathbf{N})$.*

Proof. Let U be any rectangular band and S be any semigroup satisfying an identity $ac = ab^nc$ with U as a subsemigroup of S. Then we have to show that $Dom(U,S) = U$. Take any $d \in Dom(U,S)\backslash U$. Suppose that d has a zigzag of type (*) in S over U with value d of shortest possible length m. Now

$$d = y_1a_1t_1 \text{ (by equation (*))}$$
$$= y_1a_1(a_2t_2) \text{ (as } U \text{ is a band and by equation (*))}$$
$$= y_1a_1a_2a_3^nt_2 \text{ (as } S \text{ satisfies the identity } ac = ab^nc)$$
$$= (y_1a_1)a_2a_3^{n-1}(a_3t_2)$$
$$= a_0a_2a_3^{n-1}(a_4t_3) \text{ (by equation (*))}$$
$$= a_0a_2a_3^{n-1}a_4a_5^nt_3 \text{ (as } S \text{ satisfies the identity } ac = ab^nc)$$
$$= a_0a_2a_3^{n-1}a_4a_5^{n-1}a_5t_3$$
$$\vdots$$

$$= a_0 a_2 a_3^{n-1} a_4 a_5^{n-1} \cdots a_{2m-1}^{n-1}(a_{2m-1}t_m)$$
$$= a_0 a_2 a_3^{n-1} a_4 a_5^{n-1} \cdots a_{2m-1}^{n-1} a_{2m} \text{ (by equation (*))} \in U$$

Therefore $Dom(U,S) = U$, and, hence, the theorem is proved.

□

The following corollary can be deduced from Theorem 4.

Corollary 2. *The variety of all rectangular bands is closed in the variety* $\mathcal{V} = [ac = ab^n c]$ $(n \in \mathbf{N})$ *of semigroups.*

4 Closedness of left (right) normal bands

Generally, varieties of bands that contain the variety of left (right) normal bands are not absolutely closed. Higgins had shown that the variety of left (right) normal bands is not absolutely closed in [7, Chapter 4]. So, it is worthwhile to find largest subvarieties of the variety of all semigroups in which the varieties of left (right) normal bands are closed.

In this path, we have discovered that the variety $\mathcal{V} = [axy = a^p y^q x^r]$ $(p, q, r \in \mathbf{N})$ of semigroups is one such subvariety, in which the variety of left (right) normal bands is closed.

Theorem 5. *Left normal band is closed in a semigroup satisfying an identity* $axy = a^p y^q x^r$ $(p, q, r \in \mathbf{N})$.

Proof. Let U be any left normal band and S be any semigroup satisfying an identity $axy = a^p y^q x^r$ with U as a subsemigroup of S. Then we have to show that $Dom(U,S) = U$. Take any $d \in Dom(U,S) \backslash U$. Suppose that d has a zigzag of type (*) in S over U with value d of shortest possible length m. Now

$$\begin{aligned}
d &= y_1 a_1 t_1 \text{ (by equation (*))}\\
&= (y_1 a_1 a_2) t_2 \text{ (as } U \text{ is a band and by equation (*))}\\
&= y_1^p a_2^q a_1^r t_2 \text{ (as } S \text{ satisfies the identity } axy = a^p y^q x^r)\\
&= y_1^{p-1}(y_1 a_2) a_2^{q-1} a_1^r t_2\\
&= y_1^{p-1} y_2 (a_3) a_2^{q-1} a_1^r t_2 \text{ (by equation (*))}\\
&= y_1^{p-1} y_2 (a_3 a_3 (a_2^{q-1} a_1^r)) t_2 \text{ (as } U \text{ is a band)}\\
&= y_1^{p-1} y_2 a_3 a_2^{q-1} a_1^r (a_3 t_2) \text{ (as } U \text{ is a left normal band)}\\
&= y_1^{p-1} y_2 (a_3 (a_2^{q-1} a_1^r) a_4) t_3 \text{ (by equation (*))}\\
&= y_1^{p-1} (y_2 a_3 a_4) a_2^{q-1} a_1^r t_3 \text{ (as } U \text{ is a left normal band)}\\
&= y_1^{p-1} y_2^p a_4^q a_3^r a_2^{q-1} a_1^r t_3 \text{ (as } S \text{ satisfies the identity } axy = a^p y^q x^r)\\
&= y_1^{p-1} y_2^{p-1} y_2 a_4 (a_4^{q-1} a_3^r (a_2^{q-1} a_1^r)) t_3
\end{aligned}$$

$$
\begin{aligned}
&= y_1^{p-1} y_2^{p-1} y_2 a_4 a_4^{q-1} a_2^{q-1} a_1^r a_3^r t_3 \text{ (as } U \text{ is a left normal band)}\\
&\vdots\\
&= y_1^{p-1} \cdots y_{m-1}^{p-1} (y_{m-1} a_{2m-2}) a_{2m-2}^{q-1} \cdots a_2^{q-1} a_1^r \cdots a_{2m-3}^r t_m\\
&= y_1^{p-1} \cdots y_{m-1}^{p-1} y_m (a_{2m-1}) a_{2m-2}^{q-1} \cdots a_2^{q-1} a_1^r \cdots a_{2m-3}^r t_m \text{ (by equation (*))}\\
&= y_1^{p-1} \cdots y_{m-1}^{p-1} y_m (a_{2m-1} a_{2m-1} (a_{2m-2}^{q-1} \cdots a_2^{q-1} a_1^r \cdots a_{2m-3}^r)) t_m \text{ (as } U \text{ is a band)}\\
&= y_1^{p-1} \cdots y_{m-1}^{p-1} (y_m a_{2m-1}) a_{2m-2}^{q-1} \cdots a_2^{q-1} a_1^r \cdots a_{2m-3}^r (a_{2m-1} t_m)\\
&\qquad \text{(as } U \text{ is a left normal band)}\\
&= y_1^{p-1} \cdots y_{m-1}^{p-1} y_{m-1} a_{2m-2} a_{2m-2}^{q-1} \cdots a_2^{q-1} a_1^r \cdots a_{2m-3}^r a_{2m} \text{ (by equation (*))}\\
&= y_1^{p-1} \cdots y_{m-2}^{p-1} y_{m-1}^p (a_{2m-2}^q (a_{2m-4}^{q-1} \cdots a_2^{q-1} a_1^r \cdots a_{2m-5}^r) a_{2m-3}^r) a_{2m}\\
&= y_1^{p-1} \cdots y_{m-2}^{p-1} (y_{m-1}^p a_{2m-2}^q a_{2m-3}^r) a_{2m-4}^{q-1} \cdots a_2^{q-1} a_1^r \cdots a_{2m-5}^r a_{2m}\\
&\qquad \text{(as } U \text{ is a left normal band)}\\
&= y_1^{p-1} \cdots y_{m-2}^{p-1} (y_{m-1} a_{2m-3}) a_{2m-2} a_{2m-4}^{q-1} \cdots a_2^{q-1} a_1^r \cdots a_{2m-5}^r a_{2m}\\
&\qquad \text{(as } S \text{ satisfies the identity } axy = a^p y^q x^r)\\
&= y_1^{p-1} \cdots y_{m-2}^{p-1} y_{m-2} (a_{2m-4} a_{2m-2} (a_{2m-4}^{q-1} \cdots a_2^{q-1} a_1^r \cdots a_{2m-5}^r)) a_{2m}\\
&\qquad \text{(by equation (*))}\\
&= y_1^{p-1} \cdots y_{m-2}^{p-1} y_{m-2} a_{2m-4} a_{2m-4}^{q-1} \cdots a_2^{q-1} a_1^r \cdots a_{2m-5}^r a_{2m-2} a_{2m}\\
&\qquad \text{(as } U \text{ is a left normal band)}\\
&\vdots\\
&= y_1^{p-1} y_1 a_2 a_2^{q-1} a_1^r a_4 a_6 \cdots a_{2m-2} a_{2m}\\
&= (y_1^p a_2^q a_1^r) a_4 a_6 \cdots a_{2m-2} a_{2m}\\
&= (y_1 a_1) a_2 a_4 a_6 \cdots a_{2m-2} a_{2m} \text{ (as } S \text{ satisfies the identity } axy = a^p y^q x^r)\\
&= a_0 a_2 a_4 a_6 \cdots a_{2m-2} a_{2m} \text{ (by equation (*))} \in U
\end{aligned}
$$

Therefore $Dom(U, S) = U$, and, hence, the theorem is proved. □

The following corollary can be deduced from Theorem 5.

Corollary 3. *The variety of all left(right) normal bands is closed in the variety* $\mathcal{V} = [axy = a^p y^q x^r]$ $(p, q, r \in \mathbf{N})$ *of semigroups.*

5 Closedness of left(right) quasinormal bands

Likewise the previous section, we have discovered that the variety $axy = a^p x^q a^r y$ $(p, q, r \in \mathbf{N})$ of semigroups is one such subvariety, in which the variety of left(right) quasinormal bands is closed.

Theorem 6. *Left quasinormal band is closed in a semigroup satisfying an identity* $axy = a^p x^q a^r y$ $(p, q, r \in \mathbf{N})$.

Proof. Let U be any left quasinormal band and S be any semigroup satisfying an identity $axy = a^p x^q a^r y$ with U as a subsemigroup of S. Then we have to show that $Dom(U, S) = U$. Take any $d \in Dom(U, S) \setminus U$. Suppose that d has a zigzag of type (*) in S over U with value d of shortest possible length m. Now

$$\begin{aligned}
d &= y_1 a_1 t_1 \text{ (by equation (*))} \\
&= (y_1 a_1 a_2) t_2 \text{ (as } U \text{ is a band and by equation (*))} \\
&= y_1^p a_1^q y_1^r a_2 t_2 \text{ (as } S \text{ satisfies the identity } axy = a^p x^q a^r y) \\
&= y_1^p a_1^q y_1^{r-1} (y_1 a_2) t_2 \\
&= y_1^p a_1^q y_1^{r-1} y_2 a_3 t_2 \text{ (by equation (*))} \\
&= y_1^p a_1^q y_1^{r-1} (y_2 a_3 a_4) t_3 \text{ (as } U \text{ is a band and by equation (*))} \\
&= y_1^p a_1^q y_1^{r-1} y_2^p a_3^q y_2^r a_4 t_3 \text{ (as } S \text{ satisfies the identity } axy = a^p x^q a^r y) \\
&= y_1^p a_1^q y_1^{r-1} y_2^p a_3^q y_2^{r-1} y_2 a_4 t_3 \\
&\vdots \\
&= y_1^p a_1^q y_1^{r-1} \cdots y_{m-1}^p a_{2m-3}^q y_{m-1}^{r-1} (y_{m-1} a_{2m-2}) t_m \\
&= y_1^p a_1^q y_1^{r-1} \cdots y_{m-1}^p a_{2m-3}^q y_{m-1}^{r-1} y_m a_{2m-1} t_m \text{ (by equation (*))} \\
&= y_1^p a_1^q y_1^{r-1} \cdots y_{m-1}^p a_{2m-3}^q y_{m-1}^{r-1} y_{m-1} a_{2m-2} a_{2m} \\
&\qquad \text{(as } U \text{ is a band and by equation (*))} \\
&= y_1^p a_1^q y_1^{r-1} \cdots y_{m-2}^p a_{2m-5}^q y_{m-2}^{r-1} (y_{m-1}^p a_{2m-3}^q y_{m-1}^r a_{2m-2}) a_{2m} \\
&= y_1^p a_1^q y_1^{r-1} \cdots y_{m-2}^p a_{2m-5}^q y_{m-2}^{r-1} y_{m-1} a_{2m-3} a_{2m-2} a_{2m} \\
&\qquad \text{(as } S \text{ satisfies the identity } axy = a^p x^q a^r y) \\
&\vdots \\
&= y_1^p a_1^q y_1^{r-1} (y_2 a_3) a_4 a_6 \cdots a_{2m-2} a_{2m} \\
&= y_1^p a_1^q y_1^{r-1} y_1 a_2 a_4 a_6 \cdots a_{2m-2} a_{2m} \text{ (by equation (*))} \\
&= (y_1^p a_1^q y_1^r a_2) a_4 a_6 \cdots a_{2m-2} a_{2m} \\
&= (y_1 a_1) a_2 a_4 a_6 \cdots a_{2m-2} a_{2m} \text{ (as } S \text{ satisfies the identity } axy = a^p x^q a^r y) \\
&= a_0 a_2 a_4 a_6 \cdots a_{2m-2} a_{2m} \text{ (by equation (*))} \in U
\end{aligned}$$

Therefore $Dom(U, S) = U$, and, hence, the theorem is proved. □

The following corollary can be deduced from Theorem 6.

Corollary 4. *The variety of all left(right) quasinormal bands is closed in the variety* $\mathcal{V} = [axy = a^p x^q a^r y]$ $(p, q, r \in \mathbf{N})$ *of semigroups.*

6 Conclusion

Full determination of closed varieties still remains an open problem. However, we find out some varieties which are closed in containing varieties and it also helps in determining the structure of semigroups.

7 Open problems

We conclude this article with the following open problems:

Problem 1. Is the variety of all normal bands closed in the variety of n-nilpotent extension of bands?

Problem 2. Is the variety of all left (right) semiregular bands closed in the variety of all bands?

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