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On the unitary Cayley graphs of group rings

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Abstract. Let R be a ring. The unitary Cayley graph of a ring R, denoted by $\Gamma(R)$, is a graph with vertex set R, where two vertices $u, v \in R$ are adjacent if and only if $u - v$ is a unit of R. In this paper, we investigate the unitary Cayley graph of a finite ring, called a group ring, and examine its fundamental properties. We present the conditions for adjacency, the connectivity of the graph and its basic structure. Additionally, we provide the exact value of the degree of a vertex and the distance between any two vertices within the graph.

Keywords: Unitary Cayley graph, Group ring, Connectivity, Distance.
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1 Introduction

A group ring is an algebraic structure that combines a group and a ring with simple operations. The following definition is given by Milies and Sehgal [15].

Let $(G, *)$ be a group and $(R, \oplus, \odot)$ be a ring with the unity 1_R. For clarity, we use e_G to denote the identity of the group $(G, *)$ and 0_R to denote the identity element of the group $(R, \oplus)$. Let R_G be the set of elements of the form $a_{g_1}g_1 + a_{g_2}g_2 + \cdots + a_{g_n}g_n = \sum_{g \in G} a_g g$, where $a_g \in R$. We also allow for the possibility that some of the a_{g_i} could be zero or that some g_i might repeat, so an element of R_G can be written in different ways (for example, $a_1g_1 + 0_Rg_2 = a_1g_1$ or $a_1g_1 + b_1g_1 = (a_1 \oplus b_1)g_1$). For any two elements $\alpha = \sum_{g \in G} a_g g$ and $\beta = \sum_{g \in G} b_g g$ in R_G, we define

$$\alpha +' \beta = \sum_{g \in G} (a_g \oplus b_g)g \text{ and } \alpha\beta = \sum_{g,h \in G} (a_g \odot b_h)(g * h).$$

With these operations R_G form a ring, called the group ring of G over R (or R_G is a group ring). For clarity, we use $-\alpha$ to denote an inverse element of $\alpha \in R_G$ under the operation $+'$ and so $\alpha - \beta$ means $\alpha +' (-\beta)$.

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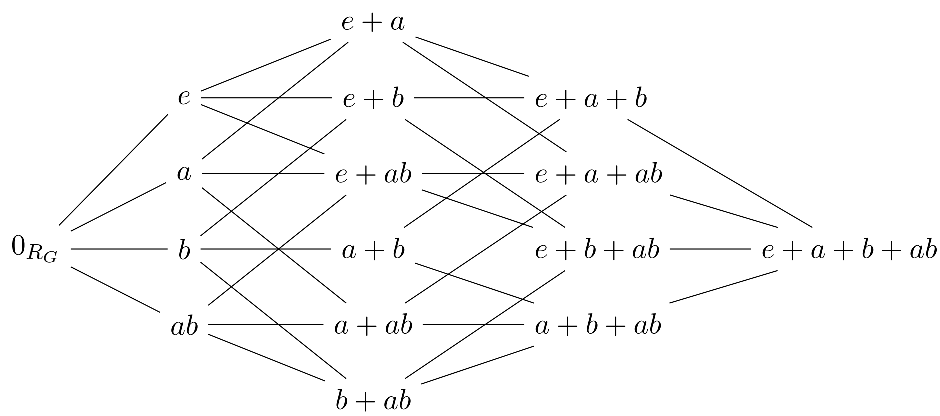

Figure 1: $\Gamma(R_G)$, where $R = (\mathbb{Z}_2, \oplus, \odot)$ and $G = V_4 = \{e, a, b, ab\}$.

An element u of a ring R is a unit if there exists $v \in R$ such that $uv = vu = 1_R$. It is straightforward to verify that $1_R e_G \left(\sum_{g\in G} a_g g\right) = \sum_{g\in G} a_g g = \left(\sum_{g\in G} a_g g\right) 1_R e_G$. Thus $1_R e_G$ is a unity element of the group ring R_G.

Group rings became widely recognized and studied after Noether's article [9] in 1929. In 2002, Milies and Sehgal [15] wrote a book called "An Introduction to Group Rings". In the section about group rings, at the end of a brief history, the authors mention Arthur Cayley's article [3], which states that if we think of the elements of a (finite) group as basic units of a hypercomplex system, the idea of a group ring appears for the first time. Currently, group rings have applications in many fields of study (see for examples [4], [7], [13]).

The unitary Cayley graph of a ring R, denoted by $\Gamma(R)$, is a graph with vertex set R and two vertices $u, v \in R$ are adjacent if and only if $u - v$ is a unit of R. The unitary Cayley graph of a ring is one of the most extensively studied structures in algebraic graph theory (see, [1, 2], [5], [8], [10–12], [14]).

Example 1. Let $R = (\mathbb{Z}_2, \oplus, \odot)$ be a ring and $G = V_4 = \{e, a, b, ab\}$ be the Klein 4-group. We get $R_G = \{0_{R_G}, e, a, b, ab, e+a, e+b, e+ab, a+b, a+ab, b+ab, e+a+b, e+a+ab, e+b+ab, a+b+ab, e+a+b+ab\}$, where $e = 1_R e_G$ is the unity and $U(R_G) = \{e, a, b, ab\}$ is the set of all unit elements of R_G. For example, we see that $e +' (-(e+a)) = e - (e+a) = (\bar{1} - \bar{1})e + (\bar{0} + \bar{1})a = a$. Thus e and $e + a$ are adjacent. The $\Gamma(R_G)$ can be pictured as Figure 1.

However, the study of the Cayley graphs of rings often focus on commutative rings. Throughout this paper, we let R be a division ring (not necessary commute), meaning that every nonzero element is a unit and no zero divisors. For more information about graphs, we refer to [6]. Additionally, we denote by $|X|$, the number of elements in the finite set X, $|R| = m$, $|G| = n$, and $\binom{n}{k}$ the binomial coefficients of integers $0 \leq k \leq n$.

In this study, we explore an algebraic graph, called the unitary Cayley graph, which originates from the work of Arthur Cayley. Moreover, the graph is associated with an algebraic structure introduced by Arthur Cayley in his article, called the group ring. This article presents several fundamental properties of the unitary Cayley graph of group rings, including the degree of a vertex, the connectivity of the graph, and the distance between two vertices.

2 Adjacent vertices and vertex degree

To obtain the results we mainly use the support of each $\alpha = \sum_{g\in G} a_g g$ of R_G that defined by $s(\alpha) = \{g \in G : a_g \neq 0_R\}$. The following two lemmas are needed in the sequel.

Lemma 1. *Let $\alpha, \beta \in R_G$. Then, the following conditions hold.*

i) $|s(\alpha +' \beta)| \leq |s(\alpha)| + |s(\beta)|$;

ii) $|s(\alpha\beta)| \geq max\{|s(\alpha)|, |s(\beta)|\}$.

Proof. Let $\alpha, \beta \in R_G$.

i) It is easy to check that, $s(\alpha +' \beta) \subseteq s(\alpha) \cup s(\beta)$.
Thus, $|s(\alpha +' \beta)| \leq |s(\alpha) \cup s(\beta)| \leq |s(\alpha)| + |s(\beta)|$.

ii) Since $s(\alpha\beta) = \{gh : g \in s(\alpha), h \in s(\beta)\}$, $|s(\alpha\beta)| = |\{gh : g \in s(\alpha), h \in s(\beta)\}|$. From G is a group, we have $gh_1 = gh_2$ if and only if $h_1 = h_2$ for all $g, h_1, h_2 \in G$. Then $|\{gh : g \in s(\alpha), h \in s(\beta)\}| \geq \max\{|s(\alpha)|, |s(\beta)|\}$. □

Lemma 2. *Let $\alpha \in R_G$. Then α is a unit if and only if $|s(\alpha)| = 1$.*

Proof. Let α be a unit. Then there exist $\alpha^{-1} \in R_G$ such that $\alpha\alpha^{-1} = 1_R e_G$. Thus, $s(\alpha\alpha^{-1}) = s(1_R e_G) = \{e_G\}$. Since $1 = |s(\alpha\alpha^{-1})| \geq \max\{|s(\alpha)|, |s(\alpha^{-1})|\}$, we get $|s(\alpha)| = |s(\alpha^{-1})| = 1$.

Conversely, assume that $|s(\alpha)| = 1$. Then we put $\alpha = a_g g$ for some $a_g \in R, g \in G$. Because R is a division ring, there exists $\beta = a_g^{-1}(-g) \in R_G$ such that $\alpha\beta = (a_g g)(a_g^{-1}(-g)) = a_g a_g^{-1}(g - g) = 1_R e_G$. Therefore, α is a unit in R_G. □

By the definition of the unitary Cayley graph, we conclude that $\alpha, \beta \in V(\Gamma(R_G))$ are adjacent if and only if $\alpha - \beta$ is a unit if and only if $|s(\alpha - \beta)| = 1$.

In the following discussion, we classify an element of R_G by $S_k = \{\alpha \in R_G : |s(\alpha)| = k\}$ for $1 \leq k \leq n$ and we put $S_0 = \{0_{R_G}\}$. Clearly, $|S_k| = (m-1)^k \frac{n!}{(n-k)!}$. In this part, the conditions for vertex adjacency and the degree of a vertex of the unitary Cayley graph $\Gamma(R_G)$ are presented.

Lemma 3. *Let $\alpha, \beta \in R_G$, where $\alpha = \sum_{g\in G} a_g g$ and $\beta = \sum_{g\in G} b_g g$. If $|s(\alpha - \beta)| = 1$, then $|(s(\alpha) \cup s(\beta)) \setminus (s(\alpha) \cap s(\beta))| \leq 1$.*

Proof. Let $|s(\alpha - \beta)| = 1$. Assume, to the contrary, that $|(s(\alpha) \cup s(\beta)) \setminus (s(\alpha) \cap s(\beta))| \geq 2$. Then there exists $g, g' \in (s(\alpha) \cup s(\beta)) \setminus (s(\alpha) \cap s(\beta))$, where $g \neq g'$, this means that $g, g' \in (s(\alpha) \cup s(\beta))$ and $g, g' \notin (s(\alpha) \cap s(\beta))$. We consider two cases:

Case 1: $g, g' \in s(\alpha)$. Thus $a_g, a_{g'} \notin \{0_R\}$. From $g, g' \notin s(\beta)$, we get $b_g = b_{g'} = 0_R$ and so $g, g' \in s(\alpha - \beta)$. Then $|s(\alpha - \beta)| \geq 2$, it contradicts to the assumption.

Case 2: $g \in s(\alpha)$ and $g' \in s(\beta)$. Then $a_g \neq 0_R$ and $b_{g'} \neq 0_R$. Since $g, g' \notin (s(\alpha) \cap s(\beta))$, it follows that $a_{g'} = b_g = 0_R$. Thus $a_g, b_{g'} \in s(\alpha - \beta)$ this implies that $|s(\alpha - \beta)| \geq 2$, a contradiction.

The case where $g, g' \in s(\beta)$ and $g' \in s(\alpha)$, $g \in s(\beta)$ can be proved in the same way. Therefore, $|(s(\alpha) \cup s(\beta)) \setminus (s(\alpha) \cap s(\beta))| \leq 1$, as required. □

The following theorem, we present a necessary and sufficient condition for the adjacent vertices in R_G.

Theorem 1. *Let $\alpha, \beta \in R_G$, where $\alpha = \sum_{g\in G} a_g g$ and $\beta = \sum_{g\in G} b_g g$. Then $\alpha, \beta \in V(\Gamma(R_G))$ are adjacent if and only if the following conditions hold;*

i) $|(s(\alpha) \cup s(\beta)) \setminus (s(\alpha) \cap s(\beta))| \leq 1$;

ii) if $s(\alpha) = s(\beta)$, then there exists a unique $g' \in s(\beta)$ such that $b_{g'} \neq a_{g'}$ and $b_g = a_g$ for all $g \neq g'$;

iii) if $|(s(\alpha) \cup s(\beta)) \setminus (s(\alpha) \cap s(\beta))| = 1$, then $b_g = a_g$ for all $g \in s(\beta) \cap s(\alpha)$.

Proof. Let $\alpha, \beta \in R_G$ such that $\alpha = \sum_{g\in G} a_g g$ and $\beta = \sum_{g\in G} b_g g$. Assume that $\alpha, \beta \in V(\Gamma(R_G))$ are adjacent. Then $\alpha - \beta$ is a unit of R_G and so $|s(\alpha - \beta)| = 1$. Clearly, *i)* is obtained by Lemma 3.

Now, suppose that $s(\alpha) = s(\beta)$. Assume, to the contrary, that $a_g = b_g$ for all $g \in s(\beta) = s(\alpha)$. Then $\alpha - \beta = \sum_{g\in G}(a_g - b_g)g = \sum_{g\in G}(a_g - a_g)g = 0_{R_G}$, this means that $s(\alpha - \beta) = \emptyset$. This contradicts our assumption that $|s(\alpha - \beta)| = 1$. Thus, there exists $g' \in s(\beta)$ such that $a_{g'} \neq b_{g'}$. In addition, if there exists $h \neq g'$ such that $a_h \neq b_h$ and $c_{g'} \neq d_{g'}$, we get $g', h \in s(\alpha - \beta)$. Then $|s(\alpha - \beta)| \geq 2$, a contradiction. Therefore the condition *ii)* holds.

Suppose that $|(s(\alpha) \cup s(\beta)) \setminus (s(\alpha) \cap s(\beta))| = 1$. Then there exists $h \in (s(\alpha) \cup s(\beta)) \setminus (s(\alpha) \cap s(\beta))$. Assume, to the contrary, that $b_g \neq a_g$ for some $g \in s(\beta) \cap s(\alpha)$. Let $h \in (s(\alpha) \cup s(\beta)) \setminus (s(\alpha) \cap s(\beta))$. Then either $h \in s(\alpha)$ or $h \in s(\beta)$. Consequently, $h \in s(\alpha - \beta)$. Choose $g \in s(\beta) \cap s(\alpha)$ such that $a_g \neq b_g$, then $g \in s(\alpha - \beta)$ and so $g \neq h$. Hence $h, g \in s(\alpha - \beta)$ which contradicts the fact that $|s(\alpha - \beta)| = 1$. Therefore *iii)* holds.

Conversely, assume the conditions hold. We shall show that $|s(\alpha - \beta)| = 1$. Since $|(s(\alpha) \cup s(\beta)) \setminus (s(\alpha) \cap s(\beta))| \leq 1$, $|(s(\alpha) \cup s(\beta)) \setminus (s(\alpha) \cap s(\beta))| \in \{0, 1\}$. We consider two cases:

Case 1: $|(s(\alpha) \cup s(\beta)) \setminus (s(\alpha) \cap s(\beta))| = 0$. Then $s(\alpha) = s(\beta)$. By using *ii)* we obtain that there exists $g' \in s(\beta) = s(\alpha)$ such that $a_{g'} \neq b_{g'}$ and $b_g = a_g$ for all $g \in s(\beta) \setminus \{g'\}$. Then $a_{g'} - b_{g'} \neq 0_R$ and $a_g - b_g = 0_R$ for all $g \in s(\beta) \setminus \{g'\}$. Consequently, $g' \in s(\alpha - \beta)$ and $g \notin s(\alpha - \beta)$ for all $g \in s(\beta) \setminus \{g'\}$. Hence $s(\alpha - \beta) = \{g'\}$. Therefore $|s(\alpha - \beta)| = 1$, as required.

Case 2: $|(s(\alpha) \cup s(\beta)) \setminus (s(\alpha) \cap s(\beta))| = 1$. Let $g' \in (s(\alpha) \cup s(\beta)) \setminus (s(\alpha) \cap s(\beta))$. Consequently, either $g' \in s(\alpha)$ or $g' \in s(\beta)$. By using *iii)*, $a_g - b_g = 0_R$ for all $g \in s(\alpha) \cap s(\beta)$ and $g' \in s(\alpha - \beta)$. Hence, $s(\alpha - \beta) = \{g'\}$ and so $|s(\alpha - \beta)| = 1$. We then obtain $\{\alpha, \beta\} \in E(\Gamma(R_G))$, as required. □

Lemma 4. *Let $\alpha \in V(\Gamma(R_G))$ and $\alpha \in S_k$, where $1 \leq k \leq n-1$. Then $N(\alpha) \subseteq S_{k-1} \cup S_k \cup S_{k+1}$, where $N(\alpha) = \{\beta \in V(\Gamma(R_G)) : \{\alpha, \beta\} \in E(\Gamma(R_G))\}$.*

Proof. Let $\beta \in N(\alpha)$. Then $\{\alpha, \beta\} \in E(\Gamma(R_G))$. By Theorem 1 we obtain $|(s(\alpha) \cup s(\beta)) \setminus (s(\alpha) \cap s(\beta))| \leq 1$ and so $|(s(\alpha) \setminus s(\beta))| \in \{0, 1\}$ or $|(s(\beta) \setminus s(\alpha))| \in \{0, 1\}$. We shall focus on the case $|(s(\alpha) \setminus s(\beta))| \in \{0, 1\}$: the other case can be showed in the same way. Now, we consider two following cases:

Case 1: $|s(\alpha) \setminus s(\beta)| = 1$. Let $g \in s(\alpha) \setminus s(\beta)$. If there exists $h \in G$ such that $h \in s(\beta)$ and $h \notin s(\alpha)$, then $g, h \in (s(\alpha) \cup s(\beta)) \setminus (s(\alpha) \cap s(\beta))$ and so $|(s(\alpha) \cup s(\beta)) \setminus (s(\alpha) \cap s(\beta))| \geq 2$, a contradiction. Thus $s(\beta) \subsetneq s(\alpha)$. From $|s(\alpha) \setminus s(\beta)| = 1$, it follows that $\beta \in S_{k-1}$.

Case 2: $|s(\alpha) \setminus s(\beta)| = 0$. We have either $s(\alpha) = s(\beta)$ or $s(\alpha) \subsetneq s(\beta)$. Clearly, if $s(\alpha) = s(\beta)$, then $\beta \in S_k$. In the case where $s(\alpha) \subsetneq s(\beta)$, we get $|(s(\alpha) \cup s(\beta)) \setminus (s(\alpha) \cap s(\beta))| = |(s(\alpha) \cup s(\beta)) \setminus s(\alpha)| \leq 1$. Since $s(\alpha) \subsetneq s(\beta)$, $|s(\beta) \setminus s(\alpha)| = 1$. Thus $\beta \in S_{k+1}$. □

Lemma 5. *Let $\alpha \in R_G$ such that $\alpha \in S_k$ and $\beta \in N(\alpha)$, where $1 \leq k \leq n-1$. Then the following conditions hold:*

i) $\beta \in S_{k-1}$ *if and only if* $s(\beta) \subsetneq s(\alpha)$*;*

ii) $\beta \in S_k$ *if and only if* $s(\beta) = s(\alpha)$*;*

iii) $\beta \in S_{k+1}$ *if and only if* $s(\alpha) \subsetneq s(\beta)$*.*

Proof. Let $\beta \in N(\alpha)$. Then $\{\alpha, \beta\} \in E(\Gamma(R_G))$.

i) Let $\beta \in S_{k-1}$. Then $|s(\alpha)| - |s(\beta)| = 1$ it follows that there exists $g \in s(\alpha)$ such that $g \notin s(\beta)$. Assume, to the contrary, that $s(\beta) \nsubseteq s(\alpha)$, then there exists $g' \in s(\beta)$ such that $g' \notin s(\alpha)$. Clearly, $g \neq g'$. Thus $\{g, g'\} \subseteq (s(\alpha) \cup s(\beta)) \setminus (s(\alpha) \cap s(\beta))$. It follows that $|(s(\alpha) \cup s(\beta)) \setminus (s(\alpha) \cap s(\beta))| \geq 2$, contrary to $\{\alpha, \beta\} \in E(\Gamma(R_G))$. Therefore $s(\beta) \subseteq s(\alpha)$.

Conversely, let $s(\beta) \subsetneq s(\alpha)$. Clearly, if $|s(\alpha) \setminus s(\beta)| \geq 2$, then $|(s(\alpha) \cup s(\beta)) \setminus (s(\alpha) \cap s(\beta))| \geq 2$, contrary to $\beta \in N(\alpha)$. Thus $|s(\alpha) \setminus s(\beta)| \leq 1$. It follows that $|s(\alpha) \setminus s(\beta)| = 1$. Therefore $\beta \in S_{k-1}$, as required.

ii) Let $\beta \in S_k$. Assume, to the contrary, that $s(\beta) \neq s(\alpha)$. Since $|s(\alpha)| = |s(\beta)|$, there exist $g, g' \in G$, where $g \neq g'$, such that $g \in s(\alpha), g' \in s(\beta)$ and $g' \notin s(\alpha), g \notin s(\beta)$. Thus $\{g, g'\} \subseteq (s(\alpha) \cup s(\beta)) \setminus (s(\alpha) \cap s(\beta))$. It follows that $|(s(\alpha) \cup s(\beta)) \setminus (s(\alpha) \cap s(\beta))| \geq 2$, contrary to $\{\alpha, \beta\} \in E(\Gamma(R_G))$. Therefore $s(\beta) = s(\alpha)$.

Conversely, let $s(\beta) = s(\alpha)$. Then $\beta \in S_k$, because $\alpha \in S_k$.

iii) Let $\beta \in S_{k+1}$. Then $|s(\beta)| - |s(\alpha)| = 1$ which gives there exists $g' \in s(\beta)$ such that $g' \notin s(\alpha)$. Assume, to the contrary, that there exists $g \in s(\alpha)$ such that $g \notin s(\beta)$. Clearly, $g \neq g'$. Thus $\{g, g'\} \subseteq (s(\alpha) \cup s(\beta)) \setminus (s(\alpha) \cap s(\beta))$ and so $|(s(\alpha) \cup s(\beta)) \setminus (s(\alpha) \cap s(\beta))| \geq 2$, contrary to $\{\alpha, \beta\} \in E(\Gamma(R_G))$.

Conversely, let $s(\alpha) \subsetneq s(\beta)$. Clearly, if $|s(\beta) \setminus s(\alpha)| \geq 2$, then $|(s(\alpha) \cup s(\beta)) \setminus (s(\alpha) \cap s(\beta))| \geq 2$, contrary to $\beta \in N(\alpha)$. Thus $|s(\beta) \setminus s(\alpha)| \leq 1$ and so $|s(\beta) \setminus s(\alpha)| = 1$. Therefore $\beta \in S_{k+1}$, as required. □

Lemma 6. *Let $\alpha \in R_G$, where $\alpha = \sum_{g \in G} a_g g \in S_k$ and $2 \leq k \leq n-1$. Then*

i) $N(\alpha) \cap S_{k-1} = \left\{ \sum_{g \in A} a_g g : A \subsetneq s(\alpha), |A| = |s(\alpha)| - 1 \right\}$;

ii) $N(\alpha) \cap S_k = \left\{ \left(\sum_{g \in s(\alpha) \setminus \{h\}} a_g g \right) + b_h h : h \in s(\alpha), b_h \in R \setminus \{0_R, a_h\} \right\}$;

iii) $N(\alpha) \cap S_{k+1} = \left\{ \left(\sum_{g \in s(\alpha)} a_g g \right) + b_h h : h \notin s(\alpha), b_h \neq 0_R \right\}$.

Proof. Let $\alpha = \sum_{g\in G} a_g g \in S_k$.

i) Let $\beta \in N(\alpha) \cap S_{k-1}$. By using Lemma 5 and Theorem 1, we get $s(\beta) \subsetneq s(\alpha)$ and $|s(\alpha) \setminus s(\beta)| = 1$. Thus $\beta = \sum_{g\in A} a_g g$ for some $A \subsetneq s(\alpha)$, where $|A| = |s(\alpha)| - 1$. Hence $\beta \in \left\{\sum_{g\in A} a_g g : A \subsetneq s(\alpha), |A| = |s(\alpha)| - 1\right\}$.

On the other hand, let $\beta = \sum_{g\in A} a_g g$ for some $A \subsetneq s(\alpha)$, where $|A| = |s(\alpha)| - 1 = k - 1$. We obtain $\alpha - \beta = a_{g'} g'$, where $g' \in s(\alpha) \setminus A$. It follows that $\beta \in N(\alpha) \cap S_{k-1}$. Therefore $N(\alpha) \cap S_{k-1} = \left\{\sum_{g\in A} a_g g : A \subsetneq s(\alpha), |A| = |s(\alpha)| - 1\right\}$.

ii) Let $\beta \in N(\alpha) \cap S_k$. By using Lemma 5, we obtain $s(\alpha) = s(\beta)$. From Theorem 1, there exists $h \in s(\alpha)$ such that $b_h \notin \{0_R, a_h\}$. Thus $\beta = \left(\sum_{g\in s(\alpha)\setminus\{h\}} a_g g\right) + b_h h$ where $h \in s(\alpha)$ and $b_h \in R \setminus \{0_R, a_h\}$. Hence
$\beta \in \left\{\left(\sum_{g\in s(\alpha)\setminus\{h\}} a_g g\right) + b_h h : h \in s(\alpha), b_h \in R \setminus \{0_R, a_h\}\right\}$.

Conversely, let $\beta = \sum_{g\in s(\alpha)\setminus\{h\}} a_g g + b_h h$, where $h \in s(\alpha)$ and $b_h \in R \setminus \{0_R, a_h\}$. Then $\beta \in S_k$. Furthermore, we obtain $\alpha - \beta = a_h h - b_h h = (a_h - b_h)h$, where $a_h - b_h \neq 0_R$, it follows that $\beta \in N(\alpha)$. Therefore $N(\alpha) \cap S_k = \{\sum_{g\in s(\alpha)\setminus\{h\}} a_g g + b_h h : h \in s(\alpha), b_h \in R \setminus \{0_R, a_h\}\}$.

iii) Let $\beta \in N(\alpha) \cap S_{k+1}$. By using Lemma 5 and Theorem 1, we get $s(\alpha) \subsetneq s(\beta)$ and $|s(\beta) \setminus s(\alpha)| = 1$. Thus $\beta = \left(\sum_{g\in s(\alpha)} a_g g\right) + b_h h$, where $h \notin s(\alpha)$ and $b_h \neq 0_R$. Hence $\beta \in \left\{\left(\sum_{g\in s(\alpha)} a_g g\right) + b_h h : h \notin s(\alpha), b_h \neq 0_R\right\}$.

On the other hand, let $\beta = \sum_{g\in s(\alpha)} a_g g + b_h h$, where $h \notin s(\alpha)$. We then obtain $\beta \in S_{k+1}$ and $\alpha - \beta = -b_h h$. Therefore $\beta \in N(\alpha) \cap S_{k+1}$. □

Theorem 2. *Let $\alpha \in R_G$ and $\alpha = \sum_{g\in G} a_g g \in S_k$. Then $d(\alpha) = n(m-1)$ for all $0 \leq k \leq n$.*

Proof. We split our proof into four cases:

Case 1: $2 \leq k \leq n-1$. By using Lemma 4, we obtain $d(\alpha) = |N(\alpha) \cap S_{k-1}| + |N(\alpha) \cap S_k| + |N(\alpha) \cap S_{k+1}|$. From $|R| = m$, $|G| = n$ and Lemma 6, we get $|N(\alpha) \cap S_{k-1}| = \binom{k}{k-1}$, $|N(\alpha) \cap S_k| = \binom{k}{1}(m-2)$ and $|N(\alpha) \cap S_{k+1}| = \binom{n-k}{1}(m-1)$. Therefore $d(\alpha) = k + k(m-2) + (n-k)(m-1) = n(m-1)$ for all $2 \leq k \leq n-1$.

Case 2: $k = 0$. By using Lemma 4, we obtain $N(\alpha) \subseteq S_1$. Thus $d(\alpha) = |N(\alpha) \cap S_{0+1}| = \binom{n-0}{1}(m-1) = n(m-1)$.

Case 3: $k = 1$. From $S_0 = \{0_{R_G}\}$ and $\{\alpha, 0_{R_G}\}$ is an edge for every $\alpha \in S_1$, we then obtain $N(\alpha) \cap S_0 = \{0_{R_G}\}$. Thus $d(\alpha) = |N(\alpha) \cap S_0| + |N(\alpha) \cap S_1| + |N(\alpha) \cap S_2| = 1 + (m-2) + (n-1)(m-1) = n(m-1)$.

Case 4: $k = n$. By using Lemma 4, we obtain $N(\alpha) \subseteq S_{n-1} \cup S_n$. Thus $d(\alpha) = |N(\alpha) \cap S_{n-1}| + |N(\alpha) \cap S_n| = n + n(m-2) = n(m-1)$. □

3 Connectedness

For each $\alpha, \beta \in R_G$ we define a relation $\sim$ by

$$\alpha \sim \beta \text{ if and only if } s(\alpha) = s(\beta).$$

It is easily seen that, $\sim$ is an equivalence relation on R_G. Consequently, the set of all distinct equivalence classes under $\sim$ forms a partition of R_G. We denote $\bar{\alpha} = \{\beta \in R_G : \beta \sim \alpha\}$ the equivalence class of α. Furthermore, we put $T_\alpha = \{a_g g : g \in s(\alpha)\}$.

In the following discussion, we investigate the connectivity of the unitary Cayley graph of group ring and show that $\Gamma(R_G)$ is a connected graph.

Lemma 7. *Let $\alpha \in R_G$ and $\alpha = \sum_{g\in G} a_g g \in S_k$. Then an induced subgraph $\Gamma(R_G)[\bar{\alpha}]$ is a connected subgraph of $\Gamma(R_G)$.*

Proof. Let $\alpha = \sum_{g\in G} a_g g$. We shall show that there is a path from α to β for every $\beta \in \bar{\alpha}$. Assume that $\beta = \sum_{g\in G} b_g g \in \bar{\alpha}$, where $\beta \neq \alpha$. Then $s(\beta) = s(\alpha)$. We let $\{a'_{h_1} h_1, a'_{h_2} h_2, \ldots, a'_{h_t} h_t\}$ be a set of all elements of T_α such that $a'_{h_i} \neq b_{h_i}$, where $b_{h_i} h_i \in T_\beta$ and we define $T' = \{h_1, h_2, \ldots, h_t\}$. Then $\alpha = \alpha_0 = \left(\sum_{g\in s(\alpha)\setminus T'} a_g g\right) + a'_{h_1} h_1 + a'_{h_2} h_2 + \cdots + a'_{h_t} h_t$. For each $1 \leq i \leq t$, we set $\alpha_i = \left(\sum_{g\in s(\alpha)\setminus T'} a_g g\right) + \left(\sum_{j=1}^{i} b_{h_j} h_j\right) + a'_{h_{i+1}} h_{i+1} + \cdots + a'_{h_t} h_t$ and so $\alpha_i \in \bar{\alpha}$. Thus $\{\alpha_0,\, \alpha_1\}, \{\alpha_i, \alpha_{i+1}\} \in E(\Gamma(R_G))$ for all $1 \leq i \leq t-1$ and $\beta = \alpha_t$. It follows that there exists a path from α to β for all $\beta \in \bar{\alpha}$. Then for any $\beta, \beta' \in \bar{\alpha}$, there exists a path from β to α and a path from α to β'. Hence there exists a path from β to β' for all $\beta, \beta' \in \bar{\alpha}$. Therefore $\Gamma(R_G)[\bar{\alpha}]$ is a connected subgraph of $\Gamma(R_G)$, as required. □

Lemma 8. *Let $\alpha, \beta \in V(\Gamma(R_G))$. If $s(\alpha) \subseteq s(\beta)$, then α, β are connected.*

Proof. Let $\alpha = \sum_{g\in s(\alpha)} a_g g$, $\beta = \sum_{g\in s(\beta)} b_g g \in R_G$ such that $s(\alpha) \subseteq s(\beta)$. Clearly, if $s(\alpha) = s(\beta)$, then α, β are connected by Lemma 7. suppose that $|s(\alpha)| = k$ for some $1 \leq k \leq n$. We consider the case where $s(\alpha) \subsetneq s(\beta)$. Assume that $|s(\beta) \setminus s(\alpha)| = t$ for some $1 \leq t \leq |s(\beta)| - 1$ then we put $T_\beta \setminus T_\alpha = \{a_{h_1} h_1, a_{h_2} h_2, \ldots, a_{h_t} h_t\}$. Let $\alpha_i = \sum_{g\in s(\alpha)} a_g g + a_{h_1} h_1 + a_{h_2} h_2 + \cdots + a_{h_t} h_t$, where $a_{h_j} h_j \in T_\beta \setminus T_\alpha$ for all $1 \leq j \leq t$. Let $\alpha = \alpha_0$. We have $\{\alpha_{i-1}, \alpha_i\} \in E(\Gamma(R_G))$ for all $1 \leq i \leq t-1$. Consequently, we obtain that $\{\alpha_{t-1}, \beta\} \in E(\Gamma(R_G))$. Therefore there exists a path from α to β and so α, β are connected, as required. □

Theorem 3. *$\Gamma(R_G)$ is a connected graph.*

Proof. Let $\alpha, \beta \in V(\Gamma(R_G))$, we shall show that there exists a path from α to β. Let $\lambda \in V(\Gamma(R_G))$ such that $s(\lambda) = G$. Thus, $s(\alpha) \subseteq s(\lambda)$ and $s(\beta) \subseteq s(\lambda)$. By Lemma 8, we obtain there exists a path from α to λ and a path from β to λ. We conclude that there exist a path from α to β for all $\alpha, \beta \in V(\Gamma(R_G))$ and so $\Gamma(R_G)$ is a connected graph, as required. □

4 Distance between vertices

Recall that the distance between two vertices $\alpha, \beta \in R_G$, denoted by $d(\alpha, \beta)$, is the length of the shortest path from α to β. In this section, we begin by presenting the relation on of $|s(\alpha - \beta)|$, $|s(\alpha) \cup s(\beta)|$, and $|T_\alpha \cap T_\beta|$ which are used to investigate the distance $d(\alpha, \beta)$.

Lemma 9. *Let $\alpha, \beta \in R_G$. Then $|s(\alpha - \beta)| = |s(\alpha) \cup s(\beta)| - |T_\alpha \cap T_\beta|$*

Proof. Let $s(\alpha,\beta) := \{g : a_g g \in T_\alpha \cap T_\beta\}$. Then $|s(\alpha,\beta)| = |T_\alpha \cap T_\beta|$. We see that if $s(\alpha-\beta) = (s(\alpha)\cup s(\beta)) \setminus s(\alpha,\beta)$, then $|s(\alpha-\beta)| = |s(\alpha)\cup s(\beta)| - |T_\alpha \cap T_\beta|$. We let $g \in s(\alpha-\beta)$. Clearly, $g \in s(\alpha)\cup s(\beta)$. If $g \in s(\alpha,\beta)$, then $a_g g \in T_\alpha \cap T_\beta$ and so $g \notin s(\alpha-\beta)$. Thus $g \notin s(\alpha-\beta)$. It follows that $s(\alpha-\beta) \subseteq (s(\alpha)\cup s(\beta)) \setminus s(\alpha,\beta)$.

Conversely, let $g \in (s(\alpha)\cup s(\beta)) \setminus s(\alpha,\beta)$ and $g \in s(\alpha)$. Thus $a_g \neq 0$. Since $g \notin T(\alpha,\beta)$, $b_g \neq a_g$, where $b_g g \in T_\beta$. Hence $a_g - b_g \neq 0$ and so $g \in s(\alpha-\beta)$. Thus $s(\alpha-\beta) = (s(\alpha)\cup s(\beta)) \setminus s(\alpha,\beta)$. Therefore $|s(\alpha-\beta)| = |s(\alpha)\cup s(\beta)| - |T_\alpha \cap T_\beta|$. □

Theorem 4. *Let $\alpha,\beta \in R_G$. Then $d(\alpha,\beta) = |s(\alpha)\cup s(\beta)| - |T_\alpha \cap T_\beta|$.*

Proof. Let $\alpha,\beta \in R_G$. We first show that there is a path from α to β of length $|s(\alpha)\cup s(\beta)| - |T_\alpha \cap T_\beta|$. We know that there exists $u \in R_G$ such that $\alpha + u = \beta$, which implies that $u = \beta - \alpha$. Moreover, we can assume that $u = \sum_{i=1}^{n} u_i$, where $u_i = a_g g$ for some $g \in G$ and for all $1 \le i \le n$. From $u_i \neq T_\alpha \cap T_\beta$ for all i we get $n = |s(\alpha)\cup s(\beta)| - |T_\alpha \cap T_\beta|$. Furthermore, we obtain $\alpha, \alpha+u_1, \alpha+u_1+u_2, \ldots, \alpha+u_1+u_2+\cdots+u_n = \beta$ is a path of length $n = |s(\alpha)\cup s(\beta)| - |T_\alpha \cap T_\beta|$ which implies that $d(\alpha,\beta) \le |s(\alpha)\cup s(\beta)| - |T_\alpha \cap T_\beta|$.

Conversely, we assume there exists a path from α to β of length p, where $p < |s(\alpha)\cup s(\beta)| - |T_\alpha \cap T_\beta|$. Let $\alpha = v_0, v_1, v_2, \ldots, v_p = \beta$ be a path from α to β of length p. By the definition of adjacency of vertices in $\Gamma(R_G)$, there exists $u_i \in R_G$ such that $v_{i-1} - v_i = u_i$ for all $1 \le i \le p$. Furthermore, we get $v_i = v_{i-1} - u_i$ for all $1 \le i \le p$ and so $v_i = v_0 - u_1 - u_2 - \cdots - u_i$ for all $1 \le i \le p$. Thus, $v_p = v_0 - \sum_{i=1}^{p} u_i$, which implies that $\alpha - \beta = \sum_{i=1}^{p} u_i$. Hence $|s(\alpha-\beta)| \le p$. By using Lemma 9, we obtain $p < |s(\alpha)\cup s(\beta)| - |T_\alpha \cap T_\beta| = |s(\alpha-\beta)| \le p$, a contradiction. Then $d(\alpha,\beta) \ge |s(\alpha)\cup s(\beta)| - |T_\alpha \cap T_\beta|$ and so $d(\alpha,\beta) = |s(\alpha)\cup s(\beta)| - |T_\alpha \cap T_\beta|$, as required. □

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