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On the bicomplex Fibonacci p quaternions

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Abstract. The paper introduces a novel bicomplex Fibonacci p quaternions, establishing algebraic properties, Honsberger identity, D'Ocagne's identity, Cassini's identity, Catalan's identity for these quaternions. The study extends previous work on bicomplex Fibonacci quaternions [1] by incorporating bicomplex Fibonacci p quaternions. These new quaternions may have implications in applied mathematics, quantum mechanics, quantum physics, Lie groups, Kinematics and differential equations.

Keywords: Fibonacci numbers, Golden mean, Bicomplex quaternions, Bicomplex Fibonacci quaternions.
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1 Introduction

The Fibonacci numbers, F_n are defined by the recurrence relation:

$$F_n = F_{n-1} + F_{n-2} \text{ for } n > 2,$$

with initial terms

$$F_1 = F_2 = 1.$$

The Fibonacci numbers, F_n and golden mean,

$$\tau = \lim_{n \longrightarrow \infty} \frac{F_n}{F_{n-1}} = \frac{1+\sqrt{5}}{2}$$

have appeared in arts, sciences, high energy physics, information and coding theory [2,3,6,7,11, 12].
The Fibonacci p-numbers [11] are defined by the recurrence relation:

$$F_p(n) = F_p(n-1) + F_p(n-p-1),$$

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with $n > p+1$, for a given integer $p = 0, 1, 2, 3, \cdots$ and initial terms

$$F_p(1) = F_p(2) = \cdots = F_p(p) = F_p(p+1) = 1.$$

The Fibonacci p-numbers, $F_p(n)$ coincide with classical Fibonacci numbers, F_n for $p = 1$. e.g. $F_1(n) = F_n$.
The complex Fibonacci numbers [10] are defined by the recurrence relation:

$$F_n^\star = F_{n-1}^\star + F_{n-2}^\star \text{ for } n \geqslant 2,$$

with initial terms

$$F_0^\star = i, F_1^\star = 1 + i,$$

where i is the imaginary unit which satisfies $i^2 = -1$, and

$$F_n^\star = F_n + iF_{n+1}.$$

In 19th century, William Rowan Hamilton [8] defined a set of quaternions as follows:

$$H = \{q = q_0 + iq_1 + jq_2 + kq_3 | q_0, q_1, q_2, q_3 \in \Re\},$$

where $i^2 = j^2 = k^2 = -1$, $ij = -ji = k$, $jk = -kj = i$, $ki = -ik = j$.
The quaternions constitute an extension of complex numbers into a four dimensional space and considered as four dimensional vectors whereas complex numbers are two dimensional vectors.
In 1963, Horadam [9] defined complex Fibonacci and Lucas quaternions as follows:

$$Q_n = F_n + iF_{n+1} + jF_{n+2} + kF_{n+3},$$

and

$$K_n = L_n + iL_{n+1} + jL_{n+2} + kL_{n+3},$$

where F_n and L_n are nth Fibonacci, and Lucas numbers, respectively and the imaginary quaternions units i, j, k have the following properties
$i^2 = j^2 = k^2 = -1$, $ij = -ji = k$, $jk = -kj = i$, $ki = -ik = j$.
The Bicomplex numbers [1] are generalization complex numbers and defined by

$$C_2 = \{q = (q_1 + iq_2) + j(q_3 + iq_4) | q_1, q_2, q_3, q_4 \in \Re\},$$

where $i^2 = j^2 = -1$, $ij = ji$.
It has four dimensional vectors with basis $\{1, i, j, ij\}$.
Three different conjugations can operate on bicomplex numbers [1] as follows:

$$\begin{aligned} q &= (q_1 + iq_2) + j(q_3 + iq_4), q \in C_2, \\ q_i^\star &= (q_1 - iq_2) + j(q_3 - iq_4), \\ q_j^\star &= (q_1 + iq_2) - j(q_3 + iq_4), \end{aligned}$$

$$q^{\star}_{ij} = (q_1 - iq_2) - j(q_3 - iq_4).$$

The Bicomplex quaternions [1] defined by

$$C_2^Q = \{Q = (q_1 + iq_2) + j(q_3 + iq_4) | q_1, q_2, q_3, q_4 \in \Re\},$$

where $i^2 = j^2 = -1$, $ij = ji$.
It has also four dimensional vectors with basis $\{1, i, j, ij\}$.
The Bicomplex Fibonacci quaternions [1] defined by

$$C_2^F = \{Q_n = (F_n + iF_{n+1}) + j(F_{n+2} + iF_{n+3}) |\, F_n, n\text{th Fibonacci numbers}\},$$

where $i^2 = j^2 = -1$, $ij = ji$.
In this paper, we introduce bicomplex Fibonacci p quaternions as follows:

$$C_2^{F_p} = \{Q_p(n) = (F_p(n) + iF_p(n+1)) + j(F_p(n+2) + iF_p(n+3))| \\ F_p(n), n\text{th Fibonacci } p\text{-numbers}\}, \tag{1}$$

where $i^2 = j^2 = -1$, $ij = ji$.

2 Bicomplex Fibonacci p quaternions

The bicomplex Fibonacci p quaternions defined by as follows:

$$C_2^{F_p} = \{Q_p(n) = (F_p(n) + iF_p(n+1)) + j(F_p(n+2) + iF_p(n+3))| \\ F_p(n), n\text{th Fibonacci } p\text{-numbers}, \}$$

where $i^2 = j^2 = -1$, $ij = ji$.
Let $Q_p(n)$ and $Q_p(m)$ be two bicomplex Fibonacci p quaternions such that

$$Q_p(n) = (F_p(n) + iF_p(n+1)) + j(F_p(n+2) + iF_p(n+3)), \tag{2}$$

and

$$Q_p(m) = (F_p(m) + iF_p(m+1)) + j(F_p(m+2) + iF_p(m+3)), \tag{3}$$

Then the addition and subtraction of two bicomplex Fibonacci p quaternions are defined as follows:

$$\begin{aligned} Q_p(n) \pm Q_p(m) &= (F_p(n) + iF_p(n+1)) + j(F_p(n+2) + iF_p(n+3)) \pm (F_p(m) + iF_p(m+1)) \\ &\quad + j(F_p(m+2) + iF_p(m+3)) \\ &= (F_p(n) \pm F_p(m)) + i(F_p(n+1) \pm F_p(m+1)) + j(F_p(n+2) \pm F_p(m+2)) \\ &\quad + ij(F_p(n+3) \pm F_p(m+3)). \end{aligned}$$

Multiplication of two bicomplex Fibonacci p quaternions are defined as follows:

$$Q_p(n) \times Q_p(m) = [(F_p(n) + iF_p(n+1)) + j(F_p(n+2) + iF_p(n+3))][(F_p(m) + iF_p(m+1))$$

$$
\begin{aligned}
&+ j(F_p(m+2) + iF_p(m+3))] \\
&= (F_p(n)F_p(m) - F_p(n+1)F_p(m+1) - F_p(n+2)F_p(m+2) \\
&\quad + F_p(n+3)F_p(m+3)) + i(F_p(n)F_p(m+1) + F_p(n+1)F_p(m) \\
&\quad - F_p(n+2)F_p(m+3) - F_p(n+3)F_p(m+2)) + j(F_p(n)F_p(m+2) \\
&\quad + F_p(n+2)F_p(m) - F_p(n+1)F_p(m+3) - F_p(n+3)F_p(m+1)) \\
&\quad + ij(F_p(n)F_p(m+3) + F_p(n+3)F_p(m) + F_p(n+1)F_p(m+2) \\
&\quad - F_p(n+2)F_p(m+1)) \\
&= Q_p(m) \times Q_p(n).
\end{aligned}
$$

The conjugation of bicomplex Fibonacci p quaternions are defined as follows:

$$(Q_p(n))_i^\star = (F_p(n) - iF_p(n+1)) + j(F_p(n+2) - iF_p(n+3)), \tag{4}$$

$$(Q_p(n))_j^\star = (F_p(n) + iF_p(n+1)) - j(F_p(n+2) + iF_p(n+3)), \tag{5}$$

$$(Q_p(n))_{ij}^\star = (F_p(n) - iF_p(n+1)) - j(F_p(n+2) - iF_p(n+3)). \tag{6}$$

Theorem 1. *Let $Q_p(n)$ and $Q_p(m)$ be two bicomplex Fibonacci p quaternions. In this case, we give the following relations between the conjugates of these quaternions:*

$$(Q_p(n)Q_p(m))_i^\star = (Q_p(m))_i^\star(Q_p(n))_i^\star = (Q_p(n))_i^\star(Q_p(m))_i^\star, \tag{7}$$

$$(Q_p(n)Q_p(m))_j^\star = (Q_p(m))_j^\star(Q_p(n))_j^\star = (Q_p(n))_j^\star(Q_p(m))_j^\star, \tag{8}$$

$$(Q_p(n)Q_p(m))_{ij}^\star = (Q_p(m))_{ij}^\star(Q_p(n))_{ij}^\star = (Q_p(n))_{ij}^\star(Q_p(m))_{ij}^\star. \tag{9}$$

Proof.

$$
\begin{aligned}
(Q_p(n)Q_p(m))_i^\star &= (F_p(n)F_p(m) - F_p(n+1)F_p(m+1) - F_p(n+2)F_p(m+2) \\
&\quad + F_p(n+3)F_p(m+3)) - i(F_p(n)F_p(m+1) + F_p(n+1)F_p(m) \\
&\quad - F_p(n+2)F_p(m+3) - F_p(n+3)F_p(m+2)) + j(F_p(n)F_p(m+2) \\
&\quad + F_p(n+2)F_p(m) - F_p(n+1)F_p(m+3) - F_p(n+3)F_p(m+1)) \\
&\quad - ij(F_p(n)F_p(m+3) + F_p(n+3)F_p(m) + F_p(n+1)F_p(m+2) \\
&\quad - F_p(n+2)F_p(m+1)) \\
&= [(F_p(m) - iF_p(m+1)) + j(F_p(m+2) - iF_p(m+3))][(F_p(n) - iF_p(n+1)) \\
&\quad + j(F_p(n+2) - iF_p(n+3))] \\
&= [(F_p(n) - iF_p(n+1)) + j(F_p(n+2) - iF_p(n+3))][(F_p(m) - iF_p(m+1)) \\
&\quad + j(F_p(m+2) - iF_p(m+3))] \\
&= (Q_p(m))_i^\star(Q_p(n))_i^\star \\
&= (Q_p(n))_i^\star(Q_p(m))_i^\star.
\end{aligned}
$$

$$(Q_p(n)Q_p(m))_j^\star = (F_p(n)F_p(m) - F_p(n+1)F_p(m+1) - F_p(n+2)F_p(m+2)$$

$$
\begin{aligned}
&+F_p(n+3)F_p(m+3))+i(F_p(n)F_p(m+1)+F_p(n+1)F_p(m)\\
&-F_p(n+2)F_p(m+3)-F_p(n+3)F_p(m+2))-j(F_p(n)F_p(m+2)\\
&+F_p(n+2)F_p(m)-F_p(n+1)F_p(m+3)-F_p(n+3)F_p(m+1))\\
&-ij(F_p(n)F_p(m+3)+F_p(n+3)F_p(m)+F_p(n+1)F_p(m+2)\\
&-F_p(n+2)F_p(m+1))\\
=&[(F_p(m)+iF_p(m+1))-j(F_p(m+2)+iF_p(m+3))][(F_p(n)+iF_p(n+1))\\
&-j(F_p(n+2)+iF_p(n+3))]\\
=&[(F_p(n)+iF_p(n+1))-j(F_p(n+2)+iF_p(n+3))][(F_p(m)+iF_p(m+1))\\
&-j(F_p(m+2)+iF_p(m+3))]\\
=&(Q_p(m))_j^\star(Q_p(n))_j^\star\\
=&(Q_p(n))_j^\star(Q_p(m))_j^\star.
\end{aligned}
$$

$$
\begin{aligned}
(Q_p(n)Q_p(m))_{ij}^\star=&(F_p(n)F_p(m)-F_p(n+1)F_p(m+1)-F_p(n+2)F_p(m+2)\\
&+F_p(n+3)F_p(m+3))-i(F_p(n)F_p(m+1)+F_p(n+1)F_p(m)\\
&-F_p(n+2)F_p(m+3)-F_p(n+3)F_p(m+2))-j(F_p(n)F_p(m+2)\\
&+F_p(n+2)F_p(m)-F_p(n+1)F_p(m+3)-F_p(n+3)F_p(m+1))\\
&+ij(F_p(n)F_p(m+3)+F_p(n+3)F_p(m)+F_p(n+1)F_p(m+2)\\
&-F_p(n+2)F_p(m+1))\\
=&[(F_p(m)-iF_p(m+1))-j(F_p(m+2)-iF_p(m+3))][(F_p(n)-iF_p(n+1))\\
&-j(F_p(n+2)-iF_p(n+3))]\\
=&[(F_p(n)-iF_p(n+1))-j(F_p(n+2)-iF_p(n+3))][(F_p(m)-iF_p(m+1))\\
&-j(F_p(m+2)-iF_p(m+3))]\\
=&(Q_p(m))_{ij}^\star(Q_p(n))_{ij}^\star\\
=&(Q_p(n))_{ij}^\star(Q_p(m))_{ij}^\star.
\end{aligned}
$$

□

Theorem 2. *Let $(Q_p(n))_i^\star$, $(Q_p(n))_j^\star$ and $(Q_p(n))_{ij}^\star$ be three kinds of conjugation of the bicomplex Fibonacci p quaternions. In this case, we give the following relations:*

$$
\begin{aligned}
(Q_p(n))(Q_p(n))_i^\star=&(F_p^2(n)+F_p^2(n+1)-F_p^2(n+2)\\
&-F_p^2(n+3))+2j(F_p(n)F_p(n+2)+F_p(n+1)F_p(n+3)),
\end{aligned}
\tag{10}
$$

$$
\begin{aligned}
(Q_p(n))(Q_p(n))_j^\star=&(F_p^2(n)-F_p^2(n+1)+F_p^2(n+2)-F_p^2(n+3))\\
&+2i(F_p(n)F_p(n+1)+F_p(n+2)F_p(n+3)),
\end{aligned}
\tag{11}
$$

$$
(Q_p(n))(Q_p(n))_{ij}^\star=(F_p^2(n)+F_p^2(n+1)+F_p^2(n+2)+F_p^2(n+3))
\tag{12}
$$

$$+ 2ij(F_p(n)F_p(n+3) - F_p(n+1)F_p(n+2)),$$

$$\begin{aligned}(Q_p(n))(Q_p(n))_i^\star + (Q_p(n-1))(Q_p(n-1))_i^\star = {} & 2(F_p^2(n) - F_p^2(n+2)) + (F_p^2(n-1) \\ & - F_p^2(n+3)) + 2j(F_p(n+1)F_p(n+3) \\ & + F_p(n-1)F_p(n+1) + 2F_p(n)F_p(n+2)),\end{aligned} \tag{13}$$

$$\begin{aligned}(Q_p(n))(Q_p(n))_j^\star + (Q_p(n-1))(Q_p(n-1))_j^\star = {} & (F_p^2(n-1) - F_p^2(n+3)) \\ & + 2i[F_p(n)(F_p(n-1) + F_p(n+1)) \\ & + F_p(n+2)(F_p(n+1) + F_p(n+3))],\end{aligned} \tag{14}$$

$$\begin{aligned}(Q_p(n))(Q_p(n))_{ij}^\star + (Q_p(n-1))(Q_p(n-1))_{ij}^\star = {} & 2(F_p^2(n) + F_p^2(n+1) + F_p^2(n+2)) \\ & + (F_p^2(n-1) + F_p^2(n+3)) \\ & + 2ij[F_p(n)(F_p(n+3) - F_p(n+1)) \\ & + F_p(n+2)(F_p(n-1) - F_p(n+1))],\end{aligned} \tag{15}$$

$$\begin{aligned}(Q_p(n))(Q_p(n))_{ij}^\star - (Q_p(n-1))(Q_p(n-1))_{ij}^\star = {} & (F_p^2(n+3) - F_p^2(n-1)) \\ & + 2ij[F_p(n)(F_p(n+3) + F_p(n+1)) \\ & - F_p(n+2)(F_p(n+1) - F_p(n-1))].\end{aligned} \tag{16}$$

Proof.

$$\begin{aligned}(Q_p(n))(Q_p(n))_i^\star &= [(F_p(n) + iF_p(n+1)) + j(F_p(n+2) + iF_p(n+3))][(F_p(n) - iF_p(n+1)) \\ &\quad + j(F_p(n+2) - iF_p(n+3))] \\ &= (F_p^2(n) + F_p^2(n+1) - F_p^2(n+2) - F_p^2(n+3)) + 2j(F_p(n)F_p(n+2) \\ &\quad + F_p(n+1)F_p(n+3)).\end{aligned}$$

$$\begin{aligned}(Q_p(n))(Q_p(n))_j^\star &= [(F_p(n) + iF_p(n+1)) + j(F_p(n+2) + iF_p(n+3))][(F_p(n) + iF_p(n+1)) \\ &\quad - j(F_p(n+2) + iF_p(n+3))] \\ &= (F_p^2(n) - F_p^2(n+1) + F_p^2(n+2) - F_p^2(n+3)) \\ &\quad + 2i(F_p(n)F_p(n+1) + F_p(n+2)F_p(n+3)).\end{aligned}$$

$$\begin{aligned}(Q_p(n))(Q_p(n))_{ij}^\star &= [(F_p(n) + iF_p(n+1)) + j(F_p(n+2) + iF_p(n+3))][(F_p(n) - iF_p(n+1)) \\ &\quad - j(F_p(n+2) - iF_p(n+3))] \\ &= (F_p^2(n) + F_p^2(n+1) + F_p^2(n+2) + F_p^2(n+3)) \\ &\quad + 2ij(F_p(n)F_p(n+3) - F_p(n+1)F_p(n+2)).\end{aligned}$$

$$\begin{aligned}(Q_p(n))(Q_p(n))_i^\star + (Q_p(n-1))(Q_p(n-1))_i^\star &= [(F_p(n) + iF_p(n+1)) + j(F_p(n+2) \\ &\quad + iF_p(n+3))][(F_p(n) - iF_p(n+1)) \\ &\quad + j(F_p(n+2) - iF_p(n+3))] + [(F_p(n-1) \\ &\quad + iF_p(n)) + j(F_p(n+1) + iF_p(n+2))][(F_p(n-1) \\ &\quad - iF_p(n)) + j(F_p(n+1) - iF_p(n+2))] \\ &= (F_p^2(n) + F_p^2(n+1) - F_p^2(n+2) - F_p^2(n+3) \\ &\quad + 2j(F_p(n)F_p(n+2) + F_p(n+1)F_p(n+3)) \\ &\quad + (F_p^2(n-1) + F_p^2(n) - F_p^2(n+1) - F_p^2(n+2) \\ &\quad + 2j(F_p(n-1)F_p(n+1) + F_p(n)F_p(n+2)) \\ &= 2(F_p^2(n) - F_p^2(n+2)) + (F_p^2(n-1) - F_p^2(n+3)) \\ &\quad + 2j(F_p(n+1)F_p(n+3) + F_p(n-1)F_p(n+1) \\ &\quad + 2F_p(n)F_p(n+2)).\end{aligned}$$

$$\begin{aligned}(Q_p(n))(Q_p(n))_j^\star + (Q_p(n-1))(Q_p(n-1))_j^\star &= [(F_p(n) + iF_p(n+1)) + j(F_p(n+2) \\ &\quad + iF_p(n+3))][(F_p(n) + iF_p(n+1)) - j(F_p(n+2) \\ &\quad + iF_p(n+3))] + [(F_p(n-1) + iF_p(n)) \\ &\quad + j(F_p(n+1) + iF_p(n+2))][(F_p(n-1) \\ &\quad + iF_p(n)) - j(F_p(n+1) + iF_p(n+2))] \\ &= (F_p^2(n) - F_p^2(n+1) + F_p^2(n+2) - F_p^2(n+3)) \\ &\quad + 2i(F_p(n)F_p(n+1) + F_p(n+2)F_p(n+3)) \\ &\quad + (F_p^2(n-1) - F_p^2(n) + F_p^2(n+1) - F_p^2(n+2)) \\ &\quad + 2i(F_p(n-1)F_p(n) + F_p(n+1)F_p(n+2)) \\ &= (F_p^2(n-1) - F_p^2(n+3)) + 2i[F_p(n)(F_p(n-1) \\ &\quad + F_p(n+1)) + F_p(n+2)(F_p(n+1) + F_p(n+3))].\end{aligned}$$

$$\begin{aligned}(Q_p(n))(Q_p(n))_{ij}^\star + (Q_p(n-1))(Q_p(n-1))_{ij}^\star &= [(F_p(n) + iF_p(n+1)) + j(F_p(n+2) \\ &\quad + iF_p(n+3))][(F_p(n) - iF_p(n+1)) - j(F_p(n+2) \\ &\quad - iF_p(n+3))] + [(F_p(n-1) + iF_p(n)) \\ &\quad + j(F_p(n+1) + iF_p(n+2))][(F_p(n-1) \\ &\quad - iF_p(n)) - j(F_p(n+1) - iF_p(n+2))] \\ &= (F_p^2(n) + F_p^2(n+1) + F_p^2(n+2) + F_p^2(n+3)) \\ &\quad + 2ij(F_p(n)F_p(n+3) - F_p(n+1)F_p(n+2)) \\ &\quad + (F_p^2(n-1) + F_p^2(n) + F_p^2(n+1) + F_p^2(n+2)) \\ &\quad + 2ij(F_p(n-1)F_p(n+2) - F_p(n)F_p(n+1)) \\ &= 2(F_p^2(n) + F_p^2(n+1) + F_p^2(n+2)) + (F_p^2(n-1)\end{aligned}$$

$$+ F_p^2(n+3)) + 2ij[F_p(n)(F_p(n+3) - F_p(n+1)) + F_p(n+2)(F_p(n-1) - F_p(n+1))].$$

$$\begin{aligned}(Q_p(n))(Q_p(n))^\star_{ij} - (Q_p(n-1))(Q_p(n-1))^\star_{ij} &= [(F_p(n) + iF_p(n+1)) + j(F_p(n+2) \\ &+ iF_p(n+3))][(F_p(n) - iF_p(n+1)) - j(F_p(n+2) \\ &- iF_p(n+3))] - [(F_p(n-1) + iF_p(n)) \\ &+ j(F_p(n+1) + iF_p(n+2))][(F_p(n-1) - iF_p(n)) \\ &- j(F_p(n+1) - iF_p(n+2))] \\ &= (F_p^2(n) + F_p^2(n+1) + F_p^2(n+2) + F_p^2(n+3)) \\ &+ 2ij(F_p(n)F_p(n+3) - F_p(n+1)F_p(n+2)) \\ &- (F_p^2(n-1) + F_p^2(n) + F_p^2(n+1) + F_p^2(n+2)) \\ &+ 2ij(F_p(n-1)F_p(n+2) - F_p(n)F_p(n+1)) \\ &= (F_p^2(n+3) - F_p^2(n-1)) + 2ij[F_p(n)(F_p(n+3) \\ &+ F_p(n+1)) - F_p(n+2)(F_p(n+1) - F_p(n-1))].\end{aligned}$$

□

Theorem 3. *Let $Q_p(n)$ be the bicomplex Fibonacci p quaternions. In this case, we give the following relations:*

$$Q_p(n) = Q_p(n-1) + Q_p(n-p-1), \tag{17}$$

$$Q_p(n) = Q_p(n-p) + Q_p(n-p-1) + Q_p(n-p-2) + \cdots + Q_p(n-2p), \tag{18}$$

$$Q_p(1) + Q_p(2) + Q_p(3) + \cdots + Q_p(n) = Q_p(n+p+1) - Q_p(p+1). \tag{19}$$

Proof.

$$\begin{aligned}Q_p(n-1)) + Q_p(n-p-1) &= F_p(n-1) + iF_p(n) + j(F_p(n+1) + iF_p(n+2)) + F_p(n-p-1) \\ &+ iF_p(n-p) + j(F_p(n-p+1) + iF_p(n-p+2)) \\ &= (F_p(n-1) + F_p(n-p-1)) + i(F_p(n) + F_p(n-p)) + j(F_p(n+1) \\ &+ F_p(n-p+1)) + ij(F_p(n+2) + F_p(n-p+2)) \\ &= (F_p(n) + iF_p(n+1) + jF_p(n+2) + ijF_p(n+3)) \\ &= Q_p(n).\end{aligned}$$

We know that

$$\begin{aligned}Q_p(n) &= Q_p(n-1) + Q_p(n-p-1), \\ Q_p(n-1) &= Q_p(n-2) + Q_p(n-p-2), \\ Q_p(n-2) &= Q_p(n-3) + Q_p(n-p-3),\end{aligned}$$

and so on with the these identities, we get

$$Q_p(n) = Q_p(n-p) + Q_p(n-p-1) + Q_p(n-p-2) + \cdots + Q_p(n-2p).$$

We know that $Q_p(n) = Q_p(n-1) + Q_p(n-p-1)$. Therefore, $Q_p(n-p-1)$=$Q_p(n) - Q_p(n-1)$. Now, replacing n by $p+2, p+3, \ldots, p+n+1$, we get,

$$\begin{aligned}
Q_p(1) &= Q_p(p+2) - Q_p(p+1), \\
Q_p(2) &= Q_p(p+3) - Q_p(p+2), \\
Q_p(3) &= Q_p(p+4) - Q_p(p+3), \\
Q_p(4) &= Q_p(p+5) - Q_p(p+4), \\
&\vdots \\
Q_p(n-1) &= Q_p(p+n) - Q_p(p+n-1), \\
Q_p(n) &= Q_p(p+n+1) - Q_p(p+n).
\end{aligned}$$

Adding the above recurrence relations, we get,

$$Q_p(1) + Q_p(2) + Q_p(3) + \cdots + Q_p(n) = Q_p(n+p+1) - Q_p(p+1).$$

□

Theorem 4. *Let $Q_p(n)$ be the bicomplex Fibonacci p quaternions. In this case, we give the following relations:*

$$\begin{aligned}
Q_p^2(n-p+1) = {} & [F_p^2(n-p+1) - F_p^2(n-p+2) - F_p^2(n-p+3) + F_p^2(n-p+4)] \\
& + 2i[F_p(n-p+1)F_p(n-p+2) - F_p(n-p+3)F_p(n-p+4)] \\
& + 2j[F_p(n-p+1)F_p(n-p+3) - F_p(n-p+2)F_p(n-p+4)] \\
& + 2ij[F_p(n-p+1)F_p(n-p+4) + F_p(n-p+2)F_p(n-p+3)],
\end{aligned}$$

$$\begin{aligned}
Q_p^2(n-p+2) = {} & [F_p^2(n-p+2) - F_p^2(n-p+3) - F_p^2(n-p+4) + F_p^2(n-p+5)] \\
& + 2i[F_p(n-p+2)F_p(n-p+3) - F_p(n-p+4)F_p(n-p+5)] \\
& + 2j[F_p(n-p+2)F_p(n-p+4) - F_p(n-p+3)F_p(n-p+5)] \\
& + 2ij[F_p(n-p+2)F_p(n-p+5) + F_p(n-p+3)F_p(n-p+4)],
\end{aligned}$$

$$\begin{aligned}
Q_p^2(n-p+1) + Q_p^2(n-p+2) = {} & [F_p^2(n-p+1) - 2F_p^2(n-p+3) + F_p^2(n-p+5)] \\
& + 2i[F_p(n-p+1)F_p(n-p+2) - F_p(n-p+4)F_p(n-p+5) \\
& - F_p(n-p+3)(F_p(n-2p+3) + F_p(n-2p+2))] \\
& - 2j[F_p(n-p+3)(F_p(n-2p+4) + F_p(n-2p+3) \\
& + F_p(n-2p+2) + F_p(n-2p+1))] \\
& + 2ij[F_p(n-p+4)(F_p(n-p+1) + F_p(n-p+3)) \\
& + F_p(n-p+2)(F_p(n-p+3) + F_p(n-p+5))],
\end{aligned}$$

$$Q_p^2(n-p+2) - Q_p^2(n-p+1) = [F_p^2(n-p+5) - 2F_p^2(n-p+4) + 2F_p^2(n-p+2)$$

$$
\begin{aligned}
&- F_p^2(n-p+1)] + 2i[F_p(n-p+2)(F_p(n-2p+2)\\
&+ F_p(n-2p+1)) - F_p(n-p+4)(F_p(n-2p+4)\\
&+ F_p(n-2p+3))] + 2j[2F_p(n-p+2)F_p(n-p+4)\\
&- F_p(n-p+3)(F_p(n-p+5) + F_p(n-p+1))]\\
&+ 2ij[F_p(n-p+2)(F_p(n-2p+4) + F_p(n-2p+3))\\
&+ F_p(n-p+4)(F_p(n-2p+2) + F_p(n-2p+1))],
\end{aligned}
$$

$Q_p(n-p+1) - iQ_p(n-p+2) + jQ_p(n-p+3) - ijQ_p(n-p+4) = [F_p(n-p+1) + F_p(n-p+3) - F_p(n-p+5) - F_p(n-p+7)] + 2j[F_p(n-p+3) + F_p(n-p+5)]$,

$Q_p(n-p+1) - iQ_p(n-p+2) - jQ_p(n-p+3) - ijQ_p(n-p+4) = [F_p(n-p+1) + F_p(n-p+3) + F_p(n-p+5) - F_p(n-p+7)] + 2iF_p(n-p+6) + 2jF_p(n-p+5) - 2ijF_p(n-p+4)$.

Proof.

$$
\begin{aligned}
Q_p^2(n-p+1) &= (F_p(n-p+1) + iF_p(n-p+2)) + jF_p(n-p+3) + ijF_p(n-p+4))^2\\
&= [F_p^2(n-p+1) - F_p^2(n-p+2) - F_p^2(n-p+3) + F_p^2(n-p+4)]\\
&\quad + 2i[F_p(n-p+1)F_p(n-p+2) - F_p(n-p+3)F_p(n-p+4)]\\
&\quad + 2j[F_p(n-p+1)F_p(n-p+3) - F_p(n-p+2)F_p(n-p+4)]\\
&\quad + 2ij[F_p(n-p+1)F_p(n-p+4) + F_p(n-p+2)F_p(n-p+3)].
\end{aligned}
$$

$$
\begin{aligned}
Q_p^2(n-p+2) &= (F_p(n-p+2) + iF_p(n-p+3)) + jF_p(n-p+4) + ijF_p(n-p+5))^2\\
&= [F_p^2(n-p+2) - F_p^2(n-p+3) - F_p^2(n-p+4) + F_p^2(n-p+5)]\\
&\quad + 2i[F_p(n-p+2)F_p(n-p+3) - F_p(n-p+4)F_p(n-p+5)]\\
&\quad + 2j[F_p(n-p+2)F_p(n-p+4) - F_p(n-p+3)F_p(n-p+5)]\\
&\quad + 2ij[F_p(n-p+2)F_p(n-p+5) + F_p(n-p+3)F_p(n-p+4)].
\end{aligned}
$$

$$
\begin{aligned}
Q_p^2(n-p+1) + Q_p^2(n-p+2) &= [F_p^2(n-p+1) - F_p^2(n-p+2) - F_p^2(n-p+3)\\
&\quad + F_p^2(n-p+4)] + 2i[F_p(n-p+1)F_p(n-p+2)\\
&\quad - F_p(n-p+3)F_p(n-p+4)] + 2j[F_p(n-p+1)F_p(n-p+3)\\
&\quad - F_p(n-p+2)F_p(n-p+4)] + 2ij[F_p(n-p+1)F_p(n-p+4)\\
&\quad + F_p(n-p+2)F_p(n-p+3)] + [F_p^2(n-p+2) - F_p^2(n-p+3)\\
&\quad - F_p^2(n-p+4) + F_p^2(n-p+5)]\\
&\quad + 2i[F_p(n-p+2)F_p(n-p+3) - F_p(n-p+4)F_p(n-p+5)]\\
&\quad + 2j[F_p(n-p+2)F_p(n-p+4) - F_p(n-p+3)F_p(n-p+5)]\\
&\quad + 2ij[F_p(n-p+2)F_p(n-p+5) + F_p(n-p+3)F_p(n-p+4)]\\
&= [F_p^2(n-p+1) - 2F_p^2(n-p+3) + F_p^2(n-p+5)]\\
&\quad + 2i[F_p(n-p+1)F_p(n-p+2) - F_p(n-p+4)F_p(n-p+5)
\end{aligned}
$$

$$
\begin{aligned}
&- F_p(n-p+3)(F_p(n-2p+3)+F_p(n-2p+2))] \\
&- 2j[F_p(n-p+3)(F_p(n-2p+4)+F_p(n-2p+3) \\
&+ F_p(n-2p+2)+F_p(n-2p+1))] \\
&+ 2ij[F_p(n-p+4)(F_p(n-p+1)+F_p(n-p+3)) \\
&+ F_p(n-p+2)(F_p(n-p+3)+F_p(n-p+5))].
\end{aligned}
$$

$$
\begin{aligned}
&Q_p^2(n-p+2)-Q_p^2(n-p+1) \\
&= (F_p(n-p+2)+iF_p(n-p+3))+jF_p(n-p+4)+ijF_p(n-p+5))^2-(F_p(n-p+1) \\
&\quad + iF_p(n-p+2))+jF_p(n-p+3)+ijF_p(n-p+4))^2 \\
&= [F_p^2(n-p+2)-F_p^2(n-p+3)-F_p^2(n-p+4)+F_p^2(n-p+5)] \\
&\quad + 2i[F_p(n-p+2)F_p(n-p+3)-F_p(n-p+4)F_p(n-p+5)] \\
&\quad + 2j[F_p(n-p+2)F_p(n-p+4)-F_p(n-p+3)F_p(n-p+5)] \\
&\quad + 2ij[F_p(n-p+2)F_p(n-p+5)+F_p(n-p+3)F_p(n-p+4)]-[F_p^2(n-p+1) \\
&\quad - F_p^2(n-p+2)-F_p^2(n-p+3)+F_p^2(n-p+4)]-2i[F_p(n-p+1)F_p(n-p+2) \\
&\quad - F_p(n-p+3)F_p(n-p+4)]-2j[F_p(n-p+1)F_p(n-p+3)-F_p(n-p+2)F_p(n-p+4)] \\
&\quad - 2ij[F_p(n-p+1)F_p(n-p+4)+F_p(n-p+2)F_p(n-p+3)] \\
&= [F_p^2(n-p+5)-2F_p^2(n-p+4)+2F_p^2(n-p+2)-F_p^2(n-p+1)]+2i[F_p(n-p+2) \\
&\quad (F_p(n-2p+2)+F_p(n-2p+1))-F_p(n-p+4)(F_p(n-2p+4)+F_p(n-2p+3))] \\
&\quad + 2j[2F_p(n-p+2)F_p(n-p+4)-F_p(n-p+3)(F_p(n-p+5)+F_p(n-p+1))] \\
&\quad + 2ij[F_p(n-p+2)(F_p(n-2p+4)+F_p(n-2p+3))+F_p(n-p+4)(F_p(n-2p+2) \\
&\quad + F_p(n-2p+1))].
\end{aligned}
$$

$$
\begin{aligned}
&Q_p(n-p+1)-iQ_p(n-p+2)+jQ_p(n-p+3)-ijQ_p(n-p+4) \\
&\qquad = [F_p(n-p+1)+iF_p(n-p+2)+jF_p(n-p+3)+ijF_p(n-p+4)] \\
&\qquad\quad - i[F_p(n-p+2)+iF_p(n-p+3)+jF_p(n-p+4)+ijF_p(n-p+5)] \\
&\qquad\quad + j[F_p(n-p+3)+iF_p(n-p+4)+jF_p(n-p+5)+ijF_p(n-p+6)] \\
&\qquad\quad - ij[F_p(n-p+4)+iF_p(n-p+5)+jF_p(n-p+6)+ijF_p(n-p+7)] \\
&\qquad = [F_p(n-p+1)+F_p(n-p+3)-F_p(n-p+5)-F_p(n-p+7)] \\
&\qquad\quad + 2j[F_p(n-p+3)+F_p(n-p+5)].
\end{aligned}
$$

$$
Q_p(n-p+1)-iQ_p(n-p+2)-jQ_p(n-p+3)-ijQ_p(n-p+4)
$$

$$
\begin{aligned}
&= [F_p(n-p+1)+iF_p(n-p+2)+jF_p(n-p+3)+ijF_p(n-p+4)] \\
&\quad - i[F_p(n-p+2)+iF_p(n-p+3)+jF_p(n-p+4)+ijF_p(n-p+5)] \\
&\quad - j[F_p(n-p+3)+iF_p(n-p+4)+jF_p(n-p+5)+ijF_p(n-p+6)] \\
&\quad - ij[F_p(n-p+4)+iF_p(n-p+5)+jF_p(n-p+6)+ijF_p(n-p+7)]
\end{aligned}
$$

$$
\begin{aligned}
&= [F_p(n-p+1) + F_p(n-p+3) + F_p(n-p+5) - F_p(n-p+7)] \\
&\quad + 2iF_p(n-p+6) + 2jF_p(n-p+5) - 2ijF_p(n-p+4).
\end{aligned}
$$

□

Theorem 5. *Binet formula for the bicomplex Fibonacci p quaternions is $Q_p(n) = k_1(x_1)^n(1 + ix_1 + jx_1^2 + ijx_1^3) + k_2(x_2)^n(1 + ix_2 + jx_2^2 + ijx_2^3) + \cdots + k_{p+1}(x_{p+1})^n(1 + ix_{p+1} + jx_{p+1}^2 + ijx_{p+1}^3)$, where $x_1, x_2, x_3, \ldots, x_{p+1}$ are roots of the characteristic equation $x^{p+1} - x^p - 1 = 0$, and $k_1, k_2, k_3, \ldots, k_{p+1}$ are constants.*

Proof. $Q_p(n) = F_p(n) + iF_p(n+1) + jF_p(n+2) + ijF_p(n+3) = k_1(x_1)^n + k_2(x_2)^n + \ldots + k_{p+1}(x_{p+1})^n) + i(k_1(x_1)^{n+1} + k_2(x_2)^{n+1} + \cdots + k_{p+1}(x_{p+1})^{n+1}) + j(k_1(x_1)^{n+2} + k_2(x_2)^{n+2} + \cdots + k_{p+1}(x_{p+1})^{n+2}) + ij(k_1(x_1)^{n+3} + k_2(x_2)^{n+3} + \cdots + k_{p+1}(x_{p+1})^{n+3})$, where $x_1, x_2, x_3, \ldots, x_{p+1}$ are roots of the characteristic equation $x^{p+1} - x^p - 1 = 0$, and $k_1, k_2, k_3, \ldots, k_{p+1}$ are constant coefficients that depends on the initial elements of the Fibonacci p-numbers.

$$
\begin{aligned}
F_p(0) &= k_1 + k_2 + k_3 + \cdots + k_{p+1} = 0, \\
F_p(1) &= k_1x_1 + k_2x_2 + k_3x_3 + \cdots + k_{p+1}x_{p+1} = 1, \\
F_p(2) &= k_1x_1^2 + k_2x_2^2 + k_3x_3^2 + \cdots + k_{p+1}x_{p+1}^2 = 1, \\
&\vdots \\
F_p(p) &= k_1x_1^p + k_2x_2^p + k_3x_3^p + \cdots + k_{p+1}x_{p+1}^p = 1.
\end{aligned}
$$

We can find the values of $k_1, k_2, k_3, \cdots, k_{p+1}$ after solving the above system of equations. Now, $Q_p(n) = k_1(x_1)^n(1+ix_1+jx_1^2+ijx_1^3)+k_2(x_2)^n(1+ix_2+jx_2^2+ijx_2^3)+\cdots+k_{p+1}(x_{p+1})^n(1+ix_{p+1}+jx_{p+1}^2+ijx_{p+1}^3)$, where $x_1, x_2, x_3, \cdots, x_{p+1}$ are roots of the characteristic equation $x^{p+1}-x^p-1 = 0$, and $k_1, k_2, k_3, \ldots, k_{p+1}$ are constants.

□

Theorem 6. *Let $n, m \geq 0$ the Honsberger identity for the bicomplex Fibonacci p quaternions $Q_p(n)$, $Q_p(m)$ is given by*

$$
\begin{aligned}
Q_p(n)Q_p(m) + Q_p(n+1)Q_p(m+1) &= [F_p(n)F_p(m) - 2F_p(n+2)F_p(m+2) \\
&+ F_p(n+4)F_p(m+4)] + i[F_p(m+1)(F_p(n) + F_p(n+2)) \\
&- F_p(m+3)(F_p(n+2) + F_p(n+4)) \\
&- F_p(m+2)(F_p(n-p+2) + F_p(n-p+1)) \\
&+ (F_p(n+1)F_p(m) - F_p(n+3)F_p(m+4))] \\
&- j[F_p(n+2)(F_p(m-p+3) + F_p(m-p+2) \\
&+ F_p(m-p+1) + F_p(m-p)) + F_p(m+2)(F_p(n-p+3) \\
&+ F_p(n-p+2) + F_p(n-p+1) + F_p(n-p))] \\
&+ ij[F_p(m+1)(F_p(n+2) + F_p(n+4)) \\
&+ F_p(m+3)(F_p(n) + F_p(n+2)) + F_p(n+3)(F_p(m) \\
&+ F_p(m+2)) + F_p(n+1)(F_p(m+2) + F_p(m+4))].
\end{aligned}
$$

Proof.

$$\begin{aligned}
Q_p(n)Q_p(m)+Q_p(n+1)Q_p(m+1) &= [(F_p(n)+iF_p(n+1))+j(F_p(n+2)+iF_p(n+3))][(F_p(m)\\
&\quad +iF_p(m+1))+j(F_p(m+2)+iF_p(m+3))]+[(F_p(n+1)\\
&\quad +iF_p(n+2))+j(F_p(n+3)+iF_p(n+4))][(F_p(m+1)\\
&\quad +iF_p(m+2))+j(F_p(m+3)+iF_p(m+4))]\\
&= (F_p(n)F_p(m)-F_p(n+1)F_p(m+1)-F_p(n+2)F_p(m+2)\\
&\quad +F_p(n+3)F_p(m+3))+i(F_p(n)F_p(m+1)+F_p(n+1)\\
&\quad F_p(m)-F_p(n+2)F_p(m+3)-F_p(n+3)F_p(m+2))\\
&\quad +j(F_p(n)F_p(m+2)+F_p(n+2)F_p(m)-F_p(n+1)\\
&\quad F_p(m+3)-F_p(n+3)F_p(m+1))+ij(F_p(n)F_p(m+3)\\
&\quad +F_p(n+3)F_p(m)+F_p(n+1)F_p(m+2)+F_p(n+2)\\
&\quad F_p(m+1))+(F_p(n+1)F_p(m+1)-F_p(n+2)F_p(m+2)\\
&\quad -F_p(n+3)F_p(m+3)+F_p(n+4)F_p(m+4))\\
&\quad +i(F_p(n+1)F_p(m+2)+F_p(n+2)F_p(m+1)\\
&\quad -F_p(n+3)F_p(m+4)-F_p(n+4)F_p(m+3))\\
&\quad +j(F_p(n+1)F_p(m+3)+F_p(n+3)F_p(m+1)\\
&\quad -F_p(n+2)F_p(m+4)-F_p(n+4)F_p(m+2))\\
&\quad +ij(F_p(n+1)F_p(m+4)+F_p(n+4)F_p(m+1)\\
&\quad +F_p(n+2)F_p(m+3)+F_p(n+3)F_p(m+2))\\
&= [F_p(n)F_p(m)-2F_p(n+2)F_p(m+2)+F_p(n+4)\\
&\quad F_p(m+4)]+i[F_p(m+1)(F_p(n)+F_p(n+2))\\
&\quad -F_p(m+3)(F_p(n+2)+F_p(n+4))-F_p(m+2)\\
&\quad (F_p(n-p+2)+F_p(n-p+1))+(F_p(n+1)F_p(m)\\
&\quad -F_p(n+3)F_p(m+4))]-j[F_p(n+2)(F_p(m-p+3)\\
&\quad +F_p(m-p+2)+F_p(m-p+1)+F_p(m-p))\\
&\quad +F_p(m+2)(F_p(n-p+3)+F_p(n-p+2)\\
&\quad +F_p(n-p+1)+F_p(n-p))]+ij[F_p(m+1)(F_p(n+2)\\
&\quad +F_p(n+4))+F_p(m+3)(F_p(n)+F_p(n+2))+F_p(n+3)\\
&\quad (F_p(m)+F_p(m+2))+F_p(n+1)(F_p(m+2)+F_p(m+4))].
\end{aligned}$$

□

Theorem 7. *Let $n,m\geq 0$ the D'Ocagne's identity for the bicomplex Fibonacci p quaternions $Q_p(n)$, $Q_p(m)$ is given by* $Q_p(n)Q_p(m+1)-Q_p(n+1)Q_p(m)=[F_p(n)F_p(m+1)-F_p(n+4)F_p(m+3)-F_p(n+1)(F_p(m)+F_p(m+2))-F_p(n+2)(F_p(m+1)+F_p(m+3))+F_p(n+3)(F_p(m+2)+F_p(m+4))]+i[F_p(m+2)(F_p(n)+F_p(n+4))-F_p(n+2)(F_p(m)+F_p(m+4))]+j[F_p(n)F_p(m+3)+F_p(n+4)F_p(m+1)-F_p(n+1)(F_p(m+2)+F_p(m+4))+F_p(n+$

$2)(F_p(m+1)+F_p(m+3))-F_p(n+3)(F_p(m)+F_p(m+2))]+ij[F_p(n)F_p(m+4)-F_p(n+4)F_p(m))]$.

Proof. $Q_p(n)Q_p(m+1) - Q_p(n+1)Q_p(m)$

$$\begin{aligned}
&= [(F_p(n) + iF_p(n+1)) + j(F_p(n+2) + iF_p(n+3))][(F_p(m+1) + iF_p(m+2)) + j(F_p(m+3) \\
&\quad + iF_p(m+4))] - [(F_p(n+1) + iF_p(n+2)) + j(F_p(n+3) + jF_p(n+4))][(F_p(m) \\
&\quad + iF_p(m+1)) + j(F_p(m+2) + jF_p(m+3))] \\
&= (F_p(n)F_p(m+1) - F_p(n+1)F_p(m+2) - F_p(n+2)F_p(m+3) + F_p(n+3)F_p(m+4)) \\
&\quad + i(F_p(n)F_p(m+2) + F_p(n+1)F_p(m+1) - F_p(n+2)F_p(m+4) - F_p(n+3)F_p(m+3)) \\
&\quad + j(F_p(n)F_p(m+3) + F_p(n+2)F_p(m+1) - F_p(n+1)F_p(m+4) - F_p(n+3)F_p(m+2)) \\
&\quad + ij(F_p(n)F_p(m+4) + F_p(n+3)F_p(m+1) + F_p(n+1)F_p(m+3) + F_p(n+2)F_p(m+2)) \\
&\quad - (F_p(n+1)F_p(m) - F_p(n+2)F_p(m+1) - F_p(n+3)F_p(m+2) + F_p(n+4)F_p(m+3)) \\
&\quad - i(F_p(n+1)F_p(m+1) + F_p(n+2)F_p(m) - F_p(n+3)F_p(m+3) - F_p(n+4)F_p(m+2)) \\
&\quad - j(F_p(n+1)F_p(m+2) + F_p(n+3)F_p(m) - F_p(n+2)F_p(m+3) - F_p(n+4)F_p(m+1)) \\
&\quad - ij(F_p(n+1)F_p(m+3) + F_p(n+4)F_p(m) + F_p(n+2)F_p(m+2) + F_p(n+3)F_p(m+1)) \\
&= [F_p(n)F_p(m+1) - F_p(n+4)F_p(m+3) - F_p(n+1)(F_p(m) + F_p(m+2)) \\
&\quad - F_p(n+2)(F_p(m+1) + F_p(m+3)) + F_p(n+3)(F_p(m+2) + F_p(m+4))] \\
&\quad + i[F_p(m+2)(F_p(n) + F_p(n+4)) - F_p(n+2)(F_p(m) + F_p(m+4))] \\
&\quad + j[F_p(n)F_p(m+3) + F_p(n+4)F_p(m+1) - F_p(n+1)(F_p(m+2) \\
&\quad + F_p(m+4)) + F_p(n+2)(F_p(m+1) + F_p(m+3)) - F_p(n+3)(F_p(m) + F_p(m+2))] \\
&\quad + ij[F_p(n)F_p(m+4) - F_p(n+4)F_p(m))].
\end{aligned}$$

□

Theorem 8. *Let $Q_p(n)$ be the bicomplex Fibonacci p quaternions. For $n \geq 1$, Cassini's identity for $Q_p(n)$ is given by* $Q_p(n-1)Q_p(n+1) - Q_p^2(n) = [F_p^2(n+2) + F_p^2(n+1) - F_p^2(n) - F_p^2(n+3) + F_p(n+1)(F_p(n-p+3) - F_p(n-p-1)) + F_p(n-p+1)(F_p(n-p+3) + F_p(n-p+2) + F_p(n-p+1) + F_p(n-p))] + i[F_p^2(n+2) - F_p^2(n) + F_p(n+1)F_p(n+4) + F_p(n-1)F_p(n+2) + F_p(n+2)F_p(n-p+2) - F_p(n)F_p(n-p)] + j[F_p^2(n+1) - F_p^2(n+2) - F_p(n-p+2)(F_p(n) - F_p(n-p)) - F_p(n+2)(F_p(n-p-1) + F_p(n-p))] + ij[F_p(n+4)F_p(n-1) - F_p(n+3)F_p(n)]$.

Proof. $Q_p(n-1)Q_p(n+1) - Q_p^2(n)$

$$\begin{aligned}
&= [(F_p(n-1) + iF_p(n)) + j(F_p(n+1) + iF_p(n+2))][(F_p(n+1) + iF_p(n+2)) + j(F_p(n+3) \\
&\quad + iF_p(n+4))] - [(F_p(n) + iF_p(n+1)) + j(F_p(n+2) + iF_p(n+3))]^2 \\
&= (F_p(n-1)F_p(n+1) - F_p(n)F_p(n+2) - F_p(n+1)F_p(n+3) + F_p(n+2)F_p(n+4)) \\
&\quad + i(F_p(n-1)F_p(n+2) + F_p(n)F_p(n+1) - F_p(n+1)F_p(n+4) - F_p(n+2)F_p(n+3)) \\
&\quad + j(F_p(n-1)F_p(n+3) - F_p(n+1)F_p(n+4) + F_p^2(n+1) - F_p^2(n+2)) \\
&\quad + ij(F_p(n-1)F_p(n+4) + F_p(n)F_p(n+3) + 2F_p(n+1)F_p(n+2)) \\
&\quad - (F_p^2(n) + F_p^2(n+3) - F_p^2(n+1) - F_p^2(n+2)) - 2i(F_p(n)F_p(n+1) - F_p(n+2)F_p(n+3))
\end{aligned}$$

$$
\begin{aligned}
&- 2j(F_p(n)F_p(n+2) - F_p(n+1)F_p(n+3)) - 2ij(F_p(n)F_p(n+3) + F_p(n+1)F_p(n+2))\\
&= [F_p^2(n+2) + F_p^2(n+1) - F_p^2(n) - F_p^2(n+3) + F_p(n+1)(F_p(n-p+3) - F_p(n-p-1))\\
&\quad + F_p(n-p+1)(F_p(n-p+3) + F_p(n-p+2) + F_p(n-p+1) + F_p(n-p))]\\
&\quad + i[F_p^2(n+2) - F_p^2(n) + F_p(n+1)F_p(n+4) + F_p(n-1)F_p(n+2) + F_p(n+2)F_p(n-p+2)\\
&\quad - F_p(n)F_p(n-p)] + j[F_p^2(n+1) - F_p^2(n+2) - F_p(n-p+2)(F_p(n) - F_p(n-p))\\
&\quad - F_p(n+2)(F_p(n-p-1) + F_p(n-p))] + ij[F_p(n+4)F_p(n-1) - F_p(n+3)F_p(n)].
\end{aligned}
$$

□

Theorem 9. *Let $Q_p(n+r)$ be the bicomplex Fibonacci p quaternions. For $n \geq 1$, Catalan's identity for $Q_p(n+r)$ is given by*

$$
\begin{aligned}
Q_p(n+r-1)Q_p(n+r+1) - Q_p^2(n+r) = {}& [F_p^2(n+r+2) + F_p^2(n+r+1) - F_p^2(n+r)\\
& - F_p^2(n+r+3) + F_p(n+r+1)(F_p(n+r-p+3)\\
& - F_p(n+r-p-1)) + F_p(n+r-p+1)(F_p(n+r\\
& -p+3) + F_p(n+r-p+2) + F_p(n+r-p+1)\\
& + F_p(n+r-p))] + i[F_p^2(n+r+2) - F_p^2(n+r)\\
& + F_p(n+r+1)F_p(n+r+4)\\
& + F_p(n+r-1)F_p(n+r+2)\\
& + F_p(n+r+2)F_p(n+r-p+2)\\
& - F_p(n+r)F_p(n+r-p)] + j[F_p^2(n+r+1)\\
& - F_p^2(n+r+2) - F_p(n+r-p+2)(F_p(n+r)\\
& - F_p(n+r-p)) - F_p(n+r+2)(F_p(n+r-p-1)\\
& + F_p(n+r-p))] + ij[F_p(n+r+4)F_p(n+r-1)\\
& - F_p(n+r+3)F_p(n+r)].
\end{aligned}
$$

Proof.

$$
\begin{aligned}
Q_p(n+r-1)Q_p(n+r+1) - Q_p^2(n+r) = {}& [(F_p(n+r-1) + iF_p(n+r)) + j(F_p(n+r+1)\\
& + iF_p(n+r+2))][(F_p(n+r+1) + iF_p(n+r+2))\\
& + j(F_p(n+r+3) + iF_p(n+r+4))] - [(F_p(n+r)\\
& + iF_p(n+r+1)) + j(F_p(n+r+2) + iF_p(n+r+3))]^2\\
= {}& (F_p(n+r-1)F_p(n+r+1) - F_p(n+r)F_p(n+r+2)\\
& - F_p(n+r+1)F_p(n+r+3) + F_p(n+r+2)F_p(n\\
& + r+4)) + i(F_p(n+r-1)F_p(n+r+2)\\
& + F_p(n+r)F_p(n+r+1) - F_p(n+r+1)F_p(n+r+4)\\
& - F_p(n+r+2)F_p(n+r+3))\\
& + j(F_p(n+r-1)F_p(n+r+3)
\end{aligned}
$$

$$
\begin{aligned}
&- F_p(n+r+1)F_p(n+r+4) + F_p^2(n+r+1) \\
&- F_p^2(n+r+2)) + ij(F_p(n+r-1)F_p(n+r+4) \\
&+ F_p(n+r)F_p(n+r+3) + 2F_p(n+r+1)F_p(n+r+2)) \\
&- (F_p^2(n+r) + F_p^2(n+r+3) - F_p^2(n+r+1) \\
&- F_p^2(n+r+2)) - 2i(F_p(n+r)F_p(n+r+1) \\
&- F_p(n+r+2)F_p(n+r+3)) \\
&- 2j(F_p(n+r)F_p(n+r+2) - F_p(n+r+1)F_p(n+r+3)) \\
&- 2ij(F_p(n+r)F_p(n+r+3) + F_p(n+r+1)F_p(n+r+2)) \\
= &[F_p^2(n+r+2) + F_p^2(n+r+1) - F_p^2(n+r) \\
&- F_p^2(n+r+3) + F_p(n+r+1)(F_p(n+r-p+3) \\
&- F_p(n+r-p-1)) + F_p(n+r-p+1)(F_p(n+r \\
&- p+3) + F_p(n+r-p+2) + F_p(n+r-p+1) \\
&+ F_p(n+r-p))] + i[F_p^2(n+r+2) - F_p^2(n+r) \\
&+ F_p(n+r+1)F_p(n+r+4) \\
&+ F_p(n+r-1)F_p(n+r+2) \\
&+ F_p(n+r+2)F_p(n+r-p+2) \\
&- F_p(n+r)F_p(n+r-p)] + j[F_p^2(n+r+1) \\
&- F_p^2(n+r+2) - F_p(n+r-p+2)(F_p(n+r) \\
&- F_p(n+r-p)) - F_p(n+r+2)(F_p(n+r-p-1) \\
&+ F_p(n+r-p))] + ij[F_p(n+r+4)F_p(n+r-1) \\
&- F_p(n+r+3)F_p(n+r)].
\end{aligned}
$$

□

3 Conclusion

In this paper, we introduce bicomplex Fibonacci p quaternions and some properties of bicomplex Fibonacci p quaternions, Honsberger identity, D'Ocagne's identity, Cassini's identity, Catalan's identity for bicomplex Fibonacci p quaternions. I hope that these results will be useful in applied mathematics, quantum mechanics, quantum physics, Lie groups, Kinematics and differential equations like bicomplex Fibonacci quaternions [1, 4, 5].

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