

# On the bicomplex Fibonacci $p$ quaternions

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**Abstract.** The paper introduces a novel bicomplex Fibonacci  $p$  quaternions, establishing algebraic properties, Honsberger identity, D’Ocagne’s identity, Cassini’s identity, Catalan’s identity for these quaternions. The study extends previous work on bicomplex Fibonacci quaternions [1] by incorporating bicomplex Fibonacci  $p$  quaternions. These new quaternions may have implications in applied mathematics, quantum mechanics, quantum physics, Lie groups, Kinematics and differential equations.

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## 1 Introduction

The Fibonacci numbers,  $F_n$  are defined by the recurrence relation:

$$F_n = F_{n-1} + F_{n-2} \text{ for } n > 2,$$

with initial terms

$$F_1 = F_2 = 1.$$

The Fibonacci numbers,  $F_n$  and golden mean,

$$\tau = \lim_{n \rightarrow \infty} \frac{F_n}{F_{n-1}} = \frac{1 + \sqrt{5}}{2}$$

have appeared in arts, sciences, high energy physics, information and coding theory [2,3,6,7,11,12].

The Fibonacci  $p$ -numbers [11] are defined by the recurrence relation:

$$F_p(n) = F_p(n-1) + F_p(n-p-1),$$

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with  $n > p + 1$ , for a given integer  $p = 0, 1, 2, 3, \dots$  and initial terms

$$F_p(1) = F_p(2) = \dots = F_p(p) = F_p(p + 1) = 1.$$

The Fibonacci  $p$ -numbers,  $F_p(n)$  coincide with classical Fibonacci numbers,  $F_n$  for  $p = 1$ . e.g.  $F_1(n) = F_n$ .

The complex Fibonacci numbers [10] are defined by the recurrence relation:

$$F_n^* = F_{n-1}^* + F_{n-2}^* \text{ for } n \geq 2,$$

with initial terms

$$F_0^* = i, F_1^* = 1 + i,$$

where  $i$  is the imaginary unit which satisfies  $i^2 = -1$ , and

$$F_n^* = F_n + iF_{n+1}.$$

In 19th century, William Rowan Hamilton [8] defined a set of quaternions as follows:

$$H = \{q = q_0 + iq_1 + jq_2 + kq_3 | q_0, q_1, q_2, q_3 \in \mathfrak{R}\},$$

where  $i^2 = j^2 = k^2 = -1$ ,  $ij = -ji = k$ ,  $jk = -kj = i$ ,  $ki = -ik = j$ .

The quaternions constitute an extension of complex numbers into a four dimensional space and considered as four dimensional vectors whereas complex numbers are two dimensional vectors.

In 1963, Horadam [9] defined complex Fibonacci and Lucas quaternions as follows:

$$Q_n = F_n + iF_{n+1} + jF_{n+2} + kF_{n+3},$$

and

$$K_n = L_n + iL_{n+1} + jL_{n+2} + kL_{n+3},$$

where  $F_n$  and  $L_n$  are  $n$ th Fibonacci, and Lucas numbers, respectively and the imaginary quaternions units  $i, j, k$  have the following properties

$i^2 = j^2 = k^2 = -1$ ,  $ij = -ji = k$ ,  $jk = -kj = i$ ,  $ki = -ik = j$ .

The Bicomplex numbers [1] are generalization complex numbers and defined by

$$C_2 = \{q = (q_1 + iq_2) + j(q_3 + iq_4) | q_1, q_2, q_3, q_4 \in \mathfrak{R}\},$$

where  $i^2 = j^2 = -1$ ,  $ij = ji$ .

It has four dimensional vectors with basis  $\{1, i, j, ij\}$ .

Three different conjugations can operate on bicomplex numbers [1] as follows:

$$\begin{aligned} q &= (q_1 + iq_2) + j(q_3 + iq_4), q \in C_2, \\ q_i^* &= (q_1 - iq_2) + j(q_3 - iq_4), \\ q_j^* &= (q_1 + iq_2) - j(q_3 + iq_4), \end{aligned}$$

$$q_{ij}^* = (q_1 - iq_2) - j(q_3 - iq_4).$$

The Bicomplex quaternions [1] defined by

$$C_2^Q = \{Q = (q_1 + iq_2) + j(q_3 + iq_4) | q_1, q_2, q_3, q_4 \in \mathfrak{R}\},$$

where  $i^2 = j^2 = -1$ ,  $ij = ji$ .

It has also four dimensional vectors with basis  $\{1, i, j, ij\}$ .

The Bicomplex Fibonacci quaternions [1] defined by

$$C_2^F = \{Q_n = (F_n + iF_{n+1}) + j(F_{n+2} + iF_{n+3}) | F_n, nth \text{ Fibonacci numbers}\},$$

where  $i^2 = j^2 = -1$ ,  $ij = ji$ .

In this paper, we introduce bicomplex Fibonacci  $p$  quaternions as follows:

$$C_2^{F_p} = \{Q_p(n) = (F_p(n) + iF_p(n+1)) + j(F_p(n+2) + iF_p(n+3)) | F_p(n), nth \text{ Fibonacci } p\text{-numbers}\}, \quad (1)$$

where  $i^2 = j^2 = -1$ ,  $ij = ji$ .

## 2 Bicomplex Fibonacci $p$ quaternions

The bicomplex Fibonacci  $p$  quaternions defined by as follows:

$$C_2^{F_p} = \{Q_p(n) = (F_p(n) + iF_p(n+1)) + j(F_p(n+2) + iF_p(n+3)) | F_p(n), nth \text{ Fibonacci } p\text{-numbers}\},$$

where  $i^2 = j^2 = -1$ ,  $ij = ji$ .

Let  $Q_p(n)$  and  $Q_p(m)$  be two bicomplex Fibonacci  $p$  quaternions such that

$$Q_p(n) = (F_p(n) + iF_p(n+1)) + j(F_p(n+2) + iF_p(n+3)), \quad (2)$$

and

$$Q_p(m) = (F_p(m) + iF_p(m+1)) + j(F_p(m+2) + iF_p(m+3)), \quad (3)$$

Then the addition and subtraction of two bicomplex Fibonacci  $p$  quaternions are defined as follows:

$$\begin{aligned} Q_p(n) \pm Q_p(m) &= (F_p(n) + iF_p(n+1)) + j(F_p(n+2) + iF_p(n+3)) \pm (F_p(m) + iF_p(m+1)) \\ &\quad + j(F_p(m+2) + iF_p(m+3)) \\ &= (F_p(n) \pm F_p(m)) + i(F_p(n+1) \pm F_p(m+1)) + j(F_p(n+2) \pm F_p(m+2)) \\ &\quad + ij(F_p(n+3) \pm F_p(m+3)). \end{aligned}$$

Multiplication of two bicomplex Fibonacci  $p$  quaternions are defined as follows:

$$Q_p(n) \times Q_p(m) = [(F_p(n) + iF_p(n+1)) + j(F_p(n+2) + iF_p(n+3))][(F_p(m) + iF_p(m+1))$$

$$\begin{aligned}
& + j(F_p(m+2) + iF_p(m+3))] \\
& = (F_p(n)F_p(m) - F_p(n+1)F_p(m+1) - F_p(n+2)F_p(m+2) \\
& \quad + F_p(n+3)F_p(m+3)) + i(F_p(n)F_p(m+1) + F_p(n+1)F_p(m) \\
& \quad - F_p(n+2)F_p(m+3) - F_p(n+3)F_p(m+2)) + j(F_p(n)F_p(m+2) \\
& \quad + F_p(n+2)F_p(m) - F_p(n+1)F_p(m+3) - F_p(n+3)F_p(m+1)) \\
& \quad + ij(F_p(n)F_p(m+3) + F_p(n+3)F_p(m) + F_p(n+1)F_p(m+2) \\
& \quad - F_p(n+2)F_p(m+1)) \\
& = Q_p(m) \times Q_p(n).
\end{aligned}$$

The conjugation of bicomplex Fibonacci  $p$  quaternions are defined as follows:

$$(Q_p(n))_i^* = (F_p(n) - iF_p(n+1)) + j(F_p(n+2) - iF_p(n+3)), \quad (4)$$

$$(Q_p(n))_j^* = (F_p(n) + iF_p(n+1)) - j(F_p(n+2) + iF_p(n+3)), \quad (5)$$

$$(Q_p(n))_{ij}^* = (F_p(n) - iF_p(n+1)) - j(F_p(n+2) - iF_p(n+3)). \quad (6)$$

**Theorem 1.** *Let  $Q_p(n)$  and  $Q_p(m)$  be two bicomplex Fibonacci  $p$  quaternions. In this case, we give the following relations between the conjugates of these quaternions:*

$$(Q_p(n)Q_p(m))_i^* = (Q_p(m))_i^*(Q_p(n))_i^* = (Q_p(n))_i^*(Q_p(m))_i^*, \quad (7)$$

$$(Q_p(n)Q_p(m))_j^* = (Q_p(m))_j^*(Q_p(n))_j^* = (Q_p(n))_j^*(Q_p(m))_j^*, \quad (8)$$

$$(Q_p(n)Q_p(m))_{ij}^* = (Q_p(m))_{ij}^*(Q_p(n))_{ij}^* = (Q_p(n))_{ij}^*(Q_p(m))_{ij}^*. \quad (9)$$

*Proof.*

$$\begin{aligned}
(Q_p(n)Q_p(m))_i^* & = (F_p(n)F_p(m) - F_p(n+1)F_p(m+1) - F_p(n+2)F_p(m+2) \\
& \quad + F_p(n+3)F_p(m+3)) - i(F_p(n)F_p(m+1) + F_p(n+1)F_p(m) \\
& \quad - F_p(n+2)F_p(m+3) - F_p(n+3)F_p(m+2)) + j(F_p(n)F_p(m+2) \\
& \quad + F_p(n+2)F_p(m) - F_p(n+1)F_p(m+3) - F_p(n+3)F_p(m+1)) \\
& \quad - ij(F_p(n)F_p(m+3) + F_p(n+3)F_p(m) + F_p(n+1)F_p(m+2) \\
& \quad - F_p(n+2)F_p(m+1)) \\
& = (F_p(m) - iF_p(m+1)) + j(F_p(m+2) - iF_p(m+3))(F_p(n) - iF_p(n+1)) \\
& \quad + j(F_p(n+2) - iF_p(n+3)) \\
& = (F_p(n) - iF_p(n+1)) + j(F_p(n+2) - iF_p(n+3))(F_p(m) - iF_p(m+1)) \\
& \quad + j(F_p(m+2) - iF_p(m+3)) \\
& = (Q_p(m))_i^*(Q_p(n))_i^* \\
& = (Q_p(n))_i^*(Q_p(m))_i^*.
\end{aligned}$$

$$(Q_p(n)Q_p(m))_j^* = (F_p(n)F_p(m) - F_p(n+1)F_p(m+1) - F_p(n+2)F_p(m+2)$$

$$\begin{aligned}
& + F_p(n+3)F_p(m+3) + i(F_p(n)F_p(m+1) + F_p(n+1)F_p(m) \\
& - F_p(n+2)F_p(m+3) - F_p(n+3)F_p(m+2)) - j(F_p(n)F_p(m+2) \\
& + F_p(n+2)F_p(m) - F_p(n+1)F_p(m+3) - F_p(n+3)F_p(m+1)) \\
& - ij(F_p(n)F_p(m+3) + F_p(n+3)F_p(m) + F_p(n+1)F_p(m+2) \\
& - F_p(n+2)F_p(m+1)) \\
& = (F_p(m) + iF_p(m+1)) - j(F_p(m+2) + iF_p(m+3))(F_p(n) + iF_p(n+1)) \\
& - j(F_p(n+2) + iF_p(n+3)) \\
& = (F_p(n) + iF_p(n+1)) - j(F_p(n+2) + iF_p(n+3))(F_p(m) + iF_p(m+1)) \\
& - j(F_p(m+2) + iF_p(m+3)) \\
& = (Q_p(m))_j^*(Q_p(n))_j^* \\
& = (Q_p(n))_j^*(Q_p(m))_j^*.
\end{aligned}$$

$$\begin{aligned}
(Q_p(n)Q_p(m))_{ij}^* & = (F_p(n)F_p(m) - F_p(n+1)F_p(m+1) - F_p(n+2)F_p(m+2) \\
& + F_p(n+3)F_p(m+3)) - i(F_p(n)F_p(m+1) + F_p(n+1)F_p(m) \\
& - F_p(n+2)F_p(m+3) - F_p(n+3)F_p(m+2)) - j(F_p(n)F_p(m+2) \\
& + F_p(n+2)F_p(m) - F_p(n+1)F_p(m+3) - F_p(n+3)F_p(m+1)) \\
& + ij(F_p(n)F_p(m+3) + F_p(n+3)F_p(m) + F_p(n+1)F_p(m+2) \\
& - F_p(n+2)F_p(m+1)) \\
& = (F_p(m) - iF_p(m+1)) - j(F_p(m+2) - iF_p(m+3))(F_p(n) - iF_p(n+1)) \\
& - j(F_p(n+2) - iF_p(n+3)) \\
& = (F_p(n) - iF_p(n+1)) - j(F_p(n+2) - iF_p(n+3))(F_p(m) - iF_p(m+1)) \\
& - j(F_p(m+2) - iF_p(m+3)) \\
& = (Q_p(m))_{ij}^*(Q_p(n))_{ij}^* \\
& = (Q_p(n))_{ij}^*(Q_p(m))_{ij}^*.
\end{aligned}$$

□

**Theorem 2.** Let  $(Q_p(n))_i^*$ ,  $(Q_p(n))_j^*$  and  $(Q_p(n))_{ij}^*$  be three kinds of conjugation of the bicomplex Fibonacci  $p$  quaternions. In this case, we give the following relations:

$$\begin{aligned}
(Q_p(n))(Q_p(n))_i^* & = (F_p^2(n) + F_p^2(n+1) - F_p^2(n+2) \\
& - F_p^2(n+3)) + 2j(F_p(n)F_p(n+2) + F_p(n+1)F_p(n+3)),
\end{aligned} \tag{10}$$

$$\begin{aligned}
(Q_p(n))(Q_p(n))_j^* & = (F_p^2(n) - F_p^2(n+1) + F_p^2(n+2) - F_p^2(n+3)) \\
& + 2i(F_p(n)F_p(n+1) + F_p(n+2)F_p(n+3)),
\end{aligned} \tag{11}$$

$$(Q_p(n))(Q_p(n))_{ij}^* = (F_p^2(n) + F_p^2(n+1) + F_p^2(n+2) + F_p^2(n+3)) \tag{12}$$

$$+ 2ij(F_p(n)F_p(n+3) - F_p(n+1)F_p(n+2)),$$

$$\begin{aligned} (Q_p(n))(Q_p(n))_i^* + (Q_p(n-1))(Q_p(n-1))_i^* &= 2(F_p^2(n) - F_p^2(n+2)) + (F_p^2(n-1) \\ &\quad - F_p^2(n+3)) + 2j(F_p(n+1)F_p(n+3) \\ &\quad + F_p(n-1)F_p(n+1) + 2F_p(n)F_p(n+2)), \end{aligned} \quad (13)$$

$$\begin{aligned} (Q_p(n))(Q_p(n))_j^* + (Q_p(n-1))(Q_p(n-1))_j^* &= (F_p^2(n-1) - F_p^2(n+3)) \\ &\quad + 2i[(F_p(n)(F_p(n-1) + F_p(n+1)) \\ &\quad + F_p(n+2)(F_p(n+1) + F_p(n+3))], \end{aligned} \quad (14)$$

$$\begin{aligned} (Q_p(n))(Q_p(n))_{ij}^* + (Q_p(n-1))(Q_p(n-1))_{ij}^* &= 2(F_p^2(n) + F_p^2(n+1) + F_p^2(n+2)) \\ &\quad + (F_p^2(n-1) + F_p^2(n+3)) \\ &\quad + 2ij[F_p(n)(F_p(n+3) - F_p(n+1)) \\ &\quad + F_p(n+2)(F_p(n-1) - F_p(n+1))], \end{aligned} \quad (15)$$

$$\begin{aligned} (Q_p(n))(Q_p(n))_{ij}^* - (Q_p(n-1))(Q_p(n-1))_{ij}^* &= (F_p^2(n+3) - F_p^2(n-1)) \\ &\quad + 2ij[F_p(n)(F_p(n+3) + F_p(n+1)) \\ &\quad - F_p(n+2)(F_p(n+1) - F_p(n-1))]. \end{aligned} \quad (16)$$

*Proof.*

$$\begin{aligned} (Q_p(n))(Q_p(n))_i^* &= [(F_p(n) + iF_p(n+1)) + j(F_p(n+2) + iF_p(n+3))][(F_p(n) - iF_p(n+1)) \\ &\quad + j(F_p(n+2) - iF_p(n+3))] \\ &= (F_p^2(n) + F_p^2(n+1) - F_p^2(n+2) - F_p^2(n+3)) + 2j(F_p(n)F_p(n+2) \\ &\quad + F_p(n+1)F_p(n+3)). \end{aligned}$$

$$\begin{aligned} (Q_p(n))(Q_p(n))_j^* &= [(F_p(n) + iF_p(n+1)) + j(F_p(n+2) + iF_p(n+3))][(F_p(n) + iF_p(n+1)) \\ &\quad - j(F_p(n+2) + iF_p(n+3))] \\ &= (F_p^2(n) - F_p^2(n+1) + F_p^2(n+2) - F_p^2(n+3)) \\ &\quad + 2i(F_p(n)F_p(n+1) + F_p(n+2)F_p(n+3)). \end{aligned}$$

$$\begin{aligned} (Q_p(n))(Q_p(n))_{ij}^* &= [(F_p(n) + iF_p(n+1)) + j(F_p(n+2) + iF_p(n+3))][(F_p(n) - iF_p(n+1)) \\ &\quad - j(F_p(n+2) - iF_p(n+3))] \\ &= (F_p^2(n) + F_p^2(n+1) + F_p^2(n+2) + F_p^2(n+3)) \\ &\quad + 2ij(F_p(n)F_p(n+3) - F_p(n+1)F_p(n+2)). \end{aligned}$$

$$\begin{aligned}
(Q_p(n))(Q_p(n))_i^* + (Q_p(n-1))(Q_p(n-1))_i^* &= [(F_p(n) + iF_p(n+1)) + j(F_p(n+2) \\
&\quad + iF_p(n+3))][(F_p(n) - iF_p(n+1)) \\
&\quad + j(F_p(n+2) - iF_p(n+3))] + [(F_p(n-1) \\
&\quad + iF_p(n)) + j(F_p(n+1) + iF_p(n+2))][(F_p(n-1) \\
&\quad - iF_p(n)) + j(F_p(n+1) - iF_p(n+2))] \\
&= (F_p^2(n) + F_p^2(n+1) - F_p^2(n+2) - F_p^2(n+3) \\
&\quad + 2j(F_p(n)F_p(n+2) + F_p(n+1)F_p(n+3)) \\
&\quad + (F_p^2(n-1) + F_p^2(n) - F_p^2(n+1) - F_p^2(n+2) \\
&\quad + 2j(F_p(n-1)F_p(n+1) + F_p(n)F_p(n+2))) \\
&= 2(F_p^2(n) - F_p^2(n+2)) + (F_p^2(n-1) - F_p^2(n+3)) \\
&\quad + 2j(F_p(n+1)F_p(n+3) + F_p(n-1)F_p(n+1) \\
&\quad + 2F_p(n)F_p(n+2)).
\end{aligned}$$

$$\begin{aligned}
(Q_p(n))(Q_p(n))_j^* + (Q_p(n-1))(Q_p(n-1))_j^* &= [(F_p(n) + iF_p(n+1)) + j(F_p(n+2) \\
&\quad + iF_p(n+3))][(F_p(n) + iF_p(n+1)) - j(F_p(n+2) \\
&\quad + iF_p(n+3))] + [(F_p(n-1) + iF_p(n)) \\
&\quad + j(F_p(n+1) + iF_p(n+2))][(F_p(n-1) \\
&\quad + iF_p(n)) - j(F_p(n+1) + iF_p(n+2))] \\
&= (F_p^2(n) - F_p^2(n+1) + F_p^2(n+2) - F_p^2(n+3) \\
&\quad + 2i(F_p(n)F_p(n+1) + F_p(n+2)F_p(n+3)) \\
&\quad + (F_p^2(n-1) - F_p^2(n) + F_p^2(n+1) - F_p^2(n+2) \\
&\quad + 2i(F_p(n-1)F_p(n) + F_p(n+1)F_p(n+2))) \\
&= (F_p^2(n-1) - F_p^2(n+3)) + 2i[(F_p(n)(F_p(n-1) \\
&\quad + F_p(n+1)) + F_p(n+2)(F_p(n+1) + F_p(n+3))].
\end{aligned}$$

$$\begin{aligned}
(Q_p(n))(Q_p(n))_{ij}^* + (Q_p(n-1))(Q_p(n-1))_{ij}^* &= [(F_p(n) + iF_p(n+1)) + j(F_p(n+2) \\
&\quad + iF_p(n+3))][(F_p(n) - iF_p(n+1)) - j(F_p(n+2) \\
&\quad - iF_p(n+3))] + [(F_p(n-1) + iF_p(n)) \\
&\quad + j(F_p(n+1) + iF_p(n+2))][(F_p(n-1) \\
&\quad - iF_p(n)) - j(F_p(n+1) - iF_p(n+2))] \\
&= (F_p^2(n) + F_p^2(n+1) + F_p^2(n+2) + F_p^2(n+3) \\
&\quad + 2ij(F_p(n)F_p(n+3) - F_p(n+1)F_p(n+2)) \\
&\quad + (F_p^2(n-1) + F_p^2(n) + F_p^2(n+1) + F_p^2(n+2) \\
&\quad + 2ij(F_p(n-1)F_p(n+2) - F_p(n)F_p(n+1))) \\
&= 2(F_p^2(n) + F_p^2(n+1) + F_p^2(n+2)) + (F_p^2(n-1)
\end{aligned}$$

$$+ F_p^2(n+3)) + 2ij[F_p(n)(F_p(n+3) - F_p(n+1)) \\ + F_p(n+2)(F_p(n-1) - F_p(n+1))].$$

$$\begin{aligned} (Q_p(n))(Q_p(n))_{ij}^* - (Q_p(n-1))(Q_p(n-1))_{ij}^* &= [(F_p(n) + iF_p(n+1)) + j(F_p(n+2) \\ &+ iF_p(n+3))] [(F_p(n) - iF_p(n+1)) - j(F_p(n+2) \\ &- iF_p(n+3))] - [(F_p(n-1) + iF_p(n)) \\ &+ j(F_p(n+1) + iF_p(n+2))] [(F_p(n-1) - iF_p(n)) \\ &- j(F_p(n+1) - iF_p(n+2))] \\ &= (F_p^2(n) + F_p^2(n+1) + F_p^2(n+2) + F_p^2(n+3) \\ &+ 2ij(F_p(n)F_p(n+3) - F_p(n+1)F_p(n+2)) \\ &- (F_p^2(n-1) + F_p^2(n) + F_p^2(n+1) + F_p^2(n+2)) \\ &+ 2ij(F_p(n-1)F_p(n+2) - F_p(n)F_p(n+1))) \\ &= (F_p^2(n+3) - F_p^2(n-1)) + 2ij[F_p(n)(F_p(n+3) \\ &+ F_p(n+1)) - F_p(n+2)(F_p(n+1) - F_p(n-1))]. \end{aligned}$$

□

**Theorem 3.** Let  $Q_p(n)$  be the bicomplex Fibonacci  $p$  quaternions. In this case, we give the following relations:

$$Q_p(n) = Q_p(n-1) + Q_p(n-p-1), \quad (17)$$

$$Q_p(n) = Q_p(n-p) + Q_p(n-p-1) + Q_p(n-p-2) + \cdots + Q_p(n-2p), \quad (18)$$

$$Q_p(1) + Q_p(2) + Q_p(3) + \cdots + Q_p(n) = Q_p(n+p+1) - Q_p(p+1). \quad (19)$$

*Proof.*

$$\begin{aligned} Q_p(n-1) + Q_p(n-p-1) &= F_p(n-1) + iF_p(n) + j(F_p(n+1) + iF_p(n+2) + F_p(n-p-1) \\ &+ iF_p(n-p) + jF_p(n-p+1) + iF_p(n-p+2)) \\ &= (F_p(n-1) + F_p(n-p-1)) + i(F_p(n) + F_p(n-p)) + j(F_p(n+1) \\ &+ F_p(n-p+1)) + ij(F_p(n+2) + F_p(n-p+2)) \\ &= (F_p(n) + iF_p(n+1) + jF_p(n+2) + ijF_p(n+3)) \\ &= Q_p(n). \end{aligned}$$

We know that

$$\begin{aligned} Q_p(n) &= Q_p(n-1) + Q_p(n-p-1), \\ Q_p(n-1) &= Q_p(n-2) + Q_p(n-p-2), \\ Q_p(n-2) &= Q_p(n-3) + Q_p(n-p-3), \end{aligned}$$

and so on with the these identities, we get

$$Q_p(n) = Q_p(n-p) + Q_p(n-p-1) + Q_p(n-p-2) + \cdots + Q_p(n-2p).$$

We know that  $Q_p(n) = Q_p(n-1) + Q_p(n-p-1)$ . Therefore,  $Q_p(n-p-1) = Q_p(n) - Q_p(n-1)$ . Now, replacing  $n$  by  $p+2, p+3, \dots, p+n+1$ , we get,

$$\begin{aligned} Q_p(1) &= Q_p(p+2) - Q_p(p+1), \\ Q_p(2) &= Q_p(p+3) - Q_p(p+2), \\ Q_p(3) &= Q_p(p+4) - Q_p(p+3), \\ Q_p(4) &= Q_p(p+5) - Q_p(p+4), \\ &\vdots \\ Q_p(n-1) &= Q_p(p+n) - Q_p(p+n-1), \\ Q_p(n) &= Q_p(p+n+1) - Q_p(p+n). \end{aligned}$$

Adding the above recurrence relations, we get,

$$Q_p(1) + Q_p(2) + Q_p(3) + \dots + Q_p(n) = Q_p(n+p+1) - Q_p(p+1).$$

□

**Theorem 4.** Let  $Q_p(n)$  be the bicomplex Fibonacci  $p$  quaternions. In this case, we give the following relations:

$$\begin{aligned} Q_p^2(n-p+1) &= [F_p^2(n-p+1) - F_p^2(n-p+2) - F_p^2(n-p+3) + F_p^2(n-p+4)] \\ &\quad + 2i[F_p(n-p+1)F_p(n-p+2) - F_p(n-p+3)F_p(n-p+4)] \\ &\quad + 2j[F_p(n-p+1)F_p(n-p+3) - F_p(n-p+2)F_p(n-p+4)] \\ &\quad + 2ij[F_p(n-p+1)F_p(n-p+4) + F_p(n-p+2)F_p(n-p+3)], \end{aligned}$$

$$\begin{aligned} Q_p^2(n-p+2) &= [F_p^2(n-p+2) - F_p^2(n-p+3) - F_p^2(n-p+4) + F_p^2(n-p+5)] \\ &\quad + 2i[F_p(n-p+2)F_p(n-p+3) - F_p(n-p+4)F_p(n-p+5)] \\ &\quad + 2j[F_p(n-p+2)F_p(n-p+4) - F_p(n-p+3)F_p(n-p+5)] \\ &\quad + 2ij[F_p(n-p+2)F_p(n-p+5) + F_p(n-p+3)F_p(n-p+4)], \end{aligned}$$

$$\begin{aligned} Q_p^2(n-p+1) + Q_p^2(n-p+2) &= [F_p^2(n-p+1) - 2F_p^2(n-p+3) + F_p^2(n-p+5)] \\ &\quad + 2i[F_p(n-p+1)F_p(n-p+2) - F_p(n-p+4)F_p(n-p+5) \\ &\quad - F_p(n-p+3)(F_p(n-2p+3) + F_p(n-2p+2))] \\ &\quad - 2j[F_p(n-p+3)(F_p(n-2p+4) + F_p(n-2p+3) \\ &\quad + F_p(n-2p+2) + F_p(n-2p+1))] \\ &\quad + 2ij[F_p(n-p+4)(F_p(n-p+1) + F_p(n-p+3)) \\ &\quad + F_p(n-p+2)(F_p(n-p+3) + F_p(n-p+5))], \end{aligned}$$

$$Q_p^2(n-p+2) - Q_p^2(n-p+1) = [F_p^2(n-p+5) - 2F_p^2(n-p+4) + 2F_p^2(n-p+2)$$

$$\begin{aligned}
& - F_p^2(n-p+1)] + 2i[F_p(n-p+2)(F_p(n-2p+2) \\
& + F_p(n-2p+1)) - F_p(n-p+4)(F_p(n-2p+4) \\
& + F_p(n-2p+3))] + 2j[2F_p(n-p+2)F_p(n-p+4) \\
& - F_p(n-p+3)(F_p(n-p+5) + F_p(n-p+1))] \\
& + 2ij[F_p(n-p+2)(F_p(n-2p+4) + F_p(n-2p+3)) \\
& + F_p(n-p+4)(F_p(n-2p+2) + F_p(n-2p+1))],
\end{aligned}$$

$$Q_p(n-p+1) - iQ_p(n-p+2) + jQ_p(n-p+3) - ijQ_p(n-p+4) = [F_p(n-p+1) + F_p(n-p+3) - F_p(n-p+5) - F_p(n-p+7)] + 2j[F_p(n-p+3) + F_p(n-p+5)],$$

$$Q_p(n-p+1) - iQ_p(n-p+2) - jQ_p(n-p+3) - ijQ_p(n-p+4) = [F_p(n-p+1) + F_p(n-p+3) + F_p(n-p+5) - F_p(n-p+7)] + 2iF_p(n-p+6) + 2jF_p(n-p+5) - 2ijF_p(n-p+4).$$

*Proof.*

$$\begin{aligned}
Q_p^2(n-p+1) &= (F_p(n-p+1) + iF_p(n-p+2) + jF_p(n-p+3) + ijF_p(n-p+4))^2 \\
&= [F_p^2(n-p+1) - F_p^2(n-p+2) - F_p^2(n-p+3) + F_p^2(n-p+4)] \\
&\quad + 2i[F_p(n-p+1)F_p(n-p+2) - F_p(n-p+3)F_p(n-p+4)] \\
&\quad + 2j[F_p(n-p+1)F_p(n-p+3) - F_p(n-p+2)F_p(n-p+4)] \\
&\quad + 2ij[F_p(n-p+1)F_p(n-p+4) + F_p(n-p+2)F_p(n-p+3)].
\end{aligned}$$

$$\begin{aligned}
Q_p^2(n-p+2) &= (F_p(n-p+2) + iF_p(n-p+3) + jF_p(n-p+4) + ijF_p(n-p+5))^2 \\
&= [F_p^2(n-p+2) - F_p^2(n-p+3) - F_p^2(n-p+4) + F_p^2(n-p+5)] \\
&\quad + 2i[F_p(n-p+2)F_p(n-p+3) - F_p(n-p+4)F_p(n-p+5)] \\
&\quad + 2j[F_p(n-p+2)F_p(n-p+4) - F_p(n-p+3)F_p(n-p+5)] \\
&\quad + 2ij[F_p(n-p+2)F_p(n-p+5) + F_p(n-p+3)F_p(n-p+4)].
\end{aligned}$$

$$\begin{aligned}
Q_p^2(n-p+1) + Q_p^2(n-p+2) &= [F_p^2(n-p+1) - F_p^2(n-p+2) - F_p^2(n-p+3) \\
&\quad + F_p^2(n-p+4)] + 2i[F_p(n-p+1)F_p(n-p+2) \\
&\quad - F_p(n-p+3)F_p(n-p+4)] + 2j[F_p(n-p+1)F_p(n-p+3) \\
&\quad - F_p(n-p+2)F_p(n-p+4)] + 2ij[F_p(n-p+1)F_p(n-p+4) \\
&\quad + F_p(n-p+2)F_p(n-p+3)] + [F_p^2(n-p+2) - F_p^2(n-p+3) \\
&\quad - F_p^2(n-p+4) + F_p^2(n-p+5)] \\
&\quad + 2i[F_p(n-p+2)F_p(n-p+3) - F_p(n-p+4)F_p(n-p+5)] \\
&\quad + 2j[F_p(n-p+2)F_p(n-p+4) - F_p(n-p+3)F_p(n-p+5)] \\
&\quad + 2ij[F_p(n-p+2)F_p(n-p+5) + F_p(n-p+3)F_p(n-p+4)] \\
&= [F_p^2(n-p+1) - 2F_p^2(n-p+3) + F_p^2(n-p+5)] \\
&\quad + 2i[F_p(n-p+1)F_p(n-p+2) - F_p(n-p+4)F_p(n-p+5)]
\end{aligned}$$

$$\begin{aligned}
& - F_p(n-p+3)(F_p(n-2p+3) + F_p(n-2p+2)) \\
& - 2j[F_p(n-p+3)(F_p(n-2p+4) + F_p(n-2p+3) \\
& + F_p(n-2p+2) + F_p(n-2p+1))] \\
& + 2ij[F_p(n-p+4)(F_p(n-p+1) + F_p(n-p+3)) \\
& + F_p(n-p+2)(F_p(n-p+3) + F_p(n-p+5))].
\end{aligned}$$

$$Q_p^2(n-p+2) - Q_p^2(n-p+1)$$

$$\begin{aligned}
& = (F_p(n-p+2) + iF_p(n-p+3)) + jF_p(n-p+4) + ijF_p(n-p+5))^2 - (F_p(n-p+1) \\
& + iF_p(n-p+2)) + jF_p(n-p+3) + ijF_p(n-p+4))^2 \\
& = [F_p^2(n-p+2) - F_p^2(n-p+3) - F_p^2(n-p+4) + F_p^2(n-p+5)] \\
& + 2i[F_p(n-p+2)F_p(n-p+3) - F_p(n-p+4)F_p(n-p+5)] \\
& + 2j[F_p(n-p+2)F_p(n-p+4) - F_p(n-p+3)F_p(n-p+5)] \\
& + 2ij[F_p(n-p+2)F_p(n-p+5) + F_p(n-p+3)F_p(n-p+4)] - [F_p^2(n-p+1) \\
& - F_p^2(n-p+2) - F_p^2(n-p+3) + F_p^2(n-p+4)] - 2i[F_p(n-p+1)F_p(n-p+2) \\
& - F_p(n-p+3)F_p(n-p+4)] - 2j[F_p(n-p+1)F_p(n-p+3) - F_p(n-p+2)F_p(n-p+4)] \\
& - 2ij[F_p(n-p+1)F_p(n-p+4) + F_p(n-p+2)F_p(n-p+3)] \\
& = [F_p^2(n-p+5) - 2F_p^2(n-p+4) + 2F_p^2(n-p+2) - F_p^2(n-p+1)] + 2i[F_p(n-p+2) \\
& (F_p(n-2p+2) + F_p(n-2p+1)) - F_p(n-p+4)(F_p(n-2p+4) + F_p(n-2p+3))] \\
& + 2j[2F_p(n-p+2)F_p(n-p+4) - F_p(n-p+3)(F_p(n-p+5) + F_p(n-p+1))] \\
& + 2ij[F_p(n-p+2)(F_p(n-2p+4) + F_p(n-2p+3)) + F_p(n-p+4)(F_p(n-2p+2) \\
& + F_p(n-2p+1))].
\end{aligned}$$

$$Q_p(n-p+1) - iQ_p(n-p+2) + jQ_p(n-p+3) - ijQ_p(n-p+4)$$

$$\begin{aligned}
& = [F_p(n-p+1) + iF_p(n-p+2) + jF_p(n-p+3) + ijF_p(n-p+4)] \\
& - i[F_p(n-p+2) + iF_p(n-p+3) + jF_p(n-p+4) + ijF_p(n-p+5)] \\
& + j[F_p(n-p+3) + iF_p(n-p+4) + jF_p(n-p+5) + ijF_p(n-p+6)] \\
& - ij[F_p(n-p+4) + iF_p(n-p+5) + jF_p(n-p+6) + ijF_p(n-p+7)] \\
& = [F_p(n-p+1) + F_p(n-p+3) - F_p(n-p+5) - F_p(n-p+7)] \\
& + 2j[F_p(n-p+3) + F_p(n-p+5)].
\end{aligned}$$

$$Q_p(n-p+1) - iQ_p(n-p+2) - jQ_p(n-p+3) - ijQ_p(n-p+4)$$

$$\begin{aligned}
& = [F_p(n-p+1) + iF_p(n-p+2) + jF_p(n-p+3) + ijF_p(n-p+4)] \\
& - i[F_p(n-p+2) + iF_p(n-p+3) + jF_p(n-p+4) + ijF_p(n-p+5)] \\
& - j[F_p(n-p+3) + iF_p(n-p+4) + jF_p(n-p+5) + ijF_p(n-p+6)] \\
& - ij[F_p(n-p+4) + iF_p(n-p+5) + jF_p(n-p+6) + ijF_p(n-p+7)]
\end{aligned}$$

$$= [F_p(n-p+1) + F_p(n-p+3) + F_p(n-p+5) - F_p(n-p+7)] \\ + 2iF_p(n-p+6) + 2jF_p(n-p+5) - 2ijF_p(n-p+4).$$

□

**Theorem 5.** Binet formula for the bicomplex Fibonacci  $p$  quaternions is  $Q_p(n) = k_1(x_1)^n(1 + ix_1 + jx_1^2 + ijx_1^3) + k_2(x_2)^n(1 + ix_2 + jx_2^2 + ijx_2^3) + \dots + k_{p+1}(x_{p+1})^n(1 + ix_{p+1} + jx_{p+1}^2 + ijx_{p+1}^3)$ , where  $x_1, x_2, x_3, \dots, x_{p+1}$  are roots of the characteristic equation  $x^{p+1} - x^p - 1 = 0$ , and  $k_1, k_2, k_3, \dots, k_{p+1}$  are constants.

*Proof.*  $Q_p(n) = F_p(n) + iF_p(n+1) + jF_p(n+2) + ijF_p(n+3) = k_1(x_1)^n + k_2(x_2)^n + \dots + k_{p+1}(x_{p+1})^n + i(k_1(x_1)^{n+1} + k_2(x_2)^{n+1} + \dots + k_{p+1}(x_{p+1})^{n+1}) + j(k_1(x_1)^{n+2} + k_2(x_2)^{n+2} + \dots + k_{p+1}(x_{p+1})^{n+2}) + ij(k_1(x_1)^{n+3} + k_2(x_2)^{n+3} + \dots + k_{p+1}(x_{p+1})^{n+3})$ , where  $x_1, x_2, x_3, \dots, x_{p+1}$  are roots of the characteristic equation  $x^{p+1} - x^p - 1 = 0$ , and  $k_1, k_2, k_3, \dots, k_{p+1}$  are constant coefficients that depends on the initial elements of the Fibonacci  $p$ -numbers.

$$\begin{aligned} F_p(0) &= k_1 + k_2 + k_3 + \dots + k_{p+1} = 0, \\ F_p(1) &= k_1x_1 + k_2x_2 + k_3x_3 + \dots + k_{p+1}x_{p+1} = 1, \\ F_p(2) &= k_1x_1^2 + k_2x_2^2 + k_3x_3^2 + \dots + k_{p+1}x_{p+1}^2 = 1, \\ &\vdots \\ F_p(p) &= k_1x_1^p + k_2x_2^p + k_3x_3^p + \dots + k_{p+1}x_{p+1}^p = 1. \end{aligned}$$

We can find the values of  $k_1, k_2, k_3, \dots, k_{p+1}$  after solving the above system of equations. Now,  $Q_p(n) = k_1(x_1)^n(1 + ix_1 + jx_1^2 + ijx_1^3) + k_2(x_2)^n(1 + ix_2 + jx_2^2 + ijx_2^3) + \dots + k_{p+1}(x_{p+1})^n(1 + ix_{p+1} + jx_{p+1}^2 + ijx_{p+1}^3)$ , where  $x_1, x_2, x_3, \dots, x_{p+1}$  are roots of the characteristic equation  $x^{p+1} - x^p - 1 = 0$ , and  $k_1, k_2, k_3, \dots, k_{p+1}$  are constants.

□

**Theorem 6.** Let  $n, m \geq 0$  the Honsberger identity for the bicomplex Fibonacci  $p$  quaternions  $Q_p(n), Q_p(m)$  is given by

$$\begin{aligned} Q_p(n)Q_p(m) + Q_p(n+1)Q_p(m+1) &= [F_p(n)F_p(m) - 2F_p(n+2)F_p(m+2) \\ &\quad + F_p(n+4)F_p(m+4)] + i[F_p(m+1)(F_p(n) + F_p(n+2)) \\ &\quad - F_p(m+3)(F_p(n+2) + F_p(n+4)) \\ &\quad - F_p(m+2)(F_p(n-p+2) + F_p(n-p+1)) \\ &\quad + (F_p(n+1)F_p(m) - F_p(n+3)F_p(m+4))] \\ &\quad - j[F_p(n+2)(F_p(m-p+3) + F_p(m-p+2)) \\ &\quad + F_p(m-p+1) + F_p(m-p)] + F_p(m+2)(F_p(n-p+3) \\ &\quad + F_p(n-p+2) + F_p(n-p+1) + F_p(n-p))] \\ &\quad + ij[F_p(m+1)(F_p(n+2) + F_p(n+4)) \\ &\quad + F_p(m+3)(F_p(n) + F_p(n+2)) + F_p(n+3)(F_p(m) \\ &\quad + F_p(m+2)) + F_p(n+1)(F_p(m+2) + F_p(m+4))]. \end{aligned}$$

*Proof.*

$$\begin{aligned}
Q_p(n)Q_p(m) + Q_p(n+1)Q_p(m+1) &= [(F_p(n) + iF_p(n+1)) + j(F_p(n+2) + iF_p(n+3))][[(F_p(m) \\
&\quad + iF_p(m+1)) + j(F_p(m+2) + iF_p(m+3))] + [(F_p(n+1) \\
&\quad + iF_p(n+2)) + j(F_p(n+3) + iF_p(n+4))][[(F_p(m+1) \\
&\quad + iF_p(m+2)) + j(F_p(m+3) + iF_p(m+4))] \\
&= (F_p(n)F_p(m) - F_p(n+1)F_p(m+1) - F_p(n+2)F_p(m+2) \\
&\quad + F_p(n+3)F_p(m+3)) + i(F_p(n)F_p(m+1) + F_p(n+1) \\
&\quad F_p(m) - F_p(n+2)F_p(m+3) - F_p(n+3)F_p(m+2)) \\
&\quad + j(F_p(n)F_p(m+2) + F_p(n+2)F_p(m) - F_p(n+1) \\
&\quad F_p(m+3) - F_p(n+3)F_p(m+1)) + ij(F_p(n)F_p(m+3) \\
&\quad + F_p(n+3)F_p(m) + F_p(n+1)F_p(m+2) + F_p(n+2) \\
&\quad F_p(m+1)) + (F_p(n+1)F_p(m+1) - F_p(n+2)F_p(m+2) \\
&\quad - F_p(n+3)F_p(m+3) + F_p(n+4)F_p(m+4)) \\
&\quad + i(F_p(n+1)F_p(m+2) + F_p(n+2)F_p(m+1) \\
&\quad - F_p(n+3)F_p(m+4) - F_p(n+4)F_p(m+3)) \\
&\quad + j(F_p(n+1)F_p(m+3) + F_p(n+3)F_p(m+1) \\
&\quad - F_p(n+2)F_p(m+4) - F_p(n+4)F_p(m+2)) \\
&\quad + ij(F_p(n+1)F_p(m+4) + F_p(n+4)F_p(m+1) \\
&\quad + F_p(n+2)F_p(m+3) + F_p(n+3)F_p(m+2)) \\
&= [F_p(n)F_p(m) - 2F_p(n+2)F_p(m+2) + F_p(n+4) \\
&\quad F_p(m+4)] + i[F_p(m+1)(F_p(n) + F_p(n+2)) \\
&\quad - F_p(m+3)(F_p(n+2) + F_p(n+4)) - F_p(m+2) \\
&\quad (F_p(n-p+2) + F_p(n-p+1)) + (F_p(n+1)F_p(m) \\
&\quad - F_p(n+3)F_p(m+4))] - j[F_p(n+2)(F_p(m-p+3) \\
&\quad + F_p(m-p+2) + F_p(m-p+1) + F_p(m-p)) \\
&\quad + F_p(m+2)(F_p(n-p+3) + F_p(n-p+2) \\
&\quad + F_p(n-p+1) + F_p(n-p))] + ij[F_p(m+1)(F_p(n+2) \\
&\quad + F_p(n+4)) + F_p(m+3)(F_p(n) + F_p(n+2)) + F_p(n+3) \\
&\quad (F_p(m) + F_p(m+2)) + F_p(n+1)(F_p(m+2) + F_p(m+4))].
\end{aligned}$$

□

**Theorem 7.** Let  $n, m \geq 0$  the D'Ocagne's identity for the bicomplex Fibonacci  $p$  quaternions  $Q_p(n)$ ,  $Q_p(m)$  is given by  $Q_p(n)Q_p(m+1) - Q_p(n+1)Q_p(m) = [F_p(n)F_p(m+1) - F_p(n+4)F_p(m+3) - F_p(n+1)(F_p(m) + F_p(m+2)) - F_p(n+2)(F_p(m+1) + F_p(m+3)) + F_p(n+3)(F_p(m+2) + F_p(m+4))] + i[F_p(m+2)(F_p(n) + F_p(n+4)) - F_p(n+2)(F_p(m) + F_p(m+4))] + j[F_p(n)F_p(m+3) + F_p(n+4)F_p(m+1) - F_p(n+1)(F_p(m+2) + F_p(m+4)) + F_p(n+3)(F_p(m+1) + F_p(m+3)) - F_p(n+2)F_p(m+2) - F_p(n+3)F_p(m+2) - F_p(n+4)F_p(m+3)] + ij[F_p(n)F_p(m+3) + F_p(n+3)F_p(m) - F_p(n+1)F_p(m+3) - F_p(n+3)F_p(m+1) + F_p(n+3)F_p(m+1) + F_p(n+1)F_p(m+2) + F_p(n+2)F_p(m+1) + F_p(n+2)F_p(m+3) + F_p(n+3)F_p(m+2)]$ .

$$2)(F_p(m+1) + F_p(m+3)) - F_p(n+3)(F_p(m) + F_p(m+2))] + ij[F_p(n)F_p(m+4) - F_p(n+4)F_p(m)].$$

*Proof.*  $Q_p(n)Q_p(m+1) - Q_p(n+1)Q_p(m)$

$$\begin{aligned} &= [(F_p(n) + iF_p(n+1)) + j(F_p(n+2) + iF_p(n+3))][(F_p(m+1) + iF_p(m+2)) + j(F_p(m+3) \\ &\quad + iF_p(m+4))] - [(F_p(n+1) + iF_p(n+2)) + j(F_p(n+3) + jF_p(n+4))][(F_p(m) \\ &\quad + iF_p(m+1)) + j(F_p(m+2) + jF_p(m+3))] \\ &= (F_p(n)F_p(m+1) - F_p(n+1)F_p(m+2) - F_p(n+2)F_p(m+3) + F_p(n+3)F_p(m+4)) \\ &\quad + i(F_p(n)F_p(m+2) + F_p(n+1)F_p(m+1) - F_p(n+2)F_p(m+4) - F_p(n+3)F_p(m+3)) \\ &\quad + j(F_p(n)F_p(m+3) + F_p(n+2)F_p(m+1) - F_p(n+1)F_p(m+4) - F_p(n+3)F_p(m+2)) \\ &\quad + ij(F_p(n)F_p(m+4) + F_p(n+3)F_p(m+1) + F_p(n+1)F_p(m+3) + F_p(n+2)F_p(m+2)) \\ &\quad - (F_p(n+1)F_p(m) - F_p(n+2)F_p(m+1) - F_p(n+3)F_p(m+2) + F_p(n+4)F_p(m+3)) \\ &\quad - i(F_p(n+1)F_p(m+1) + F_p(n+2)F_p(m) - F_p(n+3)F_p(m+3) - F_p(n+4)F_p(m+2)) \\ &\quad - j(F_p(n+1)F_p(m+2) + F_p(n+3)F_p(m) - F_p(n+2)F_p(m+3) - F_p(n+4)F_p(m+1)) \\ &\quad - ij(F_p(n+1)F_p(m+3) + F_p(n+4)F_p(m) + F_p(n+2)F_p(m+2) + F_p(n+3)F_p(m+1)) \\ &= [F_p(n)F_p(m+1) - F_p(n+4)F_p(m+3) - F_p(n+1)(F_p(m) + F_p(m+2)) \\ &\quad - F_p(n+2)(F_p(m+1) + F_p(m+3)) + F_p(n+3)(F_p(m+2) + F_p(m+4))] \\ &\quad + i[F_p(m+2)(F_p(n) + F_p(n+4)) - F_p(n+2)(F_p(m) + F_p(m+4))] \\ &\quad + j[F_p(n)F_p(m+3) + F_p(n+4)F_p(m+1) - F_p(n+1)(F_p(m+2) \\ &\quad + F_p(m+4)) + F_p(n+2)(F_p(m+1) + F_p(m+3)) - F_p(n+3)(F_p(m) + F_p(m+2))] \\ &\quad + ij[F_p(n)F_p(m+4) - F_p(n+4)F_p(m)]. \end{aligned}$$

□

**Theorem 8.** Let  $Q_p(n)$  be the bicomplex Fibonacci  $p$  quaternions. For  $n \geq 1$ , Cassini's identity for  $Q_p(n)$  is given by  $Q_p(n-1)Q_p(n+1) - Q_p^2(n) = [F_p^2(n+2) + F_p^2(n+1) - F_p^2(n) - F_p^2(n+3) + F_p(n+1)(F_p(n-p+3) - F_p(n-p-1)) + F_p(n-p+1)(F_p(n-p+3) + F_p(n-p+2) + F_p(n-p+1) + F_p(n-p))] + i[F_p^2(n+2) - F_p^2(n) + F_p(n+1)F_p(n+4) + F_p(n-1)F_p(n+2) + F_p(n+2)F_p(n-p+2) - F_p(n)F_p(n-p)] + j[F_p^2(n+1) - F_p^2(n+2) - F_p(n-p+2)(F_p(n) - F_p(n-p)) - F_p(n+2)(F_p(n-p-1) + F_p(n-p))] + ij[F_p(n+4)F_p(n-1) - F_p(n+3)F_p(n)]$ .

*Proof.*  $Q_p(n-1)Q_p(n+1) - Q_p^2(n)$

$$\begin{aligned} &= [(F_p(n-1) + iF_p(n)) + j(F_p(n+1) + iF_p(n+2))][(F_p(n+1) + iF_p(n+2)) + j(F_p(n+3) \\ &\quad + iF_p(n+4))] - [(F_p(n) + iF_p(n+1)) + j(F_p(n+2) + iF_p(n+3))]^2 \\ &= (F_p(n-1)F_p(n+1) - F_p(n)F_p(n+2) - F_p(n+1)F_p(n+3) + F_p(n+2)F_p(n+4)) \\ &\quad + i(F_p(n-1)F_p(n+2) + F_p(n)F_p(n+1) - F_p(n+1)F_p(n+4) - F_p(n+2)F_p(n+3)) \\ &\quad + j(F_p(n-1)F_p(n+3) - F_p(n+1)F_p(n+4) + F_p^2(n+1) - F_p^2(n+2)) \\ &\quad + ij(F_p(n-1)F_p(n+4) + F_p(n)F_p(n+3) + 2F_p(n+1)F_p(n+2)) \\ &\quad - (F_p^2(n) + F_p^2(n+3) - F_p^2(n+1) - F_p^2(n+2)) - 2i(F_p(n)F_p(n+1) - F_p(n+2)F_p(n+3)) \end{aligned}$$

$$\begin{aligned}
& -2j(F_p(n)F_p(n+2) - F_p(n+1)F_p(n+3)) - 2ij(F_p(n)F_p(n+3) + F_p(n+1)F_p(n+2)) \\
= & [F_p^2(n+2) + F_p^2(n+1) - F_p^2(n) - F_p^2(n+3) + F_p(n+1)(F_p(n-p+3) - F_p(n-p-1)) \\
& + F_p(n-p+1)(F_p(n-p+3) + F_p(n-p+2) + F_p(n-p+1) + F_p(n-p))] \\
& + i[F_p^2(n+2) - F_p^2(n) + F_p(n+1)F_p(n+4) + F_p(n-1)F_p(n+2) + F_p(n+2)F_p(n-p+2) \\
& - F_p(n)F_p(n-p)] + j[F_p^2(n+1) - F_p^2(n+2) - F_p(n-p+2)(F_p(n) - F_p(n-p)) \\
& - F_p(n+2)(F_p(n-p-1) + F_p(n-p))] + ij[F_p(n+4)F_p(n-1) - F_p(n+3)F_p(n)].
\end{aligned}$$

□

**Theorem 9.** Let  $Q_p(n+r)$  be the bicomplex Fibonacci  $p$  quaternions. For  $n \geq 1$ , Catalan's identity for  $Q_p(n+r)$  is given by

$$\begin{aligned}
Q_p(n+r-1)Q_p(n+r+1) - Q_p^2(n+r) = & [F_p^2(n+r+2) + F_p^2(n+r+1) - F_p^2(n+r) \\
& - F_p^2(n+r+3) + F_p(n+r+1)(F_p(n+r-p+3) \\
& - F_p(n+r-p-1)) + F_p(n+r-p+1)(F_p(n+r \\
& - p+3) + F_p(n+r-p+2) + F_p(n+r-p+1) \\
& + F_p(n+r-p))] + i[F_p^2(n+r+2) - F_p^2(n+r) \\
& + F_p(n+r+1)F_p(n+r+4) \\
& + F_p(n+r-1)F_p(n+r+2) \\
& + F_p(n+r+2)F_p(n+r-p+2) \\
& - F_p(n+r)F_p(n+r-p)] + j[F_p^2(n+r+1) \\
& - F_p^2(n+r+2) - F_p(n+r-p+2)(F_p(n+r) \\
& - F_p(n+r-p)) - F_p(n+r+2)(F_p(n+r-p-1) \\
& + F_p(n+r-p))] + ij[F_p(n+r+4)F_p(n+r-1) \\
& - F_p(n+r+3)F_p(n+r)].
\end{aligned}$$

*Proof.*

$$\begin{aligned}
Q_p(n+r-1)Q_p(n+r+1) - Q_p^2(n+r) = & [(F_p(n+r-1) + iF_p(n+r)) + j(F_p(n+r+1) \\
& + iF_p(n+r+2))][(F_p(n+r+1) + iF_p(n+r+2)) \\
& + j(F_p(n+r+3) + iF_p(n+r+4))] - [(F_p(n+r) \\
& + iF_p(n+r+1)) + j(F_p(n+r+2) + iF_p(n+r+3))]^2 \\
= & (F_p(n+r-1)F_p(n+r+1) - F_p(n+r)F_p(n+r+2) \\
& - F_p(n+r+1)F_p(n+r+3) + F_p(n+r+2)F_p(n \\
& + r+4)) + i(F_p(n+r-1)F_p(n+r+2) \\
& + F_p(n+r)F_p(n+r+1) - F_p(n+r+1)F_p(n+r+4) \\
& - F_p(n+r+2)F_p(n+r+3)) \\
& + j(F_p(n+r-1)F_p(n+r+3)
\end{aligned}$$

$$\begin{aligned}
& - F_p(n+r+1)F_p(n+r+4) + F_p^2(n+r+1) \\
& - F_p^2(n+r+2)) + ij(F_p(n+r-1)F_p(n+r+4) \\
& + F_p(n+r)F_p(n+r+3) + 2F_p(n+r+1)F_p(n+r+2)) \\
& - (F_p^2(n+r) + F_p^2(n+r+3) - F_p^2(n+r+1) \\
& - F_p^2(n+r+2)) - 2i(F_p(n+r)F_p(n+r+1) \\
& - F_p(n+r+2)F_p(n+r+3)) \\
& - 2j(F_p(n+r)F_p(n+r+2) - F_p(n+r+1)F_p(n+r+3)) \\
& - 2ij(F_p(n+r)F_p(n+r+3) + F_p(n+r+1)F_p(n+r+2)) \\
= & [F_p^2(n+r+2) + F_p^2(n+r+1) - F_p^2(n+r) \\
& - F_p^2(n+r+3) + F_p(n+r+1)(F_p(n+r-p+3) \\
& - F_p(n+r-p-1)) + F_p(n+r-p+1)(F_p(n+r \\
& - p+3) + F_p(n+r-p+2) + F_p(n+r-p+1) \\
& + F_p(n+r-p))] + i[F_p^2(n+r+2) - F_p^2(n+r) \\
& + F_p(n+r+1)F_p(n+r+4) \\
& + F_p(n+r-1)F_p(n+r+2) \\
& + F_p(n+r+2)F_p(n+r-p+2) \\
& - F_p(n+r)F_p(n+r-p)] + j[F_p^2(n+r+1) \\
& - F_p^2(n+r+2) - F_p(n+r-p+2)(F_p(n+r) \\
& - F_p(n+r-p)) - F_p(n+r+2)(F_p(n+r-p-1) \\
& + F_p(n+r-p))] + ij[F_p(n+r+4)F_p(n+r-1) \\
& - F_p(n+r+3)F_p(n+r)].
\end{aligned}$$

□

### 3 Conclusion

In this paper, we introduce bicomplex Fibonacci  $p$  quaternions and some properties of bicomplex Fibonacci  $p$  quaternions, Honsberger identity, D'Ocagne's identity, Cassini's identity, Catalan's identity for bicomplex Fibonacci  $p$  quaternions. I hope that these results will be useful in applied mathematics, quantum mechanics, quantum physics, Lie groups, Kinematics and differential equations like bicomplex Fibonacci quaternions [10-12].

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