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The relationships of Ivar(G) with inner automorphisms in S(G)-autonilpotent groups

Sara Barin[†*], **Mohammad Mehdi Nasrabadi**[‡]

[† ‡] *Department of Mathematics, University of Birjand, Birjand, Iran*
Emails: s_barin10@yahoo.com, dr.mm.Nasrabadi@gmail.com

Abstract. Bachmuth [4] defined an IA-automorphism of a group G as an automorphism of G, which induces the identity automorphism on G/G'. Ghumde and Ghate [8] introduced S(G) and Ivar(G) subgroups. In this paper, we first introduce a new series on the IA-central subgroup and verify the relationships of the members of this series. Also, we give a new definition for S(G)-autonilpotency on this series. Then, we discuss some properties of these concepts with some theorem and their corollaries. We investigate the members of Ivar(G) fixing the center element-wise. At the end of this paper, we study the conditions in which Ivar(G) relates to Inner automorphisms in S(G)-autonilpotent groups.

Keywords: IA-group, Ivar(G), Inner automorphisms, IA-central subgroup, S(G)-autonilpotent groups.
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1 Introduction

Let G be a group. Throughout the paper, p denotes a prime number. Let us denote by C_n, G', Z(G), $Z_n(G)$, $\phi(G)$, Ker(G), exp(G), rank(G), id(G), Hom(G, H), Inn(G), End(G) and Aut(G), respectively the cyclic group of order n, the commutator subgroup, the centre, the n-th term of the upper central series, the Frattini subgroup, the kernel, the exponent, the rank, the identity automorphism, the group of homomorphisms of G into an abelian group H, the Inner automorphisms, the group of endomorphisms of (G) and the full automorphism group. Let O(g) be order $g \in G$ and $G^{p^n} = \langle g^{p^n} \mid g \in G \rangle$. Also,

$$Aut_c(G) = \left\{ \alpha \in Aut(G) \;\middle|\; g^{-1}\alpha(g) \in Z(G),\ \forall\, g \in G \right\}$$

is central automorphisms group. Let $H \leqslant G$, then

$$C_{Aut(G)}(H) = \{\alpha \in Aut(G) \mid \alpha(h) = h,\ \forall\, h \in H\}.$$

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Bachmuth [4] defined an IA-automorphism of a group G as

$$IA(G) = \left\{\alpha \in Aut(G) \;\middle|\; [g,\alpha] = g^{-1}\alpha(g) \in G',\ \forall\, g \in G\right\}.$$

For any group G, $Inn(G) \leqslant IA(G)$. Hegarty [9] in 1994 introduced the absolute center

$$L(G) = \left\{g \in G \;\middle|\; g^{-1}\alpha(g) = 1,\ \forall\, \alpha \in Aut(G)\right\},$$

and absolute central automorphisms as follows:

$$Aut_l(G) = \left\{\alpha \in Aut(G) \;\middle|\; g^{-1}\alpha(g) \in L(G),\ \forall\, g \in G\right\}.$$

On the similar lines, Ghumde and Ghate [8] in 2015 introduced the IA-central subgroup

$$S(G) = \left\{g \in G \mid g^{-1}\alpha(g) = 1,\ \alpha \in IA(G)\right\},$$

and Ivar(G) group as follows:

$$Ivar(G) = \left\{\alpha \in IA(G) \mid g^{-1}\alpha(g) \in S(G),\ \forall\, g \in G\right\}.$$

For any group G, $L(G) \trianglelefteq S(G) \trianglelefteq Z(G)$. We have studied S(G) and Ivar(G) in detail in [7] and [5], respectively.

We define

$$\mathcal{E}(G) = [G, C_{IA(G)}\big(Ivar(G)\big)],$$

where

$$C_{IA(G)}\big(Ivar(G)\big) = \{\alpha \in IA(G) \mid \sigma\alpha = \alpha\sigma,\ \forall\, \sigma \in Ivar(G)\}.$$

For every group G, $G' \leqslant \mathcal{E}(G) \overset{ch}{\leqslant} G$ and Ivar(G) acts trivially on $\mathcal{E}(G)$.

Since the center of the group and abelian groups are of great importance, also, S(G), Ivar(G) and $\mathcal{E}(G)$ are new abelian groups, and all abelian groups are S(G)-autonilpotent, in this article we studied the properties of these groups.

In the following, we give some results that will be used in our proofs.

Lemma 1 ([11], Lemma 0.4)**.** *Let G be a finite nilpotent group of class 2, then*

a) $G' \leqslant Z(G)$.

b) $exp(G') = exp\big(G/Z(G)\big)$.

Lemma 2 ([15], p. 649)**.** *Let A, B and G be abelian groups, then*

a) $Hom(A, B \times G) \cong Hom(A, B) \times Hom(A, G)$.

b) $Hom(A \times B, G) \cong Hom(A, G) \times Hom(B, G)$.

c) $Hom(C_m, C_n) \cong C_d$ *where d=gcd(m,n).*

Lemma 3 ([12], Lemma 2.3)**.** *Let G be an abelian p-group, and H is cyclic group of order divisible by exp(G), then* $Hom(G, H) \cong G$.

Proposition 1 ([5], Proposition 2.12)**.** *Let G be a group, Then* $Ivar(G) \cong Hom\big(G/S(G), S(G) \cap G'\big)$. *In particular, Ivar(G) is an abelian group.*

Theorem 1 ([3], Theorem 7)**.** *Let G be a non-abelian finite p-group of class 2, and* $exp(G') = p^l$, *then* $S(G) = G'G^{p^l}$.

Because automorphisms have interesting properties, they have been the main idea of many works by mathematicians, including Akbarizadeh et. al. [1], which works on non-solvable automorphisms and Akbarizadeh et. al. [2], which is about the symmetry of graphs in the group of transitive automorphisms.

In this article, we investigate the IA-central series, its terms, and the automorphisms that are defined on its terms. In section 4, we introduce S(G)-autonilpotent groups and provide some results about the direct product of S(G)-autonilpotent groups. In section 5, we verify the conditions under which the n-th term of the IA-central series is non-trivial. Afterwards, we study some properties of Ivar(G). We verify that if members of Ivar(G) fix the center element-wise, then what conditions are held. In the last section, we consider S(G)-autonilpotent groups and investigate states in which Ivar(G) and Inn(G) relate together.

2 The terms of IA-central series

The concept of autonilpotency was defined by Parvaneh and Moghaddam [14] in 2010. They introduced the upper autocentral series as follows:

$$\langle 1 \rangle = L_0(G) \subseteq L_1(G) = L(G) \subseteq L_2(G) \subseteq \cdots \subseteq L_n(G) \subseteq \cdots,$$

where

$$\frac{L_n(G)}{L_{n-1}(G)} = L\Big(\frac{G}{L_{n-1}(G)}\Big), \qquad for\ n \geq 2$$

and $L_n(G)$ is the nth-absolute centre of G. Also, they called a group to be autonilpotent of class at most n if $L_n(G) = G$, for some positive integer n.

In this section, we first define a new series on the IA-central subgroup, and then we identify the relationships of the members of this series.

Definition 1. *We define the IA-central series of G in the following way:*

$$\langle 1 \rangle = S_0(G) \subseteq S_1(G) = S(G) \subseteq S_2(G) \subseteq \cdots \subseteq S_n(G) \subseteq \cdots,$$

where

$$S_n(G) = \{g \in G \mid [g, \alpha_1, \ldots, \alpha_n] = 1,\ \forall\, \alpha_1, \ldots, \alpha_n \in IA(G)\}, \quad n \geq 1.$$

For abelian groups, S(G)=G, so $S_n(G) = G$, for every natural number n.

Proposition 2. *Let G be any group, then for each* $g \in G$ *we have* $g \in S_n(G)$ *if and only if* $[g, \alpha] \in S_{n-1}(G)$, *for every* $\alpha \in IA(G)$.

Proof. Due to $S_n(G)$ definition and by inductive on n, the proposition is proved. □

The following corollary is an immediate consequence of the above proposition.

Corollary 1. *For each $g \in G$, we have $g \in S_n(G)$ if and only if $[S_{n-1}(G), IA(G)] = 1$.*

Proposition 3. *Let G be a group, then $S_n(G)$ is a characteristic subgroup of G.*

Proof. We prove the result by induction on n. We know that $S(G) \overset{ch}{\leqslant} G$. Let $S_i(G) \overset{ch}{\leqslant} G$, for $i = 0, 1, \ldots, n-1$ and consider $\alpha \in IA(G)$. Then $\alpha\big(S_{n-1}(G)\big) = S_{n-1}(G)$, by hypothesis. Let $s \in S_n(G)$, then by Proposition 2, $[s, \alpha] \in S_{n-1}(G)$. Therefore

$$[\beta(s), \beta\alpha\beta^{-1}] \in \beta\big(S_{n-1}(G)\big) \leqslant S_{n-1}(G),$$

for every $\beta \in Aut(G)$. Again, by Proposition 2, we have $\beta(s) \in S_n(G)$ and the result is follow. □

Theorem 2. *If the group $G = H_1 \times H_2$ is the direct product of its characteristic subgroups H_1 and H_2, then*

$$S_n(H_1 \times H_2) = S_n(H_1) \times S_n(H_2),$$

for every natural number n.

Proof. For $\alpha \in IA(H_1)$ and $\beta \in IA(H_2)$, we define the IA-automorphism $\alpha \times \beta$ of group $H_1 \times H_2$, given by $(\alpha \times \beta)\big((h_1, h_2)\big) = \big(\alpha(h_1), \beta(h_2)\big)$, for all $h_1 \in H_1$ and $h_2 \in H_2$. Now by induction on n, it is easy to check that $([h_1, \alpha_1, \ldots, \alpha_n], [h_2, \beta_1, \ldots, \beta_n]) = [(h_1, h_2), \alpha_1 \times \beta_1, \ldots, \alpha_n \times \beta_n]$, for all $\alpha_1, \ldots, \alpha_n \in IA(H_1)$ and $\beta_1, \ldots, \beta_n \in IA(H_2)$. This implies that

$$S_n(H_1) \times S_n(H_2) \subseteq S_n(H_1 \times H_2).$$

Conversely, it is easy to check that $\sigma_{|_{H_1}} \in IA(H_1)$ and $\sigma_{|_{H_2}} \in IA(H_2)$, for all $\sigma \in IA(H_1 \times H_2)$. Now by induction on n, we have

$$[(h_1, h_2), \sigma_1, \ldots, \sigma_n] = \big([h_1, \sigma_{1_{|_{H_1}}}, \ldots, \sigma_{n_{|_{H_1}}}], [h_2, \sigma_{1_{|_{H_2}}}, \ldots, \sigma_{n_{|_{H_2}}}]\big),$$

for all $h_1 \in H_1$, $h_2 \in H_2$ and $\sigma_1, \ldots, \sigma_n \in IA(H_1 \times H_2)$, which gives the result. □

In the above theorem, according to the definition of S_n, the defined homomorphism for the direct product of groups affects the structure of $S_n(H_1 \times H_2)$.

Corollary 2. *If H_1 and H_2 be two finite groups such that $(|H_1|, |H_2|) = 1$, then*

$$S_n(H_1 \times H_2) = S_n(H_1) \times S_n(H_2).$$

3 The automorphisms on the terms of IA-commutator series

In this section, after some new definitions, we give our results about the automorphisms on the IA-central series that specify the relationships of these automorphisms with IA(G), $Aut_l(G)$, Ivar(G), Inn(G), and each other. Throughout the section, j is a positive integer.

Definition 2. *The kernel of the natural homomorphism from Aut(G) to Aut($G/S_j(G)$) is called the group of S_j-automorphism and is denoted by $Aut_{S_j}(G)$.*

According to the above definition, a S_j-automorphism group acts as the identity on G modulo $S_j(G)$. Thus:

$$Aut_{S_j}(G) = \{\alpha \in Aut(G) \mid g^{-1}\alpha(g) \in S_j(G),\ \forall\, g \in G\} \trianglelefteq Aut(G).$$

Also, we have $Aut_l(G) \leqslant Aut_{S_j}(G)$, for every j.

We use the notation $IA_{S_j}(G) = IA(G) \cap Aut_{S_j}(G)$. Another definition of $IA_{S_j}(G)$ is given by

$$IA_{S_j}(G) = \{\alpha \in Aut(G) \mid g^{-1}\alpha(g) \in S_j(G) \cap G',\ \forall\, g \in G\} \trianglelefteq Aut(G).$$

According to this notation, we have $IA_{S_1}(G) = Ivar(G)$.

Proposition 4. *For any group G,*

a) $\varphi \in Aut_{S_j}(G)$ *if and only if* $[\alpha, \varphi] \in Aut_{S_j}(G)$*, for every* $\alpha \in Aut(G)$*.*

b) $\dfrac{IA(G)}{IA_{S_j}(G)} \cong \dfrac{IA(G)Aut_{S_j}(G)}{Aut_{S_j}(G)}$.

Proof. a) It is obvious by the normality of $Aut_{S_j}(G)$.

b) The result follow from the definition of $IA_{S_j}(G)$ and the third isomorphism theorem. □

Corollary 3. *For any group G,* $[Aut(G), Aut_{S_j}(G)] \leqslant Aut_{S_j}(G)$.

Theorem 3. *Let G be a group. If* $IA\big(G/S_j(G)\big) = Inn\big(G/S_j(G)\big)$*, then*

$$IA(G) \leqslant Inn(G)Aut_{S_j}(G).$$

Proof. Let $\alpha \in IA(G)$. By hypothesis, $IA\big(G/S_j(G)\big) = Inn\big(G/S_j(G)\big)$, so there exists $g \in G$ such that $\alpha(x)S_j(G) = x^g S_j(G)$, for all $x \in G$. Hence, $x^{-g}\alpha(x) = \Big(x^{-1}\big(\alpha(x)\big)^{g^{-1}}\Big)^g \in S_j(G)$, then $x^{-1}\big(\alpha(x)\big)^{g^{-1}} \in S_j(G)$, $x^{-1}g\big(\alpha(x)\big)g^{-1} \in S_j(G)$, and $x^{-1}\varphi_g^{-1}\alpha(x) \in S_j(G)$, where $\varphi_g \in Inn(G)$.

Consequently, $\varphi_g^{-1}\alpha \in Aut_{S_j}(G)$, i.e.,

$$\alpha = \varphi_g\varphi_g^{-1}\alpha \in Inn(G)Aut_{S_j}(G).$$

□

In the special case j=1, we have the following result.

Corollary 4. *Let G be a group. If* $IA\big(G/S(G)\big) = Inn\big(G/S(G)\big)$*, then*

$$IA(G) \leqslant Inn(G)Ivar(G).$$

4 S(G)-autonilpotent groups

In section 2, we saw that Parvaneh and Moghaddam [14] introduced autonilpotent groups. In this section, we define S(G)-autonilpotent groups and give our main results about S(G)-autonilpotency and their properties.

Definition 3. *A group G is said to be S(G)-autonilpotent(or IA-nilpotent) group of class at most n if $S_n(G) = G$, for some natural number n.*

For an abelian group G, we know that $S_n(G) = G$, for every positive integer n. Therefore, abelian groups are S(G)-autonilpotent.

Remark 1. *Clearly, for a group G and every positive integer n, $L_n(G) \leqslant S_n(G) \leqslant Z_n(G)$. Thus, autonilpotent groups are S(G)-autonilpotent groups and S(G)-autonilpotent groups are nilpotent, but the converse of both results is not true in general.*

Example 1. a) $\mathbb{Z}_3$ is S(G)-autonilpotent, but is not autonilpotent.

b) The group $G = \langle a, b, x \mid [a, x] = [b, x] = 1,\ [a, b] = x^k,\ k \neq 1\rangle$ where IA-automorphism α defined by $\alpha(a) = ax^k$, $\alpha(b) = bx^k$, $\alpha(x) = x$ is a nilpotent group of class 2, but is not S(G)-autonilpotent.

Theorem 2 has two corollaries, giving important properties of S(G)-autonilpotent groups and their direct product.

Corollary 5. *If $G = H_1 \times H_2$ is the direct product of its characteristic subgroups such that H_1 or H_2 is not $S(H_1)$-autonilpotent or $S(H_2)$-autonilpotent, then so is not G.*

Corollary 6. *If $G_1,\ G_2,\ \ldots,\ G_k$ are $S(G_i)$-autonilpotent groups $(1 \leq i \leq k)$ with coprime orders, then $G_1 \times G_2 \times \cdots \times G_k$ is a $S(G_1 \times G_2 \times \cdots \times G_k)$-autonilpotent group.*

Proof. According to the definition of S(G)-autonilpotent groups, there exist positive integers n_1, n_2, $\ldots$, n_k such that

$$S_{n_1}(G_1) = G_1,\ \ S_{n_2}(G_2) = G_2,\ \ \ldots,\ \ S_{n_k}(G_k) = G_k.$$

Let $n = max\{n_1, n_2, \ldots, n_k\}$, then by induction on n, we have

$$S_n(G_1 \times G_2 \times \cdots \times G_k) = G_1 \times G_2 \times \cdots \times G_k.$$

□

5 When $S_n(G) \neq \langle 1\rangle$?

Now, we examine the conditions under which $S_n(G)$ is non-trivial. We saw that for abelian groups $S_n(G) = G$, therefore, in the following, we consider non-abelian groups.

Lemma 4. *Let G be a non-trivial S(G)-autonilpotent group, then $S_n(G)$ is also non-trivial.*

Proof. Because G is an S(G)-autonilpotent group, there exists a positive integer n such that $S_n(G) = G$. We assume by way of contradiction that $S(G) = \langle 1 \rangle$, then according to $S_n(G)$ definition and by proposition 2, $S_2(G) = \langle 1 \rangle$. Thus, we have $S_n(G) = \langle 1 \rangle$ for every positive integer n, contrary to the assumption. Hence $S(G) \neq \langle 1 \rangle$. □

Theorem 4. *Let G be a group and $H \leqslant G$, then $H \leqslant S_n(G)$ if one of the following conditions holds:*

1) $Aut(G) = C_{Aut(G)}(H)$.

2) G be a finite group and H be a characteristic subgroup of prime order p such that p be the smallest prime divisor of $|Aut(G)|$.

3) H be a cyclic and characteristic subgroup of G and Aut(G) be a perfect group.

Proof. Given that $L(G) \leqslant S(G) \leqslant S_n(G)$, the proof easily follow from [13, Lemma 2.4(iv), Corollary 3.5 and 3.7], respectively. □

Theorem 5. *Let G be a group, Aut(G) be a finite p-group, and H be a finite characteristic subgroup of G such that $p \big| |H|$, then $H \cap S_n(G) \neq \langle 1 \rangle$.*

Proof. Because H is a characteristic subgroup of G, this equivalence relation yields a partition of H and each cell in the partition arising from an equivalence relation is an equivalence class. According to [13, Lemma 2.5], there is $1 \neq h_0 \in H$ such that the equivalence class is of order 1. So we have $\alpha(h_0) = h_0$, for every $\alpha \in Aut(G)$. Thus $1 \neq h_0 \in S(G) \cap H$ and this completes the proof. □

Corollary 7. *If G be a finite group such that Aut(G) is a p-group, then $S_n(G) \neq \langle 1 \rangle$.*

Theorem 6 (MacHale [10])**.** *Let G be a finite group such that Aut(G) is nilpotent. If G is not cyclic of odd order, then G contains a non-trivial element which is left fixed by every automorphism of G.*

Corollary 8. *Let G be a finite group such that Aut(G) is nilpotent, then $S_n(G) \neq \langle 1 \rangle$.*

6 The relationships of Ivar(G) in p-groups

In this section, we continue with the elementary properties of Ivar(G) and study the conditions in which Ivar(G)=Inn(G) in p-groups.

Lemma 5. *Suppose G be a non-abelian finite p-group for which G/S(G) is abelian, then*

$$\left|Hom\Big(\frac{G}{Z(G)}, G'\Big)\right| \geq \left|\frac{G}{Z(G)}\right| p^{q(r-1)}$$

where $q = rank\big(G/Z(G)\big)$ and $r = rank(G')$.

Proof. Let $exp\big(G/Z(G)\big) = p^n$ and $exp(G') = p^l$. Since $G' \leqslant S(G) \leqslant Z(G)$, so G is nilpotent of class 2. Thus, $exp(G') = exp\big(G/Z(G)\big)$, by Lemma 1. Also, $exp(G') \leq exp\big(S(G)\big)$ whence $exp\big(G/Z(G)\big)$ divides $exp\big(S(G)\big)$. Because G is a non-abelian group, so $q \geq 2$. Using the Fundamental Theorem of Finitely Generated Abelian Groups, one can write

$$G' \cong C_{p^l} \times C_{p^{\mu_2}} \times \cdots \times C_{p^{\mu_r}} \qquad and \qquad \frac{G}{Z(G)} \cong C_{p^n} \times C_{p^{\eta_2}} \times \cdots \times C_{p^{\eta_q}}.$$

Without loss of generality, one can assume

$$l \geq \mu_2 \geq \cdots \geq \mu_r > 0 \qquad and \qquad n \geq \eta_2 \geq \cdots \geq \eta_q > 0.$$

It follows from Lemmas 2, and 3 that

$$\begin{aligned} Hom\Big(\frac{G}{Z(G)}, G'\Big) &\cong Hom\Big(\frac{G}{Z(G)}, C_{p^l} \times C_{p^{\mu_2}} \times \cdots \times C_{p^{\mu_r}}\Big) \\ &\cong Hom\Big(\frac{G}{Z(G)}, C_{p^l}\Big) \times Hom\Big(\frac{G}{Z(G)}, C_{p^{\mu_2}} \times \cdots \times C_{p^{\mu_r}}\Big) \\ &\cong \frac{G}{Z(G)} \times Hom\Big(\frac{G}{Z(G)}, C_{p^{\mu_2}} \times \cdots \times C_{p^{\mu_r}}\Big). \end{aligned} \tag{1}$$

Also, for $2 \leq i \leq r$ we get

$$\begin{aligned} Hom\Big(\frac{G}{Z(G)}, C_{p^{\mu_i}}\Big) &\cong Hom(C_{p^n} \times C_{p^{\eta_2}} \times \cdots \times C_{p^{\eta_q}}, C_{p^{\mu_i}}) \\ &\cong C_{p^{min(n,\mu_i)}} \times C_{p^{min(\eta_2,\mu_i)}} \times \cdots \times C_{p^{min(\eta_q,\mu_i)}}. \end{aligned}$$

Thus,

$$\Big|Hom\Big(\frac{G}{Z(G)}, C_{p^{\mu_i}}\Big)\Big| \geq \underbrace{p \times p \times \cdots \times p}_{q\ times} = p^q,$$

consequently

$$\Big|Hom\Big(\frac{G}{Z(G)}, C_{p^{\mu_2}} \times \cdots \times C_{p^{\mu_r}}\Big)\Big| \geq p^{q(r-1)}.$$

Therefore, by relation (1) we have

$$\Big|Hom\Big(\frac{G}{Z(G)}, G'\Big)\Big| \geq \Big|\frac{G}{Z(G)}\Big| p^{q(r-1)}.$$

□

Proposition 5. *Let G be a finite group, then $G/S(G)$ is abelian if and only if $Inn(G) \leqslant Ivar(G)$.*

Proof. Suppose that G/S(G) is abelian, then $G' \leqslant S(G)$. Let $g \in G$, then for the inner automorphism φ_g induced by g and every $x \in G$, we have

$$x^{-1}\varphi_g(x) = [x, g] \in G' \leqslant S(G).$$

So, $g^{-1}\alpha(g) \in S(G)$, for every $\alpha \in Inn(G)$. Therefore, $Inn(G) \leqslant Ivar(G)$.

Conversely, suppose that $Inn(G) \leqslant Ivar(G)$. Thus, $x^{-1}\varphi_g(x) = [x, g] \in S(G)$, for the inner automorphism φ_g and $x \in G$. So, $G' \leqslant S(G)$, hence G/S(G) is abelian. □

Corollary 9. *Let G be a finite p-group. If $S(G) = \phi(G)$, then $Inn(G) \leqslant Ivar(G)$.*

Proof. Because G is a p-group, $S(G) = \phi(G) = G'G^p$. Therefore, $G' \leqslant S(G)$, and the result is follow from previous proposition. □

Corollary 10. *Let G be a non-abelian finite p-group of class 2 and $exp(G') = p^l$, then $Inn(G) \leqslant Ivar(G)$.*

Proof. The result is follow by theorem 1 and previous corollary. □

7 On members of Ivar(G) fixing the center element-wise

In this section, we will investigate the relationships of Ivar(G) and its members. We use notation $C_{Ivar(G)}\big(Z(G)\big)$ for the group of automorphisms of Ivar(G) fixing Z(G) element-wise, thus

$$C_{Ivar(G)}\big(Z(G)\big) = \{\alpha \in Ivar(G) \mid \alpha(z) = z, \ \forall \ z \in Z(G)\}.$$

The following results give some properties in which $C_{Ivar(G)}\big(Z(G)\big) = Ivar(G) = \langle 1 \rangle$.

i) $S(G) = \langle 1 \rangle$.

ii) $Z(G) \leqslant \mathcal{E}(G)$.

iii) G be an abelian group.

In this section, in the main theorem, we give sufficient and necessary conditions for the equality of $C_{Ivar(G)}\big(Z(G)\big)$ and Ivar(G).

Proposition 6. *Let G be a group, then*

$$C_{Ivar(G)}\big(Z(G)\big) \cong Hom\Big(\frac{G}{Z(G)}, S(G) \cap G'\Big).$$

Proof. We consider the map

$$\sigma : \ C_{Ivar(G)}\big(Z(G)\big) \longrightarrow Hom(G/Z(G), S(G) \cap G')$$

defined by $\sigma(\alpha) = \overline{\alpha}$ such that $\overline{\alpha}\big(gZ(G)\big) = g^{-1}\alpha(g)$, for all $g \in G$ and each $\alpha \in C_{Ivar(G)}\big(Z(G)\big)$. Now it is easy to check that σ is an isomorphism. □

In the above proposition, the given isomorphism always holds, because the group is arbitrary. This proposition directly specifies the Ivar structure in the case where $Ivar(G) = C_{Ivar(G)}$. It can also be used to obtain the Ivar(G) order for non-abelian finite p-groups where G/S(G) is abelian.

Corollary 11. *Suppose G is a non-abelian finite p-group for which $G/S(G)$ is abelian, then*

$$\big|C_{Ivar(G)}\big(Z(G)\big)\big| \geq \Big|\frac{G}{Z(G)}\Big| p^{q(r-1)},$$

where $q = rank\big(G/Z(G)\big)$ and $r = rank(G')$.

Proof. It is followed by Lemma 5 and Proposition 6. □

According to Lemma 5, if the desired homomorphism in the above corollary is also an automorphism, using this formula, the order of the group of automorphisms of a finite non-abelian p-group is given in terms of $rank\big(G/Z(G)\big)$ and $rank(G')$.

Corollary 12. *Suppose G is a non-abelian finite p-group for which $G/S(G)$ is abelian, then*

$$|Ivar(G):Inn(G)| \geq p^{q(r-1)},$$

where $q = rank\big(G/Z(G)\big)$ and $r = rank(G')$.

Proof. By corollary 11,

$$\Big|\frac{G}{Z(G)}\Big|p^{q(r-1)} = |Inn(G)|p^{q(r-1)} \leq \big|C_{Ivar(G)}\big(Z(G)\big)\big| \leq |Ivar(G)|.$$

From Proposition 5, we have $p^{q(r-1)} \leq |Ivar(G):Inn(G)|$. □

Remark 2. *Let G be a finite p-group, $g \in G$, $\alpha \in Ivar(G)$, and $exp\big(S(G)\big) = p^k$. Since $g^{-1}\alpha(g) \in S(G)$, so $\alpha(g) = gs$, for some $s \in S(G)$. Therefore, $\alpha(g^{p^k}) = g^{p^k}s^{p^k} = g^{p^k}$.*

Theorem 7 (Main Theorem)**.** *Let G be a non-abelian finite p-group, then $Ivar(G) = C_{Ivar(G)}\big(Z(G)\big)$ if and only if $Z(G)G' \subseteq G'S(G)G^{p^k}$ where $p^k = exp\big(S(G)\big)$.*

Proof. Suppose by way of contradiction that $Ivar(G) = C_{Ivar(G)}\big(Z(G)\big)$ and $Z(G)G' \nsubseteq G'S(G)G^{p^k}$. Therefore, there exists $z \in Z(G)$, which is not in $G'S(G)G^{p^k}$. Since $G/G'S(G)$ is a finite abelian group, by the Fundamental Theorem of Finitely Generated Abelian Groups, one can write

$$\frac{G}{G'S(G)} = \langle g_1G'S(G)\rangle \times \langle g_2G'S(G)\rangle \times \cdots \times \langle g_tG'S(G)\rangle,$$

for $g_1, g_2, \ldots, g_t \in G$. Hence,

$$zG'S(G) = g_1^{p^{a_1}}G'S(G)\cdots g_t^{p^{a_t}}G'S(G) \notin G'S(G)G^{p^k},$$

for some $t_1, \ldots, t_k$. Thus, $g_i^{p^{a_i}} \notin G^{p^k}$ and so $p^{a_i} < p^k$, for some i. Now, select $s' \in S(G)$ such that $O(s') = min\Big(p^k, O\big(g_iG'S(G)\big)\Big)$. Because S(G) is a finite abelian p-group and $exp\big(S(G)\big) = p^k$, this choice is possible. Define $f: G/G'S(G) \longrightarrow S(G) \cap G'$ by $g_iG'S(G) \longmapsto s'$ and $g_jG'S(G) \longmapsto 1$, for $j \neq i$. Then, f is a homomorphism. Consider the map $\psi: G \longrightarrow G$ defined by $g \longmapsto gf\big(gG'S(G)\big)$. Clearly, $\psi \in End(G)$. Because $g_i \notin S(G)$, so $\psi(s) = sf\big(sG'S(G)\big) = s \cdot 1 = s$, for $s \in S(G)$ and ψ acts trivially on elements of S(G). Since $f\big(gG'S(G)\big) \in S(G)$, we have $\psi\Big(f\big(gG'S(G)\big)\Big) = f\big(gG'S(G)\big)$, for every $g \in G$. Also, ψ is one-to-one, because if $g \in Ker(\psi)$, then $\psi(g) = gf\big(gG'S(G)\big) = 1$, therefore $f\big(gG'S(G)\big) = g^{-1}$. On the other hand,

$$f\big(gG'S(G)\big) = \psi\Big(f\big(gG'S(G)\big)\Big)$$

$$
\begin{aligned}
&= \psi(g^{-1}) \\
&= g^{-1} f\big(g^{-1}G'S(G)\big) \\
&= g^{-1}g \\
&= 1.
\end{aligned}
$$

Hence, $g = 1$. This shows that ψ is one-to-one and since G is finite, the homomorphism ψ is a bijection. Also,

$$g^{-1}\psi(g) = g^{-1}gf\big(gG'S(G)\big) = f\big(gG'S(G)\big) \in S(G) \cap G', \quad \forall\, g \in G.$$

So, $\psi \in Ivar(G)$. Furthermore,

$$
\begin{aligned}
f\big(zG'S(G)\big) &= f\big(g_1^{p^{a_1}} G'S(G) \cdots g_t^{p^{a_t}} G'S(G)\big) \\
&= f\big(g_1^{p^{a_1}} G'S(G)\big) \cdots f\big(g_t^{p^{a_t}} G'S(G)\big) \\
&= f\big(g_i^{p^{a_i}} G'S(G)\big) \\
&= (s')^{p^{a_i}}.
\end{aligned}
$$

$(s')^{p^{a_i}}$ is a non-trivial element of S(G), because if $O(s') = p^k$, then $(s')^{p^{a_i}} \neq 1$, since $p^{a_i} < p^k$ and if $O(s') = O\big(g_iG'S(G)\big)$, since $g_i^{p^{a_i}} \notin G'S(G)$, we have $\big(g_iG'S(G)\big)^{p^{a_i}} \neq 1_{G/G'S(G)}$, thus $O\big(g_iG'S(G)\big) = O(s') > p^{a_i}$, therefore $(s')^{p^{a_i}} \neq 1$. Now,

$$\psi(z) = zf\big(zG'S(G)\big) = z(s')^{p^{a_i}} \neq z,$$

for every $z \in Z(G)$. Hence, ψ does not fix Z(G) element-wise and this means that $\psi \notin C_{Ivar(G)}\big(Z(G)\big)$, which is a contradiction.

To prove the converse, assume $Z(G)G' \subseteq G'S(G)G^{p^k}$. We know that $C_{Ivar(G)}\big(Z(G)\big) \subseteq Ivar(G)$, so it is sufficient to prove that $Ivar(G) \subseteq C_{Ivar(G)}\big(Z(G)\big)$. Let $\alpha \in Ivar(G)$ and $z \in Z(G)$. By hypothesis, one can write $z = hsg^{p^k}$, for some $h \in G'$, $s \in S(G)$, and $g \in G$. According to the points mentioned in remark 2, $\alpha(g^{p^k}) = g^{p^k}$ and $\alpha(s) = s$. Also, Ivar(G) acts trivially on G', hence $\alpha(h) = h$ and so $\alpha(z) = z$. Thus $\alpha \in C_{Ivar(G)}\big(Z(G)\big)$, therefore $Ivar(G) \subseteq C_{Ivar(G)}\big(Z(G)\big)$, and the result is followed. □

Also, we have two following results, but we prove them in the next section after some propositions.

Theorem 8. *Let G be a finite S(G)-autonilpotent group of class 2, then $Ivar(G) = C_{Ivar(G)}\big(Z(G)\big)$ if and only if $Z(G) = S(G)G^{p^k}$, where $p^k = exp\big(S(G)\big)$.*

Corollary 13. *Let G be a finite S(G)-autonilpotent group of class 2 and $exp\big(S(G)\big) = p^k$. If $Z(G) = S(G)G^{p^k}$, then each IA-automorphism fixes the center element-wise.*

8 The relationships of Ivar(G) in S(G)-autonilpotent groups

Analogous to the previous section, we give some results in which Ivar(G)=Inn(G) in S(G)-autonilp-otent groups. Our main theorem of this section provides sufficient and necessary conditions for this equality.

Proposition 7. *Let G be a finite S(G)-autonilpotent group of class 2, then for each $g \in G$ and $\alpha \in IA(G)$, $[g,\alpha]^i = [g^i, \alpha]$ holds for every $i \in \mathbb{Z}^+$.*

Proof. Because G is an S(G)-autonilpotent group of class 2, thus

$$S_2(G) = \{g \in G \mid [g, \tau_1, \tau_2] = 1,\ \tau_1, \tau_2 \in IA(G)\} = G.$$

Therefore, $[g, \tau_1] \in S(G) \leqslant Z(G)$ and we have

$$\begin{aligned}
[g,\tau_1]^i &= \big(g^{-1}\tau_1(g)\big)^i \\
&= \underbrace{g^{-1}\tau_1(g)g^{-1}\tau_1(g)\cdots g^{-1}\tau_1(g)}_{i\ times} \\
&= \underbrace{g^{-1}g^{-1}\cdots g^{-1}}_{i\ times}\underbrace{\tau_1(g)\tau_1(g)\cdots\tau_1(g)}_{i\ times} \\
&= g^{-i}\tau_1(g^i) \\
&= [g^i, \tau_1].
\end{aligned}$$

□

Proposition 8. *Let G be a finite S(G)-autonilpotent p-group of class 2, then $exp\big(G/S(G)\big)$ divides $exp\big(S(G)\big)$.*

Proof. Let $exp\big(S(G)\big) = p^k$. Since G is S(G)-autonilpotent of class 2, it holds for all $g \in G$ and $\alpha \in IA(G)$, that $[g,\alpha] \in S(G)$. Thus, $[g,\alpha]^{p^k} = 1$. Now, by Proposition 7 we have

$$\begin{aligned}
[g^{p^k}, \alpha] = 1 \Longrightarrow \quad & g^{p^k} \in S(G) \\
\Longrightarrow \quad & g^{p^k}S(G) = \big(gS(G)\big)^{p^k} = S(G) \\
\Longrightarrow \quad & \big(gS(G)\big)^{p^k} = 1_{\frac{G}{S(G)}},
\end{aligned}$$

for all $gS(G) \in G/S(G)$. Thus,

$$exp\Big(\frac{G}{S(G)}\Big) \Bigm| p^k = exp\big(S(G)\big).$$

□

Proposition 9. *If G be a finite S(G)-autonilpotent group of class 2, then $Ivar(G) = IA(G)$.*

Proof. By Proposition 7 and Ivar(G) definition, for all $g \in G$ and $\alpha \in IA(G)$, $[g,\alpha] \in S(G)$. This completes the proof. □

Corollary 14. *If G be a finite $S(G)$-autonilpotent group of class 2, then $G/S(G)$ is abelian.*

Proof. It is easy by Propositions 5 and 9. □

Now, we are ready to prove the results of the previous section.
Proof of Theorem 8. Suppose $Ivar(G) = C_{Ivar(G)}\big(Z(G)\big)$. By Theorem 7, $Z(G)G' \subseteq G'S(G)G^{p^k}$. On the other hand, by corollary 14, G/S(G) is abelian, so $G' \leqslant S(G)$, hence $Z(G) \subseteq S(G)G^{p^k}$. Also, because $exp(G') \leq exp\big(S(G)\big) = p^k$, therefore $[g,h]^{p^k} = 1$ for every $g,h \in G$. Thus $[g^{p^k}, h] = 1$, by Proposiion 7. Hence $g^{p^k} \in Z(G)$, therefore $G^{p^k} \subseteq Z(G)$ whence $S(G)G^{p^k} \subseteq Z(G)$ and so $Z(G) = S(G)G^{p^k}$.

The converse holds by Theorem 7. □

Now, Corollary 13 follows from Theorem 8 and Proposition 9.

Theorem 9 (Main Theorem)**.** *Let G be a non-abelian finite $S(G)$-autonilpotent p-group of class 2 such that $exp\big(G/Z(G)\big) \leq exp\big(S(G) \cap G'\big)$, then Ivar(G)=Inn(G) if and only if $S(G) = Z(G)$ and $S(G) \cap G'$ is cyclic.*

Proof. Suppose first that Ivar(G)=Inn(G), then $|Ivar(G) : Inn(G)| = 1$. On the other hand, because G is a S(G)-autonilpotent group of class 2, by Corollary 14 G/S(G) is abelian. Therefore, by Theorem 12, $1 \geq p^{q(r-1)}$ where $q = rank\big(G/Z(G)\big)$ and $r = rank(G')$. Because G is non-abelian, so G/Z(G) is non-cyclic and $q \geq 2$. Thus, the only possible state is r=1. Hence, $S(G) \cap G' = G'$ is cyclic.

By hypothesis, Lemma 3 and Proposition 1, we have

$$\frac{G}{S(G)} \cong Ivar(G) = Inn(G) \cong \frac{G}{Z(G)},$$

so,

$$\Big|\frac{G}{S(G)}\Big| = \Big|\frac{G}{Z(G)}\Big|, |S(G)| = |Z(G)|,$$

and, $S(G) = Z(G)$.

To prove the converse, assume that S(G)=Z(G), and $S(G) \cap G'$ is cyclic. By Propositions 1 and Lemma 3, we have

$$Ivar(G) \cong Hom\Big(\frac{G}{S(G)}, S(G) \cap G'\Big) \cong \frac{G}{S(G)} = \frac{G}{Z(G)} \cong Inn(G),$$

and by Proposition 9, $Inn(G) \leqslant Ivar(G)$. Hence, Ivar(G)=Inn(G). □

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