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Some results on 1-absorbing prime ideals of commutative semirings

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Abstract. The main objective of this paper is to study the concept of 1-absorbing prime ideals in commutative semirings as a generalization of 1-absorbing prime ideals in commutative rings. We examine several characterizations and examples of 1-absorbing prime ideals and also investigate the 1-absorbing Prime Avoidance Theorem for such ideals. Moreover, we study the concept of 1-absorbing primary ideals in commutative semirings and studying a method for constructing 1-absorbing primary ideals that are not necessarily primary ideals. Finally, we examine various characterizations and examples of 1-absorbing primary ideals in commutative semirings.

Keywords: 1-absorbing prime ideals, 1-absorbing primary ideals, Semirings
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1 Introduction

In this paper, all semirings are commutative with an identity $1 \neq 0$. The concept of semirings was first introduced by Vandiver [25] in 1934, and since then, it has been extensively studied by various authors. For a comprehensive study of semirings, refer to [12]. Ideals play an important role in both semiring and ring theory, but due to the lack of additive inverses in semirings, their structure differs from that in ring theory. Over the years, the structure of prime ideals in semiring theory has gained significance, and many mathematicians have explored its applications in algebraic systems. The concept of a 2-absorbing ideal in commutative semirings, a generalization of prime ideals, was introduced by Darani in [8]. Following this, Soheilnia introduced the notion of 2-absorbing primary ideals, a generalization of primary ideals in [23].

The concept of 1-absorbing primary ideals in commutative rings was first introduced by Badawi and Celikel in [3], where they investigated many properties of 1-absorbing primary ideals

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analogous to those of primary ideals. Later, Yassine et al. studied 1-absorbing prime ideals in commutative rings in [24]. Further generalizations of 1-absorbing prime ideals and 1-absorbing primary ideals have also been introduced and studied in [4, 7, 9, 14] and [16]. The idea of 1-absorbing weakly primary ideals in commutative semirings, as a generalization of primary ideals was introduced by Saleh and Murra in [22]. Additionally, Ozkan et al. studied 1-absorbing fuzzy ideals in commutative semirings in [21]. The purpose of this paper is to examine the results related to 1-absorbing prime ideals, and 1-absorbing primary ideals from the theory of commutative rings to the theory of commutative semirings.

We begin by recalling some basic definitions of semirings that will be used throughout this study. A semiring R is said to be an entire (semidomain) if $x, y \in R$ with $xy = 0$ implies that either $x = 0$ or $y = 0$ [12]. An element x of a semiring R is a unit if and only if there exists an element y of R such that $xy = 1 = yx$. The element y is called the inverse of x in R. If such an inverse y exists for a unit x, it must be unique. The set of all units of R is denoted by $U(R)$ [12]. A semiring R is said to be a semifield if each non-zero element of R has a multiplicative inverse [12]. A non-empty subset J of R is said to be a left (right) ideal of R if J is a sub-semigroup of $(R, +)$ and $xy \in J$ ($yx \in J$) for all $y \in J, x \in R$. If J is both a left and right ideal of R, then J is known to be an ideal of R [12]. An ideal J of R is said to be a proper ideal if $J \neq R$. A proper ideal J of R is said to be principal if $J = aR$ for some $a \in R$. An ideal J of a semiring R is said to be a k-ideal if for all $x, y \in R, x + y \in J$ and $y \in J$ implies that $x \in J$ [12]. A proper ideal of R is said to be maximal if and only if it is not properly contained in any other ideals of R [12]. A semiring R is said to be local if it has only one maximal ideal [5]. Let J be an ideal of R. We denote the radical of J by $\sqrt{J} = \{x \in R : x^m \in J \text{ for some } m \in \mathbb{N}\}$. The set of all nilpotent elements of R is denoted by $Nil(R) = \{x \in R : x^m = 0 \text{ for some } m \in \mathbb{N}\}$. A semiring R is said to be Noetherian if and only if it satisfies the ascending chain conditions on ideals [12]. A proper ideal P of R is said to be prime if $x, y \in R$ such that $xy \in P$ implies that either $x \in P$ or $y \in P$ [12]. A proper ideal P of R is said to be primary if $x, y \in R$ such that $xy \in P$ implies that either $x \in P$ or $y \in \sqrt{P}$ [12]. If P is a primary ideal of R and $P = \sqrt{P}$ is a prime ideal of R, then P is a P-primary ideal of R [23]. A proper ideal J of R is said to be a 2-absorbing ideal of R if whenever $x, y, z \in R$ such that $xyz \in J$, then $xy \in J$ or $xz \in J$ or $yz \in J$ [8]. A proper ideal J of R is said to be a 2-absorbing primary ideal of R if whenever $x, y, z \in R$ such that $xyz \in J$, then $xy \in J$ or $xz \in \sqrt{J}$ or $yz \in \sqrt{J}$ [23]. Let R and R_1 be commutative semirings. A mapping $g : R \to R_1$ is said to be a homomorphism if $g(x + y) = g(x) + g(y)$ and $g(xy) = g(x)g(y)$ for all $x, y \in R$.

Throughout this paper, R represents a commutative semiring, $\mathbb{N}$ denotes the set of non-negative integers and $\mathbb{Z}$ denotes the set of integers.

2 1-absorbing prime ideals

In this section, we are going to study 1-absorbing prime ideals in commutative semirings and investigate many of their properties and characterizations. Moreover, we study the behaviour of 1-absorbing prime ideals over prime ideals, valuation semidomain, Prüfer semidomain, Dedekind semidomain and the idealization of semimodules.

Definition 1. *Let R be a commutative semiring and J be a proper ideal of R. Then J is said to*

be a 1-absorbing prime if for any non-unit elements x, y and z of R such that $xyz \in J$ implies that either $xy \in J$ or $z \in J$.

Prime ideal $\Rightarrow$ 1-absorbing prime ideal $\Rightarrow$ 2-absorbing ideal. However, the reverse implication does not hold, as shown in Example 1.

Example 1. (i) The set of all non-negative integers $\mathbb{N}$ in [12, Example 1.3] forms a commutative and an entire semiring under usual addition and multiplication. However, $\mathbb{N}$ is not a ring. Consider $R = \mathbb{N}$ and let $J = (6)$ be an ideal of R. Then J is a 2-absorbing ideal of R. While J is not a 1-absorbing prime ideal of R, since $2 \cdot 2 \cdot 3 \in J$ but neither $2 \cdot 2 \in J$ nor $3 \in J$ where $2, 3 \in R$.

(ii) Let $R = \mathbb{Z}_8$ be the set of integers modulo 8 and $J = (4)$ be an ideal of R. Then J is a 1-absorbing prime ideal of R, but it is not a prime ideal of R since $2 \cdot 2 \in J$, while $2 \notin J$.

Theorem 1. *Let R be a commutative semiring and J be a 1-absorbing prime ideal of R. Then $\sqrt{J}$ is a prime ideal of R. Also, for any non-unit element $x \in R \setminus J$ such that $(J : x) = \{r \in R : xr \in J\}$ is a prime ideal of R.*

Proof. Assume J is a 1-absorbing prime ideal of R. Let $x, y \in R$ such that $xy \in \sqrt{J}$. If y is a unit, then $x = (xy)y^{-1} \in \sqrt{J}$. We are done. Now, consider x and y are non-unit elements of R. Since $xy \in \sqrt{J}$, there exists a positive integer m such that $x^m y^m \in J$ and so $x^m x^m y^m \in J$. As J is a 1-absorbing prime ideal of R, either $x^m x^m \in J$ or $y^m \in J$, which implies $x \in \sqrt{J}$ or $y \in \sqrt{J}$. Thus, $\sqrt{J}$ is a prime ideal of R. Moreover, we show that $(J : x)$ is a prime ideal of R. Let x be a non-unit element of $R \setminus J$ and $y, z \in R$ such that $yz \in (J : x)$ implies that $xyz \in J$. If either y or z is a unit, then we are done. Assume that y and z are non-unit elements. Since J is a 1-absorbing prime ideal of R, we have either $xy \in J$ or $z \in J$, which implies $y \in (J : x)$ or $z \in (J : x)$. Therefore, $(J : x)$ is a prime ideal of R. □

Theorem 2 ([13], Theorem 2.1)**.** *A non-zero ideal J of $(\mathbb{N}, +, .)$ is a prime if and only if $J =< p >$ for some prime number p or $J =< 2, 3 >$.*

The following example shows that the converse of Theorem 1 need not be true.

Example 2. Consider an entire semiring $R = (\mathbb{N}, +, .)$ and let $J =< 3, 5 >= \{0, 3, 5, 6, 8, 9, 10, \ldots\}$ $= \mathbb{N} - \{1, 2, 4, 7\}$. Clearly, $\sqrt{J} = \mathbb{N} - \{1\} =< 2, 3 >$. Thus, by Theorem 2, $\sqrt{J}$ is a prime ideal. However, J is not a 1-absorbing prime ideal since $2 \cdot 2 \cdot 2 \in J$ but neither $2 \cdot 2 \in J$ nor $2 \in J$.

Lemma 1. *Let R be a semiring. If for every unit element x of R and every non-unit element y of R such that $x + y$ is a unit element of R. Then R is a local semiring.*

Proof. Let M and N be at least two maximal ideals of R. Then there exist elements $m \in M$ and $n \in N$ such that $m + n = 1$ implies that $1 - m = n$. Since $n \in N$, then n cannot be a unit-element of R. This leads to a contradiction. Hence, R is a local semiring. □

In the following result, we show that if every 1-absorbing prime ideal is prime, then R cannot be a local semiring. As shown in Example 1 (ii), R is a local semiring, while every 1-absorbing prime ideal need not be a prime deal.

Theorem 3. *Let J be a proper k-ideal of a semiring R. If J is a 1-absorbing prime ideal of R that is not a prime. Then R is a local semiring.*

Proof. Let J be a 1-absorbing prime ideal of R. Assume that J is not a prime ideal of R. Thus, there exist non-unit elements $x, y \in R$ with $xy \in J$ and $x, y \notin J$. Let v be a non-unit element of R. Since J is a 1-absorbing prime ideal of R, $vxy \in J$ and $y \notin J$ implies that $vx \in J$. If u is a unit element, and v is a non-unit element of R, by Lemma 1, $u+v$ is a unit element and so R is a local semiring. Assume that $u+v$ is a non-unit element of R. Since J is a 1-absorbing prime ideal, $(u+v)xy \in J$ and $y \notin J$. Then $(u+v)x = ux + vx \in J$. Since J is a k-ideal, $ux + vx \in J$ and $vx \in J$, we conclude that $ux \in J$. Thus, $x \in J$ because u is a unit element of R, which is a contradiction. Hence, $u+v$ is a unit element of R. □

Lemma 2. *Let J be a 1-absorbing prime ideal of R and K be a proper ideal of R. If x and y are non-unit elements of R such that $xyK \subseteq J$, then either $xy \in J$ or $K \subseteq J$.*

Proof. Let K be a proper ideal of R and suppose there exist non-unit elements x and y in R such that $xyK \subseteq J$. Assume that $xy \notin J$. Consider a non-unit element $k \in K$ such that $xyk \in J$. Since J is a 1-absorbing prime ideal of R and $xy \notin J$, it follows that $k \in J$. Therefore, $K \subseteq J$. □

Theorem 4. *Let J be a proper ideal of R. Then J is a 1-absorbing prime ideal of R if and only if for any proper ideals J_1, J_2 and J_3 of R with $J_1J_2J_3 \subseteq J$, then either $J_1J_2 \subseteq J$ or $J_3 \subseteq J$.*

Proof. Let J be a 1-absorbing prime ideal of R and there exist proper ideals J_1, J_2 and J_3 of R such that $J_1J_2J_3 \subseteq J$. Assume that $J_1J_2 \nsubseteq J$. Then there exist non-unit elements $j_1 \in J_1$ and $j_2 \in J_2$ such that $j_1j_2 \notin J$. Since J is a 1-absorbing prime ideal of R, $j_1j_2J_3 \subseteq J$ and $j_1j_2 \notin J$, by Lemma 2, $J_3 \subseteq J$. Conversely, let x, y and z be non-unit elements of R such that $xyz \in J$. Set $J_1 = xR$, $J_2 = yR$ and $J_3 = zR$ are proper ideals of R such that $J_1J_2J_3 \subseteq J$. By assumption, either $J_1J_2 \subseteq J$ or $J_3 \subseteq J$ implies that $xy \in J$ or $z \in J$. Thus, J is a 1-absorbing prime ideal of R. □

Recall from [5, Definition 1.3], a prime ideal P of a semiring R is said to be a divided prime ideal of R if $P \subseteq aR$ for every $a \in R \setminus P$. A semiring R is said to be divided if all of its prime ideals are divided. A semiring R is said to be chained semiring if for any $a, b \in R$ such that $a \mid b$ or $b \mid a$. Every chained semiring is divided [22].

Lemma 3. *Let J be a proper k-ideal of a divided semiring R and P be a prime ideal of R such that $P = \sqrt{J}$. If J is a 1-absorbing prime ideal of R, then J is a primary ideal of R with $P^2 \subseteq J$.*

Proof. Assume J is a 1-absorbing prime ideal of R. At first, we show that $P^2 \subseteq J$. Let $p, q \in P = \sqrt{J}$. Then there exists a positive integer m such that $p^{m-2}pp \in J$. Since J is a 1-absorbing prime ideal of R, then $p^2 \in J$. Consider $p, q, p+q \in R$ such that $p(p+q)q \in J$. Since J is a 1-absorbing prime ideal of R, either $p(p+q) = p^2 + pq \in J$ or $q \in J$. As J is a k-ideal of R, $p^2 \in J$ and $p^2 + pq \in J$ implies that $pq \in J$. Hence, $P^2 \subseteq J$. Now, we show that J is a primary ideal of R. Let $p, q \in R$ such that $pq \in J$. Assume $q \notin P$. Then $p \in P$. Since P is a divided prime ideal of R, we have $p = qr$ for some $r \in R$. Thus, $pq = q^2r \in J$. Since J is a 1-absorbing prime ideal of R and $q^2 \notin J$, we have $r \in J$. Thus, $p \in J$. Therefore, J is a primary ideal of R such that $P^2 \subseteq J$. □

Consider an entire semiring R and a quotient semifield Q. A proper ideal J of R is said to be invertible if $JJ^{-1} = R$, where $J^{-1} = \{q \in Q : qJ \subseteq R\}$. An entire semiring R is said to be a valuation semidomain if R is a chained semiring [5]. A semiring R is said to be a Prüfer semidomain if each nonzero finitely generated ideal of R is invertible [11, Definition 2.3]. An entire semiring R is said to be a Dedekind semidomain if each nonzero proper ideal of R is invertible [18, Definition 2.1].

Theorem 5 ([5], Theorem 1.2)**.** *Let P be a proper prime ideal of a valuation semiring R.*

(i) If q is a P-primary and $x \in R \setminus P$, then $q = q(x)$.

(ii) A finite nonempty product of P-primary ideals of R is a P-primary ideal. If $P \neq P^2$, then the only P-primary ideals of R are powers of P.

Theorem 6 ([11], Theorem 2.11)**.** *For a semidomain S, the following statements are equivalent:*

(i) R is Prüfer semidomain.

(ii) For any prime ideal P, R_P is a valuation semidomain.

(iii) For any maximal ideal M, R_M is a valuation semidomain.

Theorem 7. *Let J be a non-zero proper ideal of a valuation semidomain R and P be a prime ideal of R with $P = \sqrt{J}$. If J is a 1-absorbing prime ideal of R, then either $J = P$ or $J = P^2$.*

Proof. Let R be a valuation semidomain, and J be a non-zero proper ideal of R with $P = \sqrt{J}$. By [5, Proposition 1.4], every valuation semidomain is a divided semiring. Thus, by Lemma 3, J is a primary ideal of R and $P^2 \subseteq J$. Consequently, J is a P-primary ideal of R because $P = \sqrt{J}$, where P is a prime ideal of R. Since R is a valuation semidomain and J is a P-primary ideal of R, by Theorem 5, either $J = P$ or $J = P^2$. □

Theorem 8. *Let J be a non-zero proper ideal of a Prüfer semidomain R and P be an invertible ideal of R with $P = \sqrt{J}$. If J is a 1-absorbing prime ideal of R, then either $J = P$ or $J = P^2$ where $P = \sqrt{J}$ is a prime ideal of R.*

Proof. Let R be a Prüfer semidomain, and J be a non-zero proper ideal of R with $P = \sqrt{J}$. If R is a local semiring with maximal ideal M, by Theorem 6, R_M is a valuation semidomain, since R is Prüfer semidomain. Thus, by Theorem 7, either $J = P$ or $J = P^2$. Assume that R is not a local semiring. By Theorem 3, J is a prime ideal of R, and so J is a P-primary ideal of R with $P^2 \subseteq J$. Since R is a Prüfer semidomain and J is a prime ideal of R. By Theorem 6, R_J is a valuation semidomain. It follows that by Theorem 7, either $J = P$ or $J = P^2$ where P is a prime ideal of R. □

Theorem 9 ([2], Theorem 40)**.** *Let R be a commutative semiring with an identity 1, and let P be a proper ideal in R. If $\sqrt{P}$ is maximal in R, then P is primary.*

Theorem 10. *Let R be a Noetherian entire semiring that is not a semifield, and let J be a 1-absorbing prime ideal of R. If R is a Dedekind semidomain. Then $J = M$ or $J = M^2$ where M is a maximal ideal of R.*

Proof. Let R be a Noetherian entire semiring which is not a semifield. Assume that J is a 1-absorbing prime ideal R with $P = \sqrt{J}$. Since R is a Dedekind semidomain, by [18, Theorem 2.21], every nonzero prime ideal of R is a maximal ideal of R. Therefore, by Theorem 1, P is a maximal ideal of R. Consequently, by Theorem 9, J is a primary ideal of R such that $P^2 \subseteq J$. Thus, by Theorem 8, $J = M$ or $J = M^2$, where M is a maximal ideal of R. □

Lemma 4. *Let R be a semiring, J be a P-primary ideal of R and $a \in P \setminus J$ with $(P^2 : a) \subseteq J$. Then J is a 1-absorbing prime ideal of R.*

Proof. Assume J is a P-primary ideal of R, and $a \in P \setminus J$ with $(P^2 : a) \subseteq J$. Let a, b and c be non-unit elements of R such that $abc \in J$. Suppose that $ab, c \notin J$. Since J is a P-primary ideal of R, we have $c \in P \setminus J$. Thus, $ab \in (P^2 : c) \subseteq J$ which leads to a contradiction. Therefore, either $ab \in J$ or $c \in J$. Hence, J is a 1-absorbing prime ideal of R. □

In [6, Definition 1.1], a semiring R is said to be a π-regular if for any $r \in R$, there exists $s \in R$ and for some positive integer n such that $r^{n+1}s = r^n$.

Theorem 11. *Let R be a semiring and J be any non-zero proper ideal of R. The following statements hold:*

(i) If J is a 1-absorbing prime ideal of R, then R is a π-regular semiring.

(ii) If R is a local π-regular semiring with a maximal ideal $M = Nil(R) \neq \{0\}$. Suppose for all nonzero elements $m \in M$ and $n \in M \setminus Rm$ such that $(M^2 : n) \subseteq Rm$. Then J is a 1-absorbing prime ideal of R.

Proof. (i) Assume J is a 1-absorbing prime ideal of R. For any $r \in R$, if r is a unit or a nilpotent element, then r is a π-regular element of R. Now, consider r as a non-unit element of R that is not nilpotent. Let $J = r^4R$ be a nonzero proper ideal of R. Since $r^2 \cdot r \cdot r \in J$, we have $r^2 \cdot r \in J$. Hence, J is a 1-absorbing prime ideal of R and so r is a π-regular element of R. Therefore, R is a π-regular semiring.

(ii) Let R be a local π-regular semiring with a maximal ideal $M = Nil(R) \neq \{0\}$ and for all nonzero $m \in M$ and $n \in M \setminus Rm$ such that $(M^2 : n) \subseteq Rm$. Since every nonzero proper ideal J of R is M-primary ideal and for all nonzero $m \in M$, $n \in M \setminus Rm$ such that $(M^2 : n) \subseteq Rm$. By Lemma 4, every nonzero proper ideal of R is a 1-absorbing prime ideal.

□

Theorem 12. *Let P and $Nil(R)$ be divided prime ideals of a semiring R with $P \neq Nil(R)$. Then P^2 is a 1-absorbing prime ideal of R.*

Proof. Let $a_j, b_j \in P$ for each $j \in \{1, 2, ..., m\}$ with $ab = \sum_{j=1}^{m} a_j b_j \in P^2$. Assume $b \notin P$. Since $b \notin P$ and P is a divided prime ideal of R, we have $P \subseteq Rb$. Also $P \neq Nil(R)$, $Nil(R)$ is divided and $b \notin P$, we have $Nil(R) \subseteq P^2$. Since $a_j \in P$, then there exists $c_j \in R$ such that $a_j = bc_j$, where $j \in \{1, 2, ..., m\}$. Now, $ab = \sum_{j=1}^{m} a_j b_j = \sum_{j=1}^{m} bc_j b_j = bc_1b_1 + bc_2b_2 + ... + bc_mb_m \in P^2$, which implies $b(a - c_1b_1 - c_2b_2 - ... - c_mb_m) = 0 \in Nil(R) \subseteq P^2$. Since $Nil(R)$ is divided,

$b \notin P$ and $Nil(R) \subseteq P^2$, we have $b \notin Nil(R)$, it follows that $a - c_1b_1 - c_2b_2 - ... - c_mb_m = r \in Nil(R) \subseteq P^2$. Hence, $a = c_1b_1 + c_2b_2 + ... + c_mb_m + r \in P^2$. Consequently, P^2 is a P-primary ideal of R. Therefore, by Lemma 4, P^2 is a 1-absorbing prime ideal of R. □

Theorem 13. *Let R_1 and R_2 be semirings and $g : R_1 \to R_2$ be a homomorphism such that $g(1) = 1$ and $g(x)$ is a non-unit element of R_2 for every non-unit element $x \in R_1$. Then the following statements hold:*

(i) Whenever K is a 1-absorbing prime ideal of R_2, then $g^{-1}(K)$ is a 1-absorbing prime ideal of R_1.

(ii) Whenever g is surjective and J is a 1-absorbing prime ideal of R_1 with $Ker(g) \subseteq J$, then $g(J)$ is a 1-absorbing prime ideal of R_2.

Proof. (i) Assume K is a 1-absorbing prime ideal of R_2. Let x, y and z be non-unit elements of R_1 with $xyz \in g^{-1}(K)$. Then $g(xyz) = g(x)g(y)g(z) \in K$, which implies $g(x)g(y) \in K$ or $g(z) \in K$. Thus, $xy \in g^{-1}(K)$ or $z \in g^{-1}(K)$. Therefore, $g^{-1}(K)$ is a 1-absorbing prime ideal of R_1.

(ii) Assume J is a 1-absorbing prime ideal of R_1 and g is surjective such that $Ker(g) \subseteq J$. Let a, b and c be non-unit elements of R_2 such that $abc \in g(J)$. Since g is surjective, there exist non-unit elements u, v and w in R_1 with $a = g(u), b = g(v)$ and $c = g(w)$. Thus, $g(uvw) = g(u)g(v)g(w) = abc \in g(J)$. Since $Ker(g) \subseteq J$, we have $uvw \in J$, which implies $uv \in J$ or $w \in J$. Consequently, $ab \in g(J)$ or $c \in g(J)$. Hence, $g(J)$ is a 1-absorbing prime ideal of R_2.

□

The following example shows that the condition "for every non-unit element $x \in R_1$, $g(x)$ is a non-unit element of R_2" is necessary in Theorem 13.

Example 3. Consider $R_1 = \mathbb{Z}$ and $R_2 = \mathbb{Z}_p$, where p is a prime number. Let $g : R_1 \to R_2$ be a homomorphism defined by $g(r) = \frac{r}{1}$, where $r \in R_1$. Since for any prime number $p \neq s$, we have $g(s) = \frac{s}{1}$ is a unit element in R_2 but s is a non-unit element of R_1. It is obvious that $\frac{a}{d}$ is a non-unit if and only if $p \mid a$. Take $P = p^2\mathbb{Z}_p$. Let $\frac{a}{d}$, $\frac{b}{e}$ and $\frac{c}{f}$ are non-unit elements of R_2 such that $\frac{a}{d} \cdot \frac{b}{e} \cdot \frac{c}{f} \in P$. Thus, $p \mid a, p \mid b$ and $p \mid c$. Then we have $\frac{a}{d} \cdot \frac{b}{e} \in p^2\mathbb{Z}_p = P$. Therefore, P is 1-absorbing prime ideal of R_2. In addition, $g^{-1}(P) = p^2\mathbb{Z}$. Since $psp = p^2s \in g^{-1}(P)$ but neither $ps \in g^{-1}(P)$ nor $p \in g^{-1}(P)$. Therefore, $g^{-1}(P)$ is not 1-absorbing prime ideal of R_1.

Corollary 1. *Let R be a semiring, J and K be proper ideals of R such that $J \subseteq K$ and $U(R/J) = \{x + J | \, x \in U(R)\}$. Then K is a 1-absorbing prime ideal of R if and only if K/J is a 1-absorbing prime ideal of R/J.*

Proof. Suppose J and K are proper ideals of R with $J \subseteq K$. Let $g : R \to R/J$ be defined by $g(x) = x + J$. Then g is a homomorphism from R onto R/J with $g(1) = 1$ and for some non-unit

element x in R such that $f(x) \in R/J$. Since $Ker(g) = J \subseteq K$ and g is surjective, by Theorem 13 (ii), $g(K) = K/J$ is a 1-absorbing prime ideal of R/J. Conversely, if K/J is a 1-absorbing prime ideal of R/J. Then by Theorem 13 (i), $g^{-1}(K/J) = K$ is a 1-absorbing prime ideal of R. □

For the definition of semimodule and subsemimodule, refer to [12, Chapter 14]. In [10], consider R as a commutative semiring with an identity element and let M be a unitary semimodule. Then $R(+)M$ with coordinate-wise addition $(x, m) + (y, n) = (x + y, m + n)$ and multiplication $(x, m)(y, n) = (xy, xn + ym)$ forms a commutative semiring with identity known as the idealization of M. Let J be an ideal of R and N be a subsemimodule of M such that $JN \subseteq M$. Then $J(+)N$ is said to be homogeneous ideal of $R(+)M$. The nil radical of $R(+)M$ is denoted by $\sqrt{0_{R(+)M}} = \sqrt{0}(+)M$.

Theorem 14. *Let J be a k-ideal of a semiring R, N be a k-R-subsemimodule of an R-semimodule M and $J(+)N$ be a homogeneous ideal of $R(+)M$. Then $J(+)N$ is a 1-absorbing prime ideal of $R(+)M$ if and only if J is a 1-absorbing prime ideal of R such that $JM \subseteq N$ and $(N : a) = N$ for all $a \in R \setminus J$.*

Proof. Assume x, y and z are non-unit elements of R such that $xyz \in J$. Then $(x, 0_M)(y, 0_M)$ $(z, 0_M) = (xyz, 0_M) \in J(+)N$. Since $J(+)N$ is a 1-absorbing prime ideal of $R(+)M$, it follows that $(x, 0_M)(y, 0_M) \in J(+)N$ or $(z, 0_M) \in J(+)N$. Consequently, $xy \in J$ or $z \in J$. Therefore, J is a 1-absorbing prime ideal of R. Conversely, assume J is a 1-absorbing prime ideal of R such that $JM \subseteq N$ and $(N : a) = N$ for all $a \in R \setminus J$. Consider non-unit elements x, y and z of R and m_1, m_2 and m_3 of M such that $(x, m_1)(y, m_2)(z, m_3) = (xyz, xym_3 + xzm_2 + yzm_1) \in J(+)N$. Thus, $xyz \in J$ and so $xy \in J$ or $z \in J$. If $z \in J$, then $(z, m_3) \in J(+)N$. Assume that $z \notin J$. Then $xy \in J$ implies $(x, m_1)(y, m_2) = (xy, xm_2 + ym_1) \in J(+)N$. Since N is a k-R-subsemimodule of an R-semimodule M, $xym_3 + xzm_2 + yzm_1 \in N$ and $xym_3 \in N$ implies that $xzm_2 + yzm_1 \in N$. As $z \notin J$, then $xm_2 + ym_1 \in (N : z) = N$. Hence, $J(+)N$ is a 1-absorbing prime ideal of $R(+)M$. □

A more general result known as the Prime Avoidance Theorem for semirings has been studied in [20, Theorem 1.15]. Additionally, an extended version of the Prime Avoidance Theorem for semirings was studied by Kumar et al. in [17]. Now, we will examine the 1-Absorbing Prime Avoidance Theorem for 1-absorbing prime ideals in semirings. Firstly, the following lemma is necessary.

Let $J, J_1, J_2, J_3, \ldots, J_m$ be ideals of R. A covering $J \subseteq J_1 \cup J_2 \cup J_3 \cup \ldots \cup J_m$ is called an efficient if J is not contained in the union of any $m - 1$ of the ideals $J, J_1, J_2, J_3, \ldots, J_m$. In other words, $J = J_1 \cup J_2 \cup J_3 \cup \ldots \cup J_m$ is called an efficient union if none of the J_j may be excluded.

Lemma 5. *Let R be a semiring and let $J_1, J_2, J_3, \ldots, J_m$ be k-ideals of R such that $J = J_1 \cup J_2 \cup J_3 \cup \ldots \cup J_m$ be an efficient union. Then for any $i \in \{1, 2, \ldots, m\}$, we have $\bigcap_{l=1}^{m} J_l = \bigcap_{l=1,\, l\neq i}^{m} J_l$.*

Proof. Consider $l = 1$, then $J_1 \cap J_2 \cap J_3 \cap \ldots \cap J_m \subseteq J_2 \cap J_3 \cap \ldots \cap J_m$. Thus, $\cap_{l=1}^{m} J_l \subseteq \cap_{l=2}^{m} J_l$. Let $a \in \cap_{l=2}^{m} J_l$ and $b \in J \setminus \cup_{l=2}^{m} J_l$. Therefore, $a \in J$ and $b \in J_1$, then $a + b \in J$, which implies

$a+b \in J_l$. If $l \neq i$ and J_l is a k-ideal, then $b \in J_l$, a contradiction. If $l = 1$, then $a+b \in J_1$ implies that $a \in J_1$. Hence, $a \in \cap_{l=1}^{m} J_l$. □

Lemma 6. *Let J be a k-ideal of a semiring R with $J \subseteq J_1 \cup J_2 \cup J_3 \cup ... \cup J_m$ be an efficient covering, where $m \geq 2$. If for any $a \in R \setminus J_l$ and $j \neq l$ such that $J_j \nsubseteq (J_l : a)$, then no J_l for $l \in \{1, 2, \ldots, m\}$ is a 1-absorbing prime ideal of R.*

Proof. Assume that J_l is a 1-absorbing prime ideal of R for some $l \in \{1, 2, \ldots, m\}$. Since $J \subseteq J_1 \cup J_2 \cup J_3 \cup \ldots \cup J_m$ is an efficient covering, $J = (J \cap J_1) \cup (J \cap J_2) \cup (J \cap J_3) \cup \ldots \cup (J \cap J_m)$ is an efficient union. If not, then for some $j \neq l$, we have $J \cap J_j \subseteq J \cap J_l$. Thus, $J = (J \cap J_1) \cup (J \cap J_2) \cup (J \cap J_3) \cup ... \cup (J \cap J_{j-1}) \cup (J \cap J_{j+1}) \cup \ldots \cup (J \cap J_m)$ implies that $J \subseteq J_1 \cup J_2 \cup \ldots \cup J_{j-1} \cup J_{j+1} \cup \ldots \cup J_m$, a contradiction. Then there exists an element $a_l \in J \setminus J_l$ for all $l \leq m$. By Lemma 5, we have $\bigcap_{j \neq l} J \cap J_j \subseteq J \cap J_l$. If $j \neq l$ and $J_j \nsubseteq (J_l : a_l)$, then $a_l \in J \setminus J_l$. Thus, there exists $b_j \in J_j \setminus (J_l : a_l)$ for all $j \neq l$. Let $b = \prod_{j \neq l} b_j$ implies that $ba_l = \prod_{j \neq l} b_j a_l \in \cap_{j \neq l} J \cap J_j$. However, $ba_l = \prod_{j \neq l} b_j a_l \notin J \cap J_l$. If $ba_l \in J \cap J_l$. Since J_l is a 1-absorbing prime ideal of R, then by Theorem 1, $(J_l : a_l)$ is a prime ideal of R, we have $b_j \in (J_l : a_l)$ for $j \neq l$, which is not possible. Thus, $ba_l \notin J \cap J_l$, which is a contradiction to the fact that $\bigcap_{j \neq l} J \cap J_j \subseteq J \cap J_l$. Hence, J_l is not a 1-absorbing prime ideal of R. □

In the following result, we prove 1-absorbing Prime Avoidance Theorem.

Theorem 15. *Let J be an ideal of a semiring R and $J_1, J_2, J_3, \ldots, J_m$ be k-ideals of R such that $J \subseteq J_1 \cup J_2 \cup J_3 \cup \ldots \cup J_m$ and at most two of $J_j's$ are not 1-absorbing prime. If for any $a \in R \setminus J_l$ and $j \neq l$ such that $J_j \nsubseteq (J_l : a)$. Then $J \subseteq J_l$ for some l.*

Proof. Consider $J \subseteq J_1 \cup J_2 \cup J_3 \cup \ldots \cup J_m$ as a covering. Assume $J \subseteq J_{h_1} \cup J_{h_2} \cup J_{h_3} \cup \ldots \cup J_{h_n}$ is its efficient reduction. Then $1 \leq n \leq m$ and $n \neq 2$. If $n > 2$. Since the covering is efficient and for any $a \in R \setminus J_l$ and $j \neq l$ such that $J_j \nsubseteq (J_l : a)$. By Lemma 6, J_l has no 1-absorbing prime ideal of R. Thus, $n = 1$ and so $J \subseteq J_l$ for some l. □

Corollary 2. *Let R be a semiring and $J =< s_1, s_2, \ldots, s_r >$ be a finitely generated ideal of R, where $s_1, s_2, \ldots, s_r \in R$. If $J_1, J_2, J_3, \ldots, J_m$ are 1-absorbing prime ideals of R such that $J \nsubseteq J_j$ and $J_j \nsubseteq (J_l : a)$ for all $a \in R \setminus J_l$ and $j \neq l$. Then there exist elements $t_2, t_3, \ldots, t_r \in R$ such that $\gamma = s_1 + t_2 s_2 + \cdots + t_r s_r \notin \cup_{j=1}^{m} J_j$.*

Proof. We will prove this result by induction on m. If $m = 1$, then the result is obvious. Assume $m > 1$ and the result for smaller values of m. Then there exist $u_2, u_3, \ldots, u_r \in R$ such that $z = s_1 + u_2 s_2 + \cdots + u_r s_r \notin \cup_{j=1}^{m} J_j$. If $z \notin J_m$, then $z \notin \cup_{j=1}^{m} J_j$ and we are done. Assume that $z \in J_m$. If $s_2, s_3, \ldots, s_r \in J_m$, then $s_1 \in J_m$, which contradicts the assumption that $J \nsubseteq J_m$. Suppose $s_j \notin J_m$ for some j. Let $s_2 \notin J_m$. Since $J_j \nsubseteq (J_l : a)$ for all $a \in R \setminus J_l$ and $j \neq l$, then there exists $b_j \in J_j \setminus (J_m : s_2)$ for all $j \neq m$. Let $b = \prod_{j=1}^{m-1} b_j$. Then $b \in J_j$ for any $j \neq m$ but $b \notin (J_m : s_2)$. It follows that $b \in J_j \setminus (J_m : s_2)$ for any $j \neq m$. Let $\gamma = s_1 + (u_2 + b)s_2 + \cdots + u_r s_r$. We have two cases arises:

Case I: If $J \subseteq J_1 \cup J_2 \cup J_3 \cup \ldots \cup J_m$, then by the Prime Avoidance Theorem, $J \subseteq J_l$ for some l, which is a contradiction.

Case II: If $J \nsubseteq J_1 \cup J_2 \cup J_3 \cup \ldots \cup J_m$, then by the Prime Avoidance Theorem, we assume $s_2 \notin J_m$, we have $\gamma = z + bs_2 \notin \cup_{j=1}^{m} J_j$. □

3 1-absorbing primary ideals

The main objective of this section is to characterize some examples and properties of 1-absorbing primary ideals in commutative semirings.

Definition 2 ([22]). *A proper ideal J of a commutative semiring R is said to be a 1-absorbing primary ideal if $xyz \in J$, then $xy \in J$ or $z \in \sqrt{J}$, where x, y and z are non-unit elements of R.*

Primary ideals $\Rightarrow$ 1-absorbing primary ideals $\Rightarrow$ 2-absorbing primary ideals. However, the following example demonstrates that the reverse implications do not generally hold.

Example 4. (i) Consider $R = \mathbb{N}$ and $J = mn\mathbb{N}$, where m and n are prime numbers with $m \neq n$. Then J is a 2-absorbing primary ideal of R. However, it is not 1-absorbing primary ideal of R. Since $mmn \in J$ but neither $m^2 \in J$ nor $n \in \sqrt{J}$.

(ii) Consider $R = \mathbb{N} \times \mathbb{N}$, and $J = 12\mathbb{N} \times 6\mathbb{N}$. Then J is a 2-absorbing primary ideal of R by [23, Example 2.18 (i)]. But it is not 1-absorbing primary ideal of R. Since $(2,1), (2,2), (3,3) \in R$ such that $(2,1) \cdot (2,2) \cdot (3,3) \in J$ while neither $(2,1) \cdot (2,2) \in J$ nor $(3,3) \in \sqrt{J}$.

(iii) In [1, Example 1], let $\mathbb{N}$ be the set of non-negative integers under the usual operations of addition and multiplication. Then $\mathbb{N}$ is a local semiring with maximal ideal $M = \mathbb{N} - \{1\} =< 2, 3 >$. The prime ideals of $\mathbb{N}$ are 0, $p\mathbb{Z}^+$ (p is prime) and M. Now, consider $R = \mathbb{N}$, where R is semifield and $M = \mathbb{N} - \{1\} =< 2, 3 >$ is a maximal ideal of R. Let $p =< 2 >$ be a prime ideal of R. Since R is a local semiring with maximal ideal M. Consider the ideal $J = 2.M = 2 < 2, 3 >= (2^2, 2 \cdot 3)$ and $\sqrt{J} = p$. Then J is a 1-absorbing primary ideal of R (see Theorem 18). While $2 \cdot 3 \in J$ but neither $2 \in J$ nor $3 \in \sqrt{J}$. Hence, J is not a primary ideal of R.

Theorem 16. *Let R be a semiring, and J be a 1-absorbing primary ideal of R. Then $\sqrt{J}$ is a prime ideal of R.*

Proof. Let $a, b \in R$ such that $ab \in \sqrt{J}$. Then $(ab)^m \in J$ for some positive integer m. Assume a and b are non-unit elements of R, and let m be an even positive integer such that $m = 2n$ for some $n \geq 1$. Thus, $(ab)^m = (ab)^{2n} = a^n a^n b^m \in J$. Since J is a 1-absorbing primary ideal of R, it follows that either $a^n a^n = a^m \in J$ or $b^m \in J$, and so, $a \in \sqrt{J}$ or $b \in \sqrt{J}$. Therefore, $\sqrt{J}$ is a prime ideal of R. □

However, the converse of this result does not generally hold. For example, consider an ideal $I = \{0, 2, 4\}$ of $\mathbb{Z}_8$ and the idealization of semiring $R = \mathbb{Z}_8(+)I$. Let $J = \{(0,0), (0,2), (0,4)\}$ be an ideal of R. Then $\sqrt{J} = J$ and so $\sqrt{J}$ is a prime ideal of R. While, J is not 1-absorbing primary ideal of R, since $(2,0) \in R$ such that $(2,0) \cdot (2,0) \cdot (2,0) \in J$ but neither $(2,0) \cdot (2,0) \in J$ nor $(2,0) \in \sqrt{J}$.

The following result shows that R is a local semiring if J is a 1-absorbing primary ideal of R which is not a primary ideal of R (see Example 4 (iii)).

Theorem 17. *Let R be a semiring, and J be a proper k-ideal of R. If J is a 1-absorbing primary ideal, which is not a primary ideal of R. Then R is a local semiring.*

Proof. Assume that J is a 1-absorbing primary ideal, but not a primary ideal of R. Then there exist non-unit elements a and b in R such that $ab \in J$ but either $a \notin J$ or $b \notin \sqrt{J}$. Let c be a non-unit element of R. Since J is a 1-absorbing primary ideal and $cab \in J$ with $b \notin \sqrt{J}$, then $ca \in J$. Now, consider d to be a unit element of R, and assume $c+d$ is a non-unit element of R. As J is a 1-absorbing primary ideal, $(c+d)ab \in J$, and $b \notin \sqrt{J}$, we have $(c+d)a = ca + da \in J$. Since $ca + da \in J$ and $ca \in J$, which implies $da \in J$, as J is a k-ideal of R. However, d is a unit element of R and $da \in J$, we have $a \in J$, which leads to a contradiction. Hence, $c + d$ is a unit element of R. Thus, by Lemma 1, R is a local semiring. □

Recall from [12], let R be a semiring and a be a non-zero non-unit element of R. Then a is said to be irreducible if $a = bc$ such that either b or c is a unit of R, where $b, c \in R$. An element $r \in R$ is said to be prime if, whenever $r \mid st$, then either $r \mid s$ or $r \mid t$, where $s, t \in R$.

Lemma 7. *Every non-zero prime element of a local semiring R is an irreducible element of R.*

Proof. Assume p is a prime element of R, and for some elements $r, s \in R$ such that $p = rs$. Consider $p \mid r$, as p is a prime element of R. Now, we show that s is a unit of R. Thus, there exists element $t \in R$ such that $r = pt$. Then $p = pts$, which implies $p - pts = p(1 - ts) = 0$. If s is a non-unit element of R and R is local semiring, then $1 - ts$ is a unit of R implies that $p = 0$, which is a contradiction. Hence, s is a unit of R. Therefore, p is an irreducible element of R. □

The following result presents a technique to construct 1-absorbing primary ideals that are not themselves primary ideals.

Theorem 18. *Let M be a maximal ideal of a local semiring R. If $a \in M$ is a non-zero prime element of R, and $M \neq aR$. Then aM is a 1-absorbing primary ideal of R which is not a primary ideal of R.*

Proof. First, we prove that aM is not a primary ideal of R. Observe that $\sqrt{aM} = aR$. As $M \neq aR$, then there exists $m \in M \setminus aR$ such that $am \in aM$. Since a is a non-zero prime element of R. Thus, by Lemma 7, a is an irreducible element, which implies $a \notin aM$. Since $\sqrt{aM} = aR$ and $m \in M \setminus aR$, we have $m \notin \sqrt{aM}$. Therefore, aM cannot be a primary ideal of R. Next, we prove that aM is a 1-absorbing primary ideal of R. Let x, y and z be non-unit elements of R such that $xyz \in aM$. Assume $xy \notin aM$. This implies $a \nmid x$ and $a \nmid y$. Since $a \mid xyz$ while $a \nmid xy$, it follows that $a \mid z$. Consequently, $z \in aM$. □

Theorem 19. *Let J be a 1-absorbing primary ideal of a semiring R which is not a primary ideal. Then there exists a non-unit element b in R and an irreducible element a in R such that $ab \in J$ but $a \notin J$ or $b \notin \sqrt{J}$. Moreover, if c and d are non-unit elements of R with $cd \in J$ and $c \notin J$ or $d \notin \sqrt{J}$, then c is an irreducible element of R.*

Proof. Assume that J is not a primary ideal of R. Let a and b be non-unit elements of R such that $ab \in J$ but $a \notin J$ or $b \notin \sqrt{J}$. Consider that a is not an irreducible element of R. Then there exist non-unit elements $e, f \in R$ such that $a = ef$. Thus, $ab = efb \in J$. Since J is a 1-absorbing primary ideal of R and $efb \in J$ with $b \notin \sqrt{J}$, it follows that $ef = a \in J$, which is a contradiction. Therefore, a is an irreducible element of R. □

Theorem 20. *Let M be a maximal ideal of a local semiring R and P be a prime ideal of R with $P \subseteq M$. Then PM is a 1-absorbing primary ideal of R.*

Proof. First, note that $\sqrt{PM} = P$. Let x, y and z be non-unit elements of M with $xyz \in PM$. If $x \in P$ or $y \in P$, then clearly $xy \in PM$. So, assume that $x \notin P$ or $y \notin P$, which implies $xy \notin P$. Since $xyz \in PM \subseteq P$ and $xy \notin P$, we have $z \in P = \sqrt{PM}$. Therefore, PM is a 1-absorbing primary ideal of R. □

In the following example, we show that a local semiring R with maximal ideal M and for some non principal prime ideal P of R, then PM is a 1-absorbing primary ideal of R that is not primary.

Example 5. In [15, Example 2.5], $R = \mathbb{Z}_2[x, y, z]$, where $\mathbb{Z}_2$ is a semifield. Let $M = (x, y, z)$ be a maximal ideal of R and $P = (x, y)$ be a prime ideal of R. Then R_M is a local semiring with maximal ideal M_M, and P_M is a prime ideal of R_M. Thus, by Theorem 20, PM is a 1-absorbing primary ideal of R. Now, $PM = (x, y)(x, y, z) = (x^2, xy, xz, yx, y^2, yz)$, and $\sqrt{PM} = (x, y) = P$. Since $xz \in PM$ but $x \notin PM$ or $z \notin \sqrt{PM}$. Hence, PM is not a primary ideal of R.

Theorem 21. *Let R be a semiring and J be a 1-absorbing primary ideal of R. If there exists a non-unit element $a \in R \setminus J$ such that $(J : a) = \{r \in R | \, ra \in J\}$. Then $(J : a)$ is a primary ideal of R.*

Proof. Assume J is a 1-absorbing primary ideal of R and a is a non-unit element of R with $a \in R \setminus J$. Consider elements $b, c \in R$ such that $bc \in (J : a)$, which implies $abc \in J$. Assume b and c are non-unit elements of R. If $b \notin (J : a)$, then $ab \notin J$. Since J is a 1-absorbing primary ideal of R, $abc \in J$ and $ab \notin J$, we have $c \in \sqrt{J} \subseteq \sqrt{(J : a)}$. Thus, $(J : a)$ is a primary ideal of R. □

The following result shows that if R is a divided semiring, then J is a 1-absorbing primary ideal of R if and only if J is a primary ideal of R.

Theorem 22. *Let J be a proper ideal of a divided semiring R. Then J is a 1-absorbing primary ideal of R if and only if J is a primary ideal of R.*

Proof. Suppose J is a 1-absorbing primary ideal of R. Let $a, b \in R$ with $ab \in J$ and $b \notin \sqrt{J}$. Assume a and b are non-unit elements of R. By Theorem 16, $\sqrt{J}$ is a prime ideal of R, and $b \notin \sqrt{J}$, we have $a \in \sqrt{J}$. Since R is divided, we have $b \mid a$. Thus, there exists an element $c \in R$ such that $a = bc$. Since $b \notin \sqrt{J}$ and $a \in \sqrt{J}$, then c is a non-unit element of R. Since J is a 1-absorbing primary ideal of R, $ab = bcb \in J$ and $b \notin \sqrt{J}$, we have $bc = a \in J$. Hence, J is a primary ideal of R. The converse is straightforward. □

Theorem 23. *Let M be a maximal ideal of a divided semiring R. If M is not a principal prime ideal of R, then every nonzero prime ideal is not a principal ideal of R.*

Proof. Let $b \in R$ such that $P = bR$ be a non-zero prime ideal of R. Then b is a non-zero prime element of R and $M \neq bR$. Thus, by Theorem 18, bM is a 1-absorbing primary ideal of R that is not a primary ideal, and by Theorem 22, which is a contradiction. □

Theorem 24. *Let P be a prime ideal of a divided entire semiring R. Then for any positive integer $m \geq 1$, P^m is a primary ideal of R. Moreover, P^m is a 1-absorbing primary ideal of R for any positive integer $m \geq 1$.*

Proof. If $m = 1$, then the result is obvious. Now, we consider $m \geq 2$ and $a, b \in R$ such that $ab \in P^m$. Then $ab = p_1q_1 + \cdots + p_iq_i$, where $p_1, p_2, \ldots, p_i \in P$ and $q_1, q_2, \ldots, q_i \in P^{m-1}$ for some positive integer $i \geq 1$. Assume $b \notin P$. Since R is divided, we have $b \mid p_j$ for every j, where $1 \leq j \leq i$. Thus, $p_j = br_j$ for some $r_j \in P$. Then $ab = br_1q_1 + \cdots + br_iq_i$ which implies $b(a - (r_1q_1 + \cdots + r_iq_i)) = 0$. Since R is an entire semiring, we have $a = r_1q_1 + \cdots + r_iq_i \in P^m$. Hence, P^m is a primary ideal of R. Since every primary ideal of R is a 1-absorbing primary ideal of R. Therefore, P^m is a 1-absorbing primary ideal of R for all $m \geq 2$. □

The following result shows that the converse of Theorem 16 holds when R is a Dedekind semidomain.

Theorem 25. *Let J be a non-zero proper ideal of a commutative Dedekind semidomain R. Then J is a 1-absorbing primary ideal of R if and only if $\sqrt{J}$ is a prime ideal of R.*

Proof. Assume J is a 1-absorbing primary ideal of R. Then by Theorem 16, $\sqrt{J}$ is a prime ideal of R. Conversely, let $\sqrt{J}$ be a prime ideal of R. Since R is a Dedekind semidomain, then by [18, Theorem 2.21], every nonzero prime ideal of R is a maximal ideal of R. Thus, $\sqrt{J}$ is a maximal ideal of R and so J is a primary ideal of R by Theorem 9. Therefore, J is a 1-absorbing primary ideal of R. □

Theorem 26. *Let J be a proper k-ideal of a semiring R that is not a local semiring. Then J is a 1-absorbing primary ideal of R if and only if J is a primary ideal of R. Especially, if R_1 and R_2 are semirings such that $R = R_1 \times R_2$. Then J is a 1-absorbing primary ideal of R if and only if J is a primary ideal of R.*

Proof. Let J be a 1-absorbing primary ideal of R. Assume $a, b \in R$ such that $ab \in J$ and that both a and b are non-unit elements in R. Suppose $b \notin J$. Since R is not a local semiring, by Lemma 1, there exists a non-unit element $c \in R$ and a unit element $d \in R$ such that $c + d$ is also a non-unit element of R. Since J is a 1-absorbing primary ideal of R, $cab \in J$ and $b \notin J$ implies that $ca \in J$. Moreover, since J is a 1-absorbing primary ideal of R, $(c + d)ab \in J$ and $b \notin J$, we have $(c + d)a = ca + da \in J$. As J is a k-ideal of R, $ca \in J$ and $ca + da \in J$, which implies $da \in J$. Since d is a unit element of R and $da \in J$, we conclude that $a \in J$. Hence, J is a primary ideal of R. Conversely, if J is a primary ideal of R, then J is also a 1-absorbing primary ideal of R. □

Theorem 27. *Let $R = R_1 \times R_2$ where R_j is a commutative semiring for all $j \in \{1, 2\}$ and J_1 is an ideal of R_1, and J_2 is an ideal of R_2 such that $J = J_1 \times J_2$ is an ideal of R. Then the following statements hold:*

(i) J_1 is a primary ideal of R_1 if and only if $J_1 \times R_2$ is a primary ideal of R.

(ii) J_2 is a primary ideal of R_2 if and only if $R_1 \times J_2$ is a primary ideal of R.

Proof. **(i).** Assume that J_1 is a primary ideal of R_1. Let $(x, y), (z, w) \in R$ such that $(x, y)(z, w) \in J_1 \times R_2$. Therefore, $xz \in J_1$. Since J_1 is a primary ideal of R_1, $x \in J_1$ or $z \in \sqrt{J_1}$. Thus, $(x, y) \in J_1 \times R_2$ or $(z, w) \in \sqrt{J_1 \times R_2} = \sqrt{J_1} \times R_2$. Hence, $J_1 \times R_2$ is a primary ideal of R. Conversely, the proof is trivial.
(ii). The proof is similar to (i). □

Theorem 28. *Let $R = R_1 \times R_2$ be a commutative semiring with identity 1_{R_j} for each $j \in \{1, 2\}$. If J_1 is an ideal of R_1, and J_2 is an ideal of R_2 such that $J = J_1 \times J_2$ is an ideal of R. Then the following statements are equivalent:*

(i) J is a primary ideal of R.

(ii) $J = J_1 \times R_2$ for some primary ideal J_1 of R_1 or $J = R_1 \times J_2$ for some primary ideal J_2 of R_2.

Proof. Let $J = J_1 \times J_2$ be a primary ideal of R, where J_1 is an ideal of R_1 and J_2 is an ideal of R_2. Assume that $J = J_1 \times R_2$ for some proper ideal J_1 of R_1. Now, we show that J_1 is a primary ideal. If J_1 is a primary ideal of R_1, then by Theorem 27, $J = J_1 \times R_2$ for some primary ideal J_1 of R_1. Assume that J_1 is not a primary ideal of R_1. Then there exist elements $x, y \in R_1$ such that $xy \in J_1$ but neither $x \in J_1$ nor $y \in \sqrt{J_1}$. Thus, $(x, 1_{R_2})(y, 1_{R_2}) \in J_1 \times R_2$ implies that $(x, 1_{R_2}) \in J_1 \times R_2$ or $(y, 1_{R_2}) \in \sqrt{J_1 \times R_2} = \sqrt{J_1} \times R_2$. Consequently, $x \in J_1$ or $y \in \sqrt{J_1}$, which gives a contradiction. Hence, J_1 is a primary ideals of R_1. Therefore, $J = J_1 \times R_2$ for some primary ideal J_1 of R_1. The converse is straightforward. □

From Theorem 26 and Theorem 28, we conclude the following result.

Theorem 29. *Let $R = R_1 \times R_2$ be a commutative semiring with identity 1_{R_j} for each $j \in \{1, 2\}$. Suppose J_1 is an ideal of R_1 and J_2 is an ideal of R_2 such that $J = J_1 \times J_2$ is an ideal of R. Then the following statements are equivalent:*

(i) J is a 1-absorbing primary ideal of R.

(ii) J is a primary ideal of R.

(iii) $J = J_1 \times R_2$ for some primary ideal J_1 of R_1 or $J = R_1 \times J_2$ for some primary ideal J_2 of R_2.

This example shows that the intersection of two 1-absorbing primary ideals needs not be 1-absorbing primary ideal in R.

Example 6. Consider the semiring $R = \mathbb{N}$ with $J = 2\mathbb{N}$ and $K = 3\mathbb{N}$ as 1-absorbing primary ideals of R. However, their intersection $J \cap K = 2\mathbb{N} \cap 3\mathbb{N} = 6\mathbb{N}$ is not a 1-absorbing primary ideal of R, since $2 \cdot 2 \cdot 3 \in 6\mathbb{N}$ while neither $2 \cdot 2 \in 6\mathbb{N}$ nor $3 \in \sqrt{6\mathbb{N}}$.

Let J be a 1-absorbing primary ideal of R and $P = \sqrt{J}$ is a prime ideal of R. Then J is said to be a P-1-absorbing primary ideal of R.

Theorem 30. *Let R be a semiring and $J_1, J_2, \ldots, J_m$ be P-1-absorbing primary ideals of R. Then $J = \cap_{j=1}^{m} J_j$ is a P-1-absorbing primary ideal of R.*

Proof. First, note that $\sqrt{J} = P$. Let x, y and z be non-unit elements of R such that $xyz \in J$ and $xy \notin J$. Assume that $xy \notin J_j$ for some $j \in \{1, 2, \ldots, m\}$. Since J_j is P-1-absorbing primary ideal of R, $xyz \in J_j$ and $xy \notin J_j$, we have $z \in J_j$. □

Theorem 31. *Let R be a commutative semiring and let J be a 1-absorbing primary ideal of R. If for some non-unit elements x and y of R and a proper ideal K of R such that $xyK \subseteq J$, then $xy \in J$ or $K \subseteq \sqrt{J}$.*

Proof. Let J be a 1-absorbing primary ideal of R. Assume that $xyK \subseteq J$ but $xy \notin J$ or $K \nsubseteq \sqrt{J}$. Then there exists an element $k \in K$ such that $k \notin \sqrt{J}$. As a result, $xyk \in J$ but neither $xy \in J$ nor $k \in \sqrt{J}$, leading to a contradiction. □

Theorem 32. *Let J be a proper ideal of a semiring R. Then J is a 1-absorbing primary ideal of R if and only if $J_1J_2J_3 \subseteq J$ implies that $J_1J_2 \subseteq J$ or $J_3 \subseteq \sqrt{J}$, where J_1, J_2 and J_3 are proper ideals of R.*

Proof. Assume J is a 1-absorbing primary ideal of R. Let J_1, J_2 and J_3 be proper ideals of R such that $J_1J_2J_3 \subseteq J$. If $J_1J_2 \nsubseteq J$, then there exist non-unit elements $x \in J_1$ and $y \in J_2$ such that $xy \notin J$. Since $xyJ_3 \subseteq J$ and $xy \notin J$, by Theorem 31, $J_3 \subseteq \sqrt{J}$. Conversely, let x, y and z be non-unit elements of R with $xyz \in J$. Assume that $xy \notin J$. Take $J_1 = xR$, $J_2 = yR$ and $J_3 = zR$. Then $J_1J_2J_3 \subseteq J$ and $J_1J_2 \nsubseteq J$. It follows that $J_3 = zR \subseteq \sqrt{J}$ implies that $z \in \sqrt{J}$. Thus, J is a 1-absorbing primary ideal of R □

Theorem 33. *Let $g : R \to R_1$ be a semiring homomorphism. Assume that for every non-unit element x in R, $g(x)$ is a non-unit element of R_1. If J is a 1-absorbing primary ideal of R_1, then $g^{-1}(J)$ is a 1-absorbing primary ideal of R.*

Proof. Consider x, y and z are non-unit elements of R such that $xyz \in g^{-1}(J)$. Then $g(xyz) = g(x)g(y)g(z) \in J$. Since J is a 1-absorbing primary ideal of R_1 and $g(x)g(y)g(z) \in J$. Thus, either $g(x)g(y) \in J$ or $g(z) \in \sqrt{J}$. This implies that either $xy \in g^{-1}(J)$ or $z \in \sqrt{g^{-1}(J)} = g^{-1}\sqrt{J}$. Thus, $g^{-1}(J)$ is a 1-absorbing primary ideal of R. □

Recall from [19], let R be a semiring and Y an indeterminate. The set of all polynomials over R, denoted by $R[Y]$, consists of all formal expressions of the form $b_0 + b_1Y + \cdots + b_mY^m$, where $b_0, b_1, \ldots, b_m \in R$. For a polynomial $g = \sum_{j=1}^{n} r_j y^j \in R[Y]$, the ideal $< r_1, r_2, \ldots, r_n >$ of R generated by the coefficients of g is called the content of g, denoted by $c(g)$. In [19, Theorem 3] it states that the following statements are equivalent for a semiring R: (i). R is a subtractive semiring. (ii). If $g, h \in R[Y]$ and $deg(h) = n$, then $c(g)^{n+1}c(h) = c(g)^n c(gh)$. For an ideal J of R, such that $J[Y] = \{g(Y) \in R[Y] : c(g) \subseteq J\}$. For further details on polynomial semirings, refer to [18].

Theorem 34. *Let J be an ideal of a semiring R. Then J is a 1-absorbing primary ideal of R if and only if $J[X]$ is a 1-absorbing primary ideal of $R[X]$.*

Proof. Let J be a 1-absorbing primary ideal of R. Assume that r, s and t are non-unit elements of $R[X]$ such that $rst \in J[X]$ and $t \notin \sqrt{J[X]}$. It is obvious that $\sqrt{J[X]} = \sqrt{J}[X]$. Since $t \notin \sqrt{J[X]}$, we have $c(t) \nsubseteq \sqrt{J}$. However, $c(rst) \subseteq J$. Thus by Theorem [19, Theorem

3], $c(rs)c(t)^{deg(r)+deg(s)+1} = c(t)^{deg(r)+deg(s)}c(rst) \subseteq J$. Since $c(t) \nsubseteq \sqrt{J}$, we conclude that $c(t)^{deg(r)+deg(s)+1} \nsubseteq \sqrt{J}$. As J is a 1-absorbing primary ideal of R, then by Theorem 32, $c(rs) \subseteq J$. Thus, $rs \in J[X]$. Hence, $J[X]$ is a 1-absorbing primary ideal of $R[X]$. Conversely, let $J[X]$ be a 1-absorbing primary ideal of $R[X]$. Consider the injection $j : R \to R[X]$ defined by $j(x) = x$ for every $x \in R$. Then by Theorem 33, $j^{-1}(J[X]) = J$ is a 1-absorbing primary ideal of R. □

Theorem 35. *Let J be an ideal of a semiring R and N be a k-R-subsemimodule of an R-semimodule M and $J(+)N$ be a homogeneous ideal of $R(+)M$. Then $J(+)N$ is a 1-absorbing primary ideal of $R(+)M$ if and only if J is a 1-absorbing primary ideal of R such that $\sqrt{J}M \subseteq N$ and $(N : a) = N$ for all $a \notin \sqrt{J}$.*

Proof. Assume that x, y and z are non-unit elements of R such that $xyz \in J$. Then $(x, 0_M)(y, 0_M)(z, 0_M) = (xyz, 0_M) \in J(+)N$. Since $J(+)N$ is a 1-absorbing primary ideal of $R(+)M$ and for some positive integers n, then $(x, 0_M)(y, 0_M) \in J(+)N$ or $(z, 0_M)^n = (z^n, 0_M) \in J(+)N$. Thus, $xy \in J$ or $z^n \in J$. Hence, J is a 1-absorbing primary ideal of R. Conversely, assume J is a 1-absorbing primary ideal of R, $\sqrt{J}M \subseteq N$ and $(N : a) = N$ for all $a \notin \sqrt{J}$. Consider non-unit elements x, y and z of R and non-unit elements m_1, m_2 and m_3 of M such that $(x, m_1)(y, m_2)(z, m_3) = (xyz, xym_3 + xzm_2 + yzm_1) \in J(+)N$. Then $xyz \in J$, so either $xy \in J$ or $z \in \sqrt{J}$. If $z \in \sqrt{J}$, then $(z^n, m_3) \in JM \subseteq N$ for some positive integers n. Thus, $(z, m_3)^{n+1} = (z^{n+1}, (n + 1)z^n m_3) \in J(+)N$. If $z \notin \sqrt{J}$ and $xy \in J$, then $(x, m_1)(y, m_2) = (xy, xm_2 + ym_1) \in J(+)N$. Since N is a k-R-subsemimodule of an R-semimodule M, $xym_3 + xzm_2 + yzm_1 \in N$ and $xym_3 \in N$ implies $xzm_2 + yzm_1 \in N$. Since $z \notin \sqrt{J}$, then $xm_2 + ym_1 \in (N : z) = N$. Hence, $J(+)N$ is a 1-absorbing prime ideal of $R(+)M$. □

4 Conclusion

This article studies the concept of 1-absorbing prime ideal, and 1-absorbing primary ideal of commutative semirings. Every prime ideal is a 1-absorbing prime ideal, the reverse is not necessarily true (see Example 1). However, Theorem 3 shows that if R is not a local semiring, then every 1-absorbing prime ideal is a prime ideal. The behaviour of 1-absorbing prime ideals is further examined in the context of valuation semidomain, Prüfer semidomain, Dedekind semidomain and the idealization of semimodules. We show that the condition "for every non-unit element $x \in R_1$, $g(x)$ is a non-unit element of R_2" is necessary in Theorem 13 (see Example 3). Additionally, we prove the 1-absorbing Prime Avoidance Theorem for semirings (see Theorem 15). Similarly, every primary ideal is a 1-absorbing primary ideal of R, the converse does not hold in general (see Example 4). Theorem 22 and Theorem 26 provide conditions under which a 1-absorbing primary ideal is a primary ideal of R. Theorem 18 introduces a method for constructing 1-absorbing primary ideals that are not primary ideals themselves. Furthermore, Theorem 25 shows that the converse of Theorem 16 is true, when R is a Dedekind semidomain. Despite some limitations in the current study, future work will aim to enhance the findings by exploring various applications in several areas.

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