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Convergence and stability analysis of fractional integral residual minimization method for fractional differential equations and system of fractional differential equations

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Abstract. In this article, we propose a Fractional Integral Residual Minimization Method (FIRMM) to solve Fractional Differential Equations (FDEs) with the Caputo derivative. We provide a detailed and rigorous study of convergence analysis and stability analysis of FIRMM under suitable assumptions. Also, we extend the method FIRMM to solve a class of system of Caputo FDEs with a detailed and rigorous study on convergence and stability analysis under suitable assumptions. The efficacy of our proposed method is established through numerical experiments. The advantages and limitations of FIRMM are analyzed.

Keywords: Fractional differential equations, semi-analytical method, volterra integral equations, special functions, initial value problems.
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1 Introduction

Fractional differential equations generalise the idea of classical differentiation and integration as occurring in a fractional order, allowing for the modelling of systems in which current dynamics are contingent upon not only the current state, but also the history. By generalising the derivative operator to fractional orders, these equations can demonstrate memory and hereditary effects [6, 7].

Unlike integer order models, fractional differential equations employ integral operator choices with weak singular kernels, such as the Caputo or Riemann–Liouville derivatives, which enable past states to

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be weighted according to a generalised rate of memory kernel. Fractional Differential Equations (FDEs) have been established as valuable tools for modelling processes with memory and hereditary effects in fields such as physics, biology, and engineering. Specifically, equations written in the Caputo sense provide a framework for using classical initial conditions. The fractional glucose decay model demonstrates the use of fractional differential equations, as they include their memory for glucose kinetics, which results in an accurate representation of the blood glucose clearance.

Analytical solutions to fractional differential equations are very scarce, mostly restricted to isolated or relatively linear cases, and yet real world phenomena such as the behaviour of polymeric viscoelastic materials and complex biological processes require adaptable models using history-dependent dynamics. Thus, strong numerical methods have to be developed to approximate fractional derivatives and integrals when dealing with nonlinear or higher-dimensional systems that do not have closed-form solutions.

An analytical method that decomposes the solution into an infinite series and solves nonlinear terms through Adomian polynomials called the Adomian decomposition method [1], an iterative method that uses time-dependent Lagrange multipliers to construct successive approximations of the solution called the Variational Iteration Method [11,12], which employs time dependent Lagrange multipliers, and their various transform enhanced versions, such as the Laplace Variational Iteration Method [30]. A significant number of these methods incorporate adaptive schemes in which the Lagrange multiplier is updated at each iteration to improve convergence. Moreover, they developed spectral and collocation methods using orthogonal polynomials [4].

However, these adaptive approaches may involve considerable algorithmic complexity and tuning. Recently, for solving FDEs, transform based and neural network-based methods such as the Laplace Residual Power Series Method [16], and the Fractional Residual Power Series Method [14] have been used. These methods have a limited ability to capture memory effects. However, methods such as Physics-Informed Neural FDEs [28] and feedforward neural schemes for fractal fractional systems [2] introduced flexibility and adaptability to capture memory effects, but required expensive training phases.

Recent advances also include fractional radial basis function neural networks for solving fractional differential equations [15], numerical pricing of American/European options using time-fractional Black–Scholes models [20], and numerical methods for generalised tempered-type integrodifferential equations [22].

Tempered fractional Jacobi collocation methods have been used to have spectral accuracy for tempered diffusion models, which have a higher convergence rate due to orthogonal polynomials [32]. The kernel reproducing method coupled with shifted Chebyshev or Bernoulli basis functions, where its iterative schemes are effective in handling nonlocal boundary and initial conditions while stability and higher-order accuracy are maintained [23]. Moreover, the fractional order nonlinear system of Volterra integro-differential equations has been solved through a Chebyshev pseudospectral method [25]. Recently, a numerical framework for population dynamics models was developed for systems of time-FDEs [19].

In this article, we redefine the FDE as a Volterra integral equation and approximate its solution by expanding it on a fractional polynomial basis, which are chosen to be orthogonal with respect to the weighted inner product. The residual is then minimised in a weighted norm that reflects the weak singularity of the integral kernel. The FIRMM distinguishes itself by using a fixed multiplier fractional polynomial bases of the form $t^{n\alpha}, n \in \mathbb{N}$, inherently weighted to match the weak singularity of the Volterra kernel $t^{\alpha-1}$. Unlike neural or adaptive Variational Iteration Method-type methods [28,31], FIRMM uses a residual orthogonality condition to determine the solution coefficients by minimising the residual in a fractional weighted space.

Existing spectral/residual methods such as Jacobi Galerkin [5], Chebyshev pseudospectral [26], FC-polynomials Tau approach [27], and Müntz-collocation [13] approximate Volterra FDEs using orthogonal polynomials or power bases, but lack three key features that FIRMM provides: (i) fractional monomials $t^{n\alpha}$ specifically matched to the kernel singularity $t^{\alpha-1}$, (ii) direct Galerkin projection of the residual in the kernel-weighted space $L^2_{w_\alpha}$ with $w_\alpha(t) = t^{\alpha-1}$, and (iii) simple implementation without orthogonal polynomial computation. A fast sparse spectral Volterra scheme using Jacobi polynomials [10] and a Sparse Spectral Method for Volterra Integral Equations using Orthogonal Polynomials on the Triangle [9] approximate Volterra type FDEs with polynomial or fractional bases but do not employ fractional monomials $t^{n\alpha}$ with kernel weighted orthogonal projection as in FIRMM.

The FIRMM fills these gaps through a kernel-adapted basis $\{t^{n\alpha}\}$ that naturally handles the Volterra singularity, weighted Galerkin projection making the residual orthogonal to the trial space.

Fractional models generally do not appear as a single equation, but as systems of coupled FDEs with Caputo derivatives. Such systems arise naturally in epidemic models [3, 17] and exhibit even greater analytical and numerical challenges due to memory. Motivated by these challenges, we extend our FIRMM to the class of system of Caputo FDEs. The extension follows the same as in the single equation. This generalisation allows our FIRMM to apply effectively to a wide range of real-world fractional models.

This paper is organised as follows: In Section 2 we propose our method. In Section 3, the convergence and stability of our proposed method is discussed. Numerical experiments are given in Section 4. In Section 5, FIRMM is extended to the system of FDEs. The numerical experiment for the system is discussed in Section 6. The glucose decay model is solved using FIRMM and is compared with real-life data in Section 7.

2 Fractional integral residual minimization method

Consider a time-fractional differential equation in Caputo form

$$ {}^{C}D_t^{\alpha} y(t) = f(t, y(t)), \quad 0 < \alpha < 1, \quad y(0) = y_0. \tag{1} $$

Taking integration on both sides and applying Riemann-Liouville fractional Integral ($I^{\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} f(s) ds$ [21]), equation (1) is equivalent to the Volterra integral equation

$$ y(t) = y_0 + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s, y(s))\, ds. \tag{2} $$

At this stage, we approximate the solution by a finite series on a fractional monomial basis adapted to the kernel weight

$$ y(t) \approx y_N(t) = \sum_{n=0}^{N} a_n \phi_n(t), \quad \phi_n(t) = t^{n\alpha}. \tag{3} $$

The weight $w_\alpha(t) = t^{\alpha-1}$ is chosen to reflect the weak singularity of the Volterra kernel and to define the weighted inner product

$$ \langle g, h \rangle_\alpha = \int_0^T g(t) h(t) t^{\alpha-1}\, dt. $$

Unlike Jacobi polynomials, the fractional monomials $t^{\alpha-1}$ are not mutually orthogonal with respect to this weight; instead, FIRMM enforces orthogonality of the residual to the trial space in the weighted inner product.

The initial condition implies $a_0 = y_0$. Substitute the approximation (3) into the integral equation (2) to define the residual:

$$R(t;\{a_n\}) = \sum_{n=0}^{N} a_n \phi_n(t) - y_0 - \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f\Big(s, \sum_{n=0}^{N} a_n \phi_n(s)\Big)\, ds. \tag{4}$$

The coefficients $\{a_n\}$ are determined by requiring that the residual $R(t)$ is orthogonal (in the weighted inner product) to the subspace spanned by $\{\phi_n\}$:

$$\int_0^T R(t;\{a_n\})\, \phi_m(t)\, w_\alpha(t)\, dt = 0, \quad m = 0,1,\ldots,N. \tag{5}$$

This procedure leads to a system of $N+1$ nonlinear algebraic equations:

$$\int_0^T \left[\sum_{n=0}^{N} a_n \phi_n(t) - y_0 - \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f\left(s, \sum_{n=0}^{N} a_n \phi_n(s)\right) ds \right] \phi_m(t) t^{\alpha-1}\, dt = 0, \tag{6}$$

for $m = 0,1,\ldots,N$. This system of equations is solved for the coefficients $a_0, a_1, \ldots, a_N$.

Under the assumptions of the Theorems 1 & 2, the Volterra integral equation admits a unique solution y on $[0,T]$. The FIRMM approximation is defined as the unique element of the finite-dimensional space V_N satisfying the Galerkin-type conditions (5). For sufficiently small $\kappa = \frac{LT^\alpha}{\Gamma(\alpha+1)} < 1$, the nonlinear projection operator V_N onto is a contraction in a neighbourhood of the exact solution, so the discrete nonlinear system has a unique solution in that neighbourhood. This provides a local guarantee of non-singularity of the Jacobian at the converged solution, in the sense that the discrete solution is isolated.

3 Convergence and stability analysis

The following section presents the convergence and stability analysis of the FIRMM when applied to the time-fractional initial value problem (1), which is equivalent to the Volterra integral equation (2). Here $L^2_{w_\alpha}(0,T)$ denotes the weighted space with norm

$$\|v\|^2_{L^2_{w_\alpha}(0,T)} = \int_0^T |v(t)|^2 t^{\alpha-1}\, dt.$$

Theorem 1 (Convergence of FIRMM)**.** *If*

1. *$f(t,y)$ is Lipschitz continuous in y uniformly in t, i.e., $|f(t,y_1) - f(t,y_2)| \le L|y_1 - y_2|, \quad \forall t \in [0,T], \forall y_1, y_2 \in \mathbb{R}$, for some constant $L > 0$,*
2. *the exact solution y belongs to a weighted fractional Sobolev space $H^s_{w_\alpha}(0,T)$ with $s > 0$, where the weight $w_\alpha(t) = t^{\alpha-1}$, and there exist constants $C > 0$, $p > 0$ such that the best approximation error in $V_N = span\{t^{n\alpha} : 0 \le n \le N\}$ satisfies*

$$\inf_{v \in V_N} \|y - v\|_{L^2_{w_\alpha}(0,T)} \le C N^{-p}$$

and

3. *the time interval is chosen so that* $\kappa := \frac{LT^\alpha}{\Gamma(\alpha+1)} < 1$,

then the approximate solution $y_N(t)$ *obtained from the projection condition* (5) *satisfies* $\|y-y_N\|_{L^2_{w_\alpha}(0,T)} \leq \frac{C}{1-\kappa} \inf_{v\in V_N} \|y-v\|_{L^2_{w_\alpha}(0,T)}$. *In particular, as* $N \to \infty$, $y_N(t)$ *converges to the exact solution* $y(t)$ *in the weighted* L^2 *norm.*

Proof. Let K be the integral operator defined by

$$(Ky)(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s,y(s))\, ds.$$

Then the exact solution of (1) satisfies $y = y_0 + K(y)$. We approximate $y(t)$ by $y_N(t) = \sum_{n=0}^N a_n \phi_n(t)$, and by the projection condition (5),

$$\int_0^T \Big[y_N(t) - y_0 - K(y_N)(t) \Big] \phi_m(t) w_\alpha(t)\, dt = 0, \quad m = 0,1,\ldots,N.$$

Let the error function be $e(t) = y(t) - y_N(t)$, Therefore we have

$$\begin{aligned} e(t) &= y(t) - y_N(t) \\ &= y(t) - y_0 - K(y_N) - R_N, \text{where, } R_N = {}^C D_t^\alpha y_N(t) - f(t, y_N(t)) \\ &= y_0 + K(y) - y_0 - K(y_N) - R_N \\ &\leq \frac{L}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} |e(s)|\, ds + |R(t)|, \text{ (by using the Lipschitz continuity of } f) \end{aligned}$$

where $R(t)$ refers to the residual of the projection, L is Lipschitz constant. By applying the fractional Grönwall inequality [21], it follows that

$$|e(t)| \leq |R(t)|\, E_\alpha\Big(\frac{LT^\alpha}{\Gamma(\alpha+1)}\Big).$$

We know that the residual $|R(t)|$ is bounded by the error $\inf_{v\in V_N} \|y-v\|_{L^2_{w_\alpha}(0,T)}$. Taking the weighted L^2 norm over $[0,T]$, we deduce

$$\|y - y_N\|_{L^2_{w_\alpha}(0,T)} \leq \frac{1}{1-\kappa} \inf_{v\in V_N} \|y-v\|_{L^2_{w_\alpha}(0,T)},$$

with $\kappa = \frac{LT^\alpha}{\Gamma(\alpha+1)} < 1$. □

Theorem 2 (Stability of FIRMM)**.** *If*

1. *the function* $f(t,y)$ *satisfies the Lipschitz condition* $|f(t,y_1) - f(t,y_2)| \leq L|y_1 - y_2|$, $\forall t \in [0,T]$, $\forall y_1, y_2 \in \mathbb{R}$,
2. *the integral operator* $(Ty)(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s,y(s))\, ds$ *is contractive in the sense that* $\|T(y_1) - T(y_2)\|_{L^\infty(0,T)} \leq \kappa \|y_1 - y_2\|_{L^\infty(0,T)}$, *with* $\kappa = \frac{LT^\alpha}{\Gamma(\alpha+1)} < 1$, *and*
3. *the residual minimization is performed in the weighted space* $L^2_{w_\alpha}(0,T)$ *with weight* $w_\alpha(t) = t^{\alpha-1}$,

also, if $\tilde{y}(t)$ is the solution corresponding to a perturbed problem (e.g., with a perturbed initial condition or a perturbation in f), then there exists a constant $C > 0$, depending only on T, α, and κ, such that

$$\|y_N - \tilde{y}_N\|_{L^2_{w_\alpha}(0,T)} \leq C\big(|\Delta y_0| + \|\varepsilon\|_{L^2_{w_\alpha}(0,T)}\big),$$

where $\varepsilon(t)$ is the magnitude of the perturbation, then the method is stable.

Proof. Assume that the exact solution $y(t)$ satisfies $y = y_0 + K(y)$, and a perturbed solution $\tilde{y}(t)$ satisfies $\tilde{y} = \tilde{y}_0 + K(\tilde{y}) + \varepsilon$, where $\varepsilon(t)$ is a small perturbation. The numerical approximations $y_N(t)$ and $\tilde{y}_N(t)$ are obtained by applying the projection condition (5), we get

$$\int_0^T \Big[y_N(t) - y_0 - K(y_N)(t)\Big]\, \phi_m(t) w_\alpha(t)\, dt = 0,$$

and

$$\int_0^T \Big[\tilde{y}_N(t) - \tilde{y}_0 - K(\tilde{y}_N)(t)\Big]\, \phi_m(t) w_\alpha(t)\, dt = 0, \quad m = 0, 1, \ldots, N.$$

Let $e_N(t) = y_N(t) - \tilde{y}_N(t)$. Subtracting the two projection equations and using the Lipschitz property of f, we get

$$|e_N(t)| \leq |\Delta y_0| + \frac{L}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} |e_N(s)|\, ds + |\varepsilon_N(t)|,$$

where $\Delta y_0 = y_0 - \tilde{y}_0$ and $\varepsilon_N(t)$ represent the effect of the perturbation of the data in the projection. By implementing the fractional Grönwall inequality (see [21]), we derive that

$$\|e_N\|_{L^\infty(0,T)} \leq \frac{|\Delta y_0| + \|\varepsilon\|_{L^\infty(0,T)}}{1-\kappa}.$$

Using the continuous embedding $L^\infty(0,T) \to L^2_{w_\alpha}(0,T)$,

$$\|v\|_{L^2_{w_\alpha}(0,T)} \leq C_\alpha \|v\|_{L^\infty(0,T)}, \quad C_\alpha = \Big(\int_0^T t^{\alpha-1}\, dt\Big)^{1/2} = \sqrt{\frac{T^\alpha}{\alpha}},$$

and we obtain

$$\|e_N\|_{L^2_{w_\alpha}(0,T)} \leq \frac{C_\alpha}{1-\kappa}\big(|\Delta y_0| + \|\varepsilon\|_{L^\infty(0,T)}\big).$$

Thus, there exists a constant $C > 0$, depending only on T, α, and κ, such that

$$\|y_N - \tilde{y}_N\|_{L^2_{w_\alpha}(0,T)} \leq C\big(|\Delta y_0| + \|\varepsilon\|_{L^2_{w_\alpha}(0,T)}\big).$$

□

4 Numerical illustrations

We provide three examples for FDE and a problem for System of FDE which are solved using MATLAB software to validate our suggested method.

Problem 1 (Fractional relaxation equation)**.** *Consider the linear FDE ${}^C D_t^\alpha y(t) = -\lambda\, y(t), \quad y(0) = y_0$ with $0 < \alpha < 1$, $\lambda = 1$, and $y_0 = 1$.*

Soln. As shown in the analysis, the Volterra formulation is

$$y(t) = y_0 - \frac{\lambda}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} y(s)\, ds. \tag{7}$$

First-order approximation:

Assume an approximation $y(t) \approx y_0 + a_1 t^\alpha$. The enforcer of residual minimization produces $a_1 = -\frac{\lambda y_0}{\Gamma(\alpha+1)}$. Thus, the approximate solution is

$$y(t) \approx y_0 \left[1 - \frac{\lambda t^\alpha}{\Gamma(\alpha+1)}\right].$$

The exact solution is given by

$$y(t) = y_0 E_\alpha(-\lambda t^\alpha) = y_0 \left[1 - \frac{\lambda t^\alpha}{\Gamma(\alpha+1)} + \frac{\lambda^2 t^{2\alpha}}{\Gamma(2\alpha+1)} - \cdots\right].$$

Our first-order approximation regenerates the first two terms of the series exactly.

Second-order approximation:

We approximate the solution by a second-order fractional polynomial:

$$y(t) \approx y_0 + a_1 t^\alpha + a_2 t^{2\alpha}. \tag{8}$$

Substitute (8) into (7):

$$\begin{aligned} y_0 + a_1 t^\alpha + a_2 t^{2\alpha} = y_0 &- \frac{\lambda}{\Gamma(\alpha)} y_0 \int_0^t (t-s)^{\alpha-1} ds \\ &- \frac{\lambda}{\Gamma(\alpha)} a_1 \int_0^t (t-s)^{\alpha-1} s^\alpha ds \\ &- \frac{\lambda}{\Gamma(\alpha)} a_2 \int_0^t (t-s)^{\alpha-1} s^{2\alpha} ds. \end{aligned}$$

Using the identity $\int_0^t (t-s)^{\alpha-1} s^\beta\, ds = \frac{\Gamma(\beta+1)\Gamma(\alpha)}{\Gamma(\alpha+\beta+1)} t^{\alpha+\beta}$, we have

$$\int_0^t (t-s)^{\alpha-1} ds = \frac{t^\alpha}{\alpha}, \quad \int_0^t (t-s)^{\alpha-1} s^\alpha ds = \frac{\Gamma(\alpha+1)\Gamma(\alpha)}{\Gamma(2\alpha+1)} t^{2\alpha},$$

and the third integral (of order $t^{3\alpha}$) is neglected in our second-order approximation. Matching coefficients for the t^α term yields:

$$\begin{aligned} a_1 &= -\frac{\lambda y_0}{\Gamma(\alpha)} \cdot \frac{1}{\alpha} \\ &= -\frac{\lambda\, y_0}{\alpha \Gamma(\alpha)} \end{aligned}$$

Recalling that $\Gamma(\alpha+1) = \alpha\Gamma(\alpha)$, gives

$$a_1 = -\frac{\lambda\, y_0}{\Gamma(\alpha+1)}.$$

Next, matching the $t^{2\alpha}$ term gives:

$$\begin{aligned} a_2 &= -\frac{\lambda a_1}{\Gamma(\alpha)} \cdot \frac{\Gamma(\alpha+1)\Gamma(\alpha)}{\Gamma(2\alpha+1)} \\ &= \frac{\lambda^2 y_0}{\Gamma(\alpha+1)} \cdot \frac{1}{\Gamma(\alpha)} \cdot \frac{\Gamma(\alpha+1)\Gamma(\alpha)}{\Gamma(2\alpha+1)} \\ &= \frac{\lambda^2 y_0}{\Gamma(2\alpha+1)}. \end{aligned}$$

Thus, the second-order approximate solution is

$$y(t) \approx y_0 \left[1 - \frac{\lambda t^\alpha}{\Gamma(\alpha+1)} + \frac{\lambda^2 t^{2\alpha}}{\Gamma(2\alpha+1)}\right] \tag{9}$$

Note that the exact solution is given by

$$y(t) = y_0 E_\alpha(-\lambda t^\alpha) = y_0 \left[1 - \frac{\lambda t^\alpha}{\Gamma(\alpha+1)} + \frac{\lambda^2 t^{2\alpha}}{\Gamma(2\alpha+1)} - \cdots\right],$$

so the above second-order approximation reproduces the first two nonzero terms.

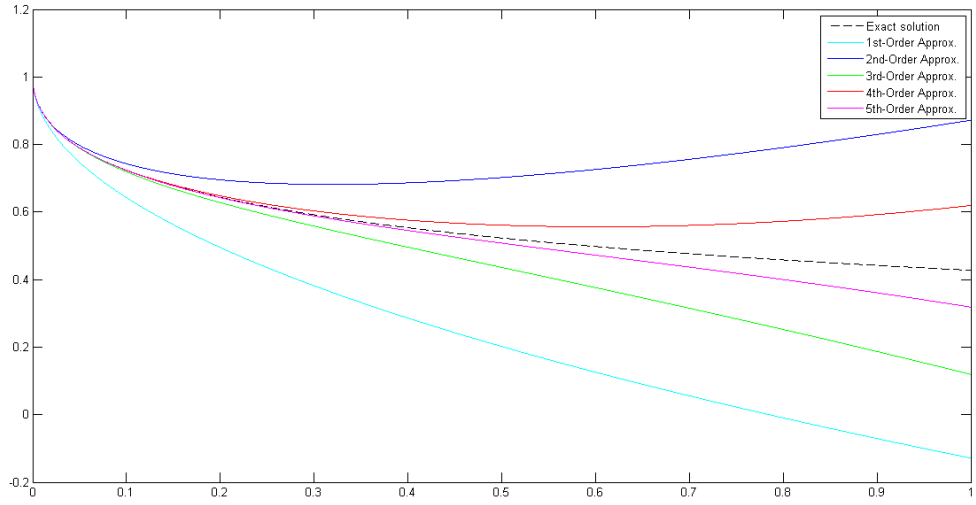

Figure 1: Solution graph for Problem 1

Figure 1 shows a comparison of the exact solution and the approximation solution produced by FIRMM (up to 5th order) for the Problem 1.

Problem 2 (Fractional logistic equation)**.** *Consider the non-linear FDE*

$${}^C D_t^\alpha y(t) = y(t)\,[1 - y(t)], \quad y(0) = y_0, \tag{10}$$

with $y_0 = 0.1$, $0 < \alpha < 1$ and, for simplicity, let $r = 1$. An exact solution is not known for this problem. However, the approximate solution is given by

$$y(t) \approx \frac{y_0}{1 - y_0 + y_0 E_\alpha(-t^\alpha)},$$

where $E_\alpha(-t^\alpha)$ is the one-parameter Mittag-Leffler function.

Soln. Its integral form is

$$y(t) = y_0 + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} y(s)[1 - y(s)]\, ds. \tag{11}$$

First-order approximation:
Assume a two-term approximation $y(t) \approx 0.1 + a_1 t^\alpha$. Expanding the non-linear term to first order (neglecting higher-order terms) yields $y(s)[1 - y(s)] \approx 0.1 \times 0.9 = 0.09$. Then, matching the coefficient of t^α (as in Example 1) gives

$$a_1 = \frac{0.09}{\Gamma(\alpha + 1)}.$$

Thus, the approximate solution is

$$y(t) \approx 0.1 + \frac{0.09}{\Gamma(\alpha + 1)} t^\alpha.$$

For $\alpha = 1$, this reproduces the first-order term of the Taylor expansion of the exact logistic solution.
Second-order approximation:
Assume a second-order approximation:

$$y(t) \approx y_0 + a_1 t^\alpha + a_2 t^{2\alpha}.$$

We proceed to expand the non-linear term. In first order (ignoring terms of order higher than t^α in the product) we write

$$y(s)[1 - y(s)] \approx y_0 (1 - y_0) + a_1 (1 - 2y_0) s^\alpha.$$

Substitute this approximation into (11):

$$y_0 + a_1 t^\alpha + a_2 t^{2\alpha} = y_0 + \frac{1}{\Gamma(\alpha)} \left[y_0(1 - y_0) \int_0^t (t-s)^{\alpha-1} ds + a_1 (1 - 2y_0) \int_0^t (t-s)^{\alpha-1} s^\alpha ds \right].$$

Using the previous integrals,

$$\int_0^t (t-s)^{\alpha-1} ds = \frac{t^\alpha}{\alpha}, \quad \int_0^t (t-s)^{\alpha-1} s^\alpha ds = \frac{\Gamma(\alpha+1)\Gamma(\alpha)}{\Gamma(2\alpha+1)} t^{2\alpha},$$

we obtain

$$a_1 t^\alpha + a_2 t^{2\alpha} = \frac{y_0(1 - y_0)}{\Gamma(\alpha)} \frac{t^\alpha}{\alpha} + \frac{a_1 (1 - 2y_0) \Gamma(\alpha+1)\Gamma(\alpha)}{\Gamma(\alpha)\Gamma(2\alpha+1)} t^{2\alpha}.$$

Matching coefficients for t^α gives:

$$a_1 = \frac{y_0(1 - y_0)}{\alpha \Gamma(\alpha)} = \frac{y_0(1 - y_0)}{\Gamma(\alpha + 1)}.$$

Matching coefficients for $t^{2\alpha}$ yields:

$$a_2 = \frac{a_1(1-2y_0)\Gamma(\alpha+1)}{\Gamma(2\alpha+1)} = \frac{y_0(1-y_0)(1-2y_0)}{\Gamma(2\alpha+1)}.$$

Thus, the second-order approximate solution is

$$y(t) \approx y_0 + \frac{y_0(1-y_0)}{\Gamma(\alpha+1)}t^{\alpha} + \frac{y_0(1-y_0)(1-2y_0)}{\Gamma(2\alpha+1)}t^{2\alpha} \tag{12}$$

For the classical case ($\alpha = 1$), this recovers the first two terms of the Taylor series of the exact logistic solution.

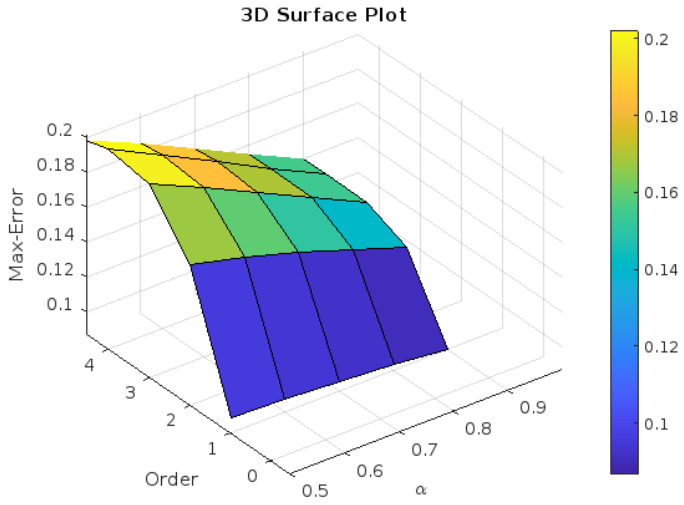

Figure 2: Maximum Error Plot for Problem 2

Figure 2 shows the maximum error comparison between exact and FIRMM approximation (up to 5th order) for the Problem 2 for various value of α.

Problem 3. *We consider the nonlinear fractional initial value problem*

$${}^{C}D^{2\alpha}u(x) = u(x)^2 + 1, \qquad 0 < \alpha \leq 1, \tag{13}$$

subject to the initial conditions

$$u(0) = 0, \qquad {}^{C}D^{\alpha}u(0) = 1. \tag{14}$$

For the classical case $\alpha = 1$, the exact solution of (13)–(14) is $u(x) = \tan x$. This problem has been studied previously using the Variational Iteration Method (VIM) [29].

Soln. Assume a third-order approximation:

$$U_F(x) = a_1x^{\alpha} + a_2x^{2\alpha} + a_3x^{3\alpha},$$

where, the initial condition ${}^{C}D^{\alpha}u(0) = 1$ yields $a_1 = \frac{1}{\Gamma(1+\alpha)}$. The Caputo fractional derivative of order 2α is computed term-wise as

$$ {}^{C}D^{2\alpha}U_F(x) = a_2\Gamma(1+2\alpha) + a_3\frac{\Gamma(1+3\alpha)}{\Gamma(1+\alpha)}x^{\alpha}.$$

The residual function is defined by $R(x) = {}^{C}D^{2\alpha}U_F(x) - \big(U_F(x)^2 + 1\big)$. The unknown coefficients a_2 and a_3 are determined by enforcing the FIRMM orthogonality conditions $\int_0^1 R(x)x^{\alpha-1}dx = 0$, $\int_0^1 R(x)$ $x^{2\alpha-1}dx = 0$ which are solved numerically. For comparison, we use the second-order truncated VIM approximation reported in [29],

$$U_{\mathrm{VIM}}(x) = \frac{x^{\alpha}}{\Gamma(1+\alpha)} + \frac{x^{2\alpha}}{\Gamma(1+2\alpha)} + \frac{\Gamma(1+2\alpha)}{\Gamma(1+\alpha)^2\Gamma(1+4\alpha)}x^{4\alpha}.$$

Both FIRMM($n = 3$) and VIM(2) involve comparable truncation. FIRMM determines two unknown coefficients via residual projection, while VIM employs two correction iterations. The numerical results are obtained on the interval $[0,1]$. The maximum absolute error is defined as

$$E_{\max} = \max_{x\in[0,1]} |U(x) - \tan x|.$$

For $\alpha = 0.90$, FIRMM yields a maximum error of 3.93271×10^{-1}, while VIM attains 2.81615×10^{-1}. For $\alpha = 0.99$, the corresponding errors are 1.53585×10^{-1} (FIRMM) and 1.40248×10^{-1} (VIM). In the classical limit $\alpha = 1$, FIRMM produces an error of 1.34263×10^{-1}, compared with 1.26153×10^{-1} for VIM. These results show that FIRMM provides a stable and systematically improvable approximation, with accuracy comparable to VIM at similar truncation depth. Table 1 presents a pointwise comparison of FIRMM($n = 3$) and VIM with the exact solution (α=1), showing that both methods provide accurate approximations, with VIM yielding slightly smaller maximum errors at the same truncation depth. Figures 3 and 4 further illustrate that FIRMM produces smooth and stable approximations whose accuracy improves systematically, while VIM exhibits marginally lower errors due to the inclusion of higher-order fractional terms.

Table 1: Comparison of FIRMM($n = 3$) and VIM with the exact solution $\tan x$

α	x	Exact	FIRMM	Error(F)	VIM	Error(V)
1.00	0.25	0.25534	0.27173	0.01639	0.28158	0.02623
	0.50	0.54630	0.61163	0.06533	0.63021	0.08391
	0.75	0.93160	1.05675	0.12515	1.05762	0.12602
	1.00	1.55741	1.64411	0.08670	0.02593	0.02593

5 FIRMM for systems of FDEs

Let $\{y_i(t)\}_{i=1}^{m}$ satisfy the Caputo–fractional system

$$ {}^{C}D_t^{\alpha}y_i(t) = f_i\big(t, y_1(t), \ldots, y_m(t)\big), \quad i = 1,2,\ldots,m, \quad y_i(0) = y_{i0}. \tag{15}$$

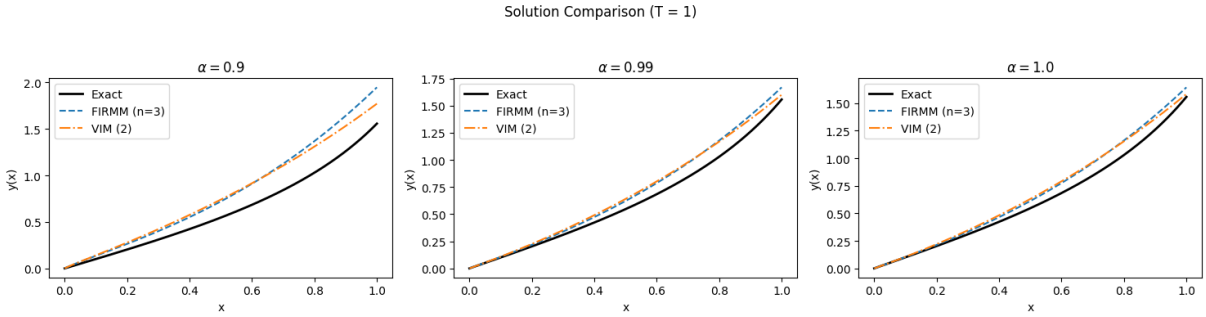

Figure 3: Comparison of FIRMM($n = 3$), VIM, and exact solution for $\alpha = 0.90, 0.99$, and 1.0

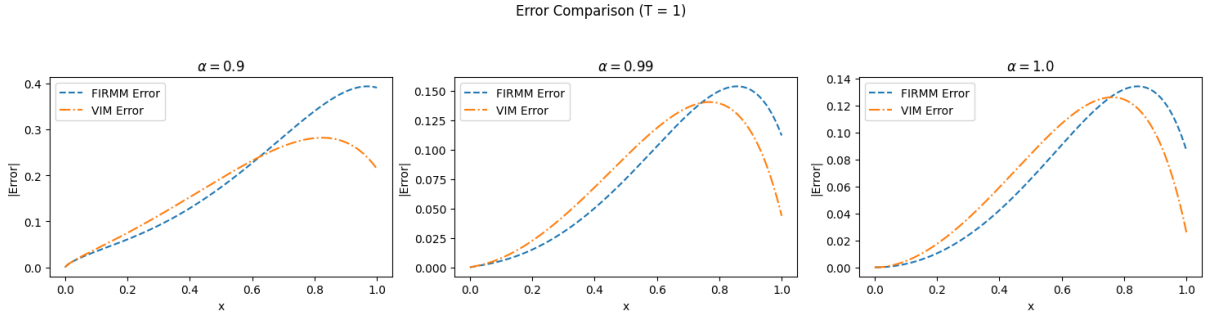

Figure 4: Absolute error comparison of FIRMM($n = 3$) and VIM(2) on $[0, 1]$

Using the Riemann–Liouville fractional integral operator $I^{\alpha}[g](t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} g(s)\, ds$, we reformulate (15) as the Volterra integral system:

$$y_i(t) = y_{i0} + I^{\alpha}\left[f_i\left(t, y_1(t), \ldots, y_m(t)\right)\right]. \tag{16}$$

We approximate each component by a finite fractional polynomial expansion:

$$y_{iN}(t) = \sum_{n=0}^{N} a_{in}\, t^{n\alpha}, \qquad a_{i0} = y_{i0}. \tag{17}$$

Define the residual for each i:

$$R_i(t; \{a_{jn}\}) = y_{iN}(t) - y_{i0} - \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f_i\left(s, y_{1N}(s), \ldots, y_{mN}(s)\right) ds. \tag{18}$$

Enforcing orthogonality of each residual with the weighted basis $\phi_j(t) = t^{j\alpha}$ in the weighted inner product $\langle u, v \rangle_{\alpha} = \int_0^T u(t) v(t) t^{\alpha-1}\, dt$, we get:

$$\int_0^T R_i(t)\, \phi_j(t)\, t^{\alpha-1}\, dt = 0, \quad i = 1, \ldots, m, \;\; j = 0, 1, \ldots, N. \tag{19}$$

This yields a system of $(m \times (N+1))$ nonlinear algebraic equations for the coefficients $\{a_{in}\}$. Solving them provides the FIRMM approximation $\{y_{iN}(t)\}$.

Theorem 3 (Convergence of FIRMM for the Caputo-fractional system)**.** *Let the vector solution* $\mathbf{y}(t) = (y_1(t),\dots,y_m(t))^\top$ *satisfy the Caputo-fractional system*

$$ {}^{C}D_t^{\alpha} y_i(t) = f_i\big(t, y_1(t),\dots,y_m(t)\big), \quad i = 1,2,\dots,m, \quad y_i(0) = y_{i0}, $$

on $t \in [0,T]$*, with* $0 < \alpha \le 1$*. Suppose*

1. $\mathbf{f}(t,\mathbf{y})$ *is Lipschitz in* $\mathbf{y}$ *uniformly in* t*:*

$$ \|\mathbf{f}(t,\mathbf{y}) - \mathbf{f}(t,\mathbf{z})\| \le L\|\mathbf{y}-\mathbf{z}\|, \quad \forall t \in [0,T],\ \mathbf{y},\mathbf{z} \in \mathbb{R}^m. $$

2. *the exact solution* $\mathbf{y}(t)$ *admits a best-approximation in* $V_N = \operatorname{span}\{t^{n\alpha} : 0 \le n \le N\}$ *satisfying* $\inf_{v\in V_N} \|\mathbf{y}-v\|_{L^2_{w_\alpha}(0,T)} \le CN^{-p}$.

3. *the contractivity constant* $\kappa = L\dfrac{T^\alpha}{\Gamma(\alpha+1)} < 1$.

then the approximate solution $y_N(t)$ *satisfies* $\|y - y_N\|_{L^2_{w_\alpha}(0,T)} \le \frac{C}{1-\kappa} N^{-p}$*. Hence, as* $N \to \infty$*,* $\mathbf{y}_N \to \mathbf{y}$ *in the weighted norm* $L^2_{w_\alpha}(0,T)$*.*

Proof. For each i, error $e_i = y_i - y_{iN}$ satisfies

$$ |e_i(t)| \le \frac{L}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \|\mathbf{e}(s)\|\, ds + |R_i(t)|. $$

Squaring/summing: $\|\mathbf{e}(t)\|^2 \le 2L^2 I_\alpha[\|\mathbf{e}\|^2](t) + 2\|R(t)\|^2$. Vector Grönwall gives

$$ \|\mathbf{e}(t)\| \le \frac{1}{1-\kappa} \max_{0\le s\le T} \|R(s)\|. $$

Weighted L^2 norm and projection orthogonality yield the estimate. □

Theorem 4 (Stability of FIRMM for the Caputo-fractional system)**.** *Let the system*

$$ {}^{C}D_t^{\alpha} y_i(t) = f_i\big(t, y_1(t),\dots,y_m(t)\big), \quad y_i(0) = y_{i0}, \quad i = 1,2,\dots,m, $$

be approximated using the FIRMM method, yielding an approximation $\mathbf{y}_N(t)$*. Suppose that a perturbed problem*

$$ {}^{C}D_t^{\alpha} \widetilde{y}_i(t) = \widetilde{f}_i\big(t, \widetilde{y}_1(t),\dots,\widetilde{y}_m(t)\big), \quad \widetilde{y}_i(0) = \widetilde{y}_{i0}, $$

is also solved using FIRMM, yielding $\widetilde{\mathbf{y}}_N(t)$*. Morever, assume that*

1. $\mathbf{f}$ *and* $\widetilde{\mathbf{f}}$ *satisfy a uniform Lipschitz condition:*

$$ \|\mathbf{f}(t,\mathbf{y}) - \mathbf{f}(t,\mathbf{z})\|,\ \|\widetilde{\mathbf{f}}(t,\mathbf{y}) - \widetilde{\mathbf{f}}(t,\mathbf{z})\| \le L\|\mathbf{y}-\mathbf{z}\|. $$

2. *the perturbation satisfies*

$$ \|\mathbf{f}(t,\mathbf{y}) - \widetilde{\mathbf{f}}(t,\mathbf{y})\| \le \|\varepsilon(t)\|, \quad \|\mathbf{y}_0 - \widetilde{\mathbf{y}}_0\| \le \|\Delta\mathbf{y}_0\|. $$

3. *the contractivity constant* $\kappa = L\dfrac{T^{\alpha}}{\Gamma(\alpha+1)} < 1.$

Then the FIRMM approximations satisfy the stability estimate:

$$\|\mathbf{y}_N - \widetilde{\mathbf{y}}_N\|_{L^2_{w_\alpha}(0,T)} \leq \frac{1}{1-\kappa}\left(\|\Delta\mathbf{y}_0\| + \|\varepsilon\|_{L^2_{w_\alpha}(0,T)}\right).$$

Proof. Let $\mathbf{e}_N = \mathbf{y}_N - \tilde{\mathbf{y}}_N$. Then

$$\|\mathbf{e}_N(t)\| \leq |\Delta\mathbf{y}_0| + \frac{L}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\|\mathbf{e}_N(s)\|\,ds + \|\tilde{\varepsilon}_N(t)\|.$$

Vector Grönwall and embedding $L^\infty \to L^2_{w_\alpha}$ give the estimate. □

6 Numerical experiment

Problem 4. *Consider the coupled linear fractional system of order* $\alpha = 0.5$*:*

$$\begin{cases} {}^{C}D_t^{0.5} y_1(t) = 1.329340t, & y_1(0) = 0, \\ {}^{C}D_t^{0.5} y_2(t) = 1.661675t^2, & y_2(0) = 0. \end{cases}$$

Soln. Exact solutions for this system is given by

$$y_1(t) = t^{1.5}, \quad y_2(t) = t^{2.5}.$$

Approximation of the solution of above system by FIRMM is

$$y_{1,3}(t) = \sum_{n=0}^{3} a_{1n}t^{0.5n}, \quad y_{2,3}(t) = \sum_{n=0}^{3} a_{2n}t^{0.5n},$$

with $a_{10} = y_1(0) = 0$, $a_{20} = y_2(0) = 0$.
The residuals for minimizations are defined as:

$$R_i(t) = y_{i,3}(t) - y_i(0) - \frac{1}{\Gamma(0.5)}\int_0^t (t-s)^{-0.5} f_i(s)ds, \quad i = 1,2.$$

For each i, enforcing orthogonality with basis functions $\phi_j(t) = t^{0.5j}$ for $j = 0,1,2,3$:

$$\int_0^2 R_i(t)t^{0.5j}t^{-0.5}dt = 0 \quad i = 1,2.$$

Thus we have

$$\begin{cases} \int_0^2 (a_{11} + a_{12}t^{0.5} + (a_{13}-1)t)dt = 0, \\ \int_0^2 (a_{11}t^{0.5} + a_{12}t + (a_{13}-1)t^{1.5})dt = 0, \\ \int_0^2 (a_{11}t + a_{12}t^{1.5} + (a_{13}-1)t^2)dt = 0, \\ \int_0^2 (a_{11}t^{1.5} + a_{12}t^2 + (a_{13}-1)t^{2.5})dt = 0 \end{cases}$$

and

$$\begin{cases} \int_0^2 (a_{21} + a_{22}t^{0.5} + a_{23}t - t^2)dt = 0, \\ \int_0^2 (a_{21}t^{0.5} + a_{22}t + a_{23}t^{1.5} - t^{2.5})dt = 0, \\ \int_0^2 (a_{21}t + a_{22}t^{1.5} + a_{23}t^2 - t^3)dt = 0, \\ \int_0^2 (a_{21}t^{1.5} + a_{22}t^2 + a_{23}t^{2.5} - t^3)dt = 0. \end{cases}$$

Therefore for $y_1(t)$, the coefficients are

$$a_{11} = 0, \quad a_{12} = 0, \quad a_{13} = 1, \text{ so, } y_{1,3}(t) = t^{1.5}$$

and for $y_2(t)$, the coefficients are

$$a_{21} = \frac{6}{7}, \quad a_{22} = -\frac{20\sqrt{2}}{7}, \quad a_{23} = \frac{30}{7}, \text{ so, } y_{2,3}(t) = \frac{6}{7}t^{0.5} - \frac{20\sqrt{2}}{7}t + \frac{30}{7}t^{1.5}.$$

Table 2: Comparison between FIRMM and the exact solution

t	y_1^{exact}	$y_{1,3}^{\text{FIRMM}}$	Error_1	y_2^{exact}	$y_{2,3}^{\text{FIRMM}}$	Error_2
0	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000
0.2	0.08944	0.08944	0.00000	0.01789	-0.04147	0.05936
0.5	0.35355	0.35355	0.00000	0.17678	0.10102	0.07576
1.0	1.00000	1.00000	0.00000	1.00000	1.10225	0.10225
1.5	1.83712	1.83712	0.00000	2.75568	2.86223	0.10655
2.0	2.82843	2.82843	0.00000	5.65685	5.25279	0.40406

The FIRMM method with $N = 3$ reproduces the $y_1(t)$ but not $y_2(t)$. The same is clearly shown in Figure 5 and in Table 2.

7 Application: fractional glucose decay model

The Caputo derivative of order $\alpha, (0 < \alpha < 1)$ models the memory-dependent decay rate of glucose concentration, such that any previous glucose level has the potential to continue influencing, α indicates the extent of the history dependent transport, which represents delayed insulin action, while the rate constant k indicated the effective metabolic clearance efficiency, making the fractional glucose decay model, particularly good at describing how the body's prior glycemic states modulate current glucose.

Table 3: Parameters for fractional glucose decay model

Parameter	**Value**	**Unit**
G_0 (initial glucose)	137.2	mg/dL
k (decay rate)	0.0067 ± 0.0003	min^{-1}
α (fractional order)	0.86 ± 0.02	–
t (time)	–	min

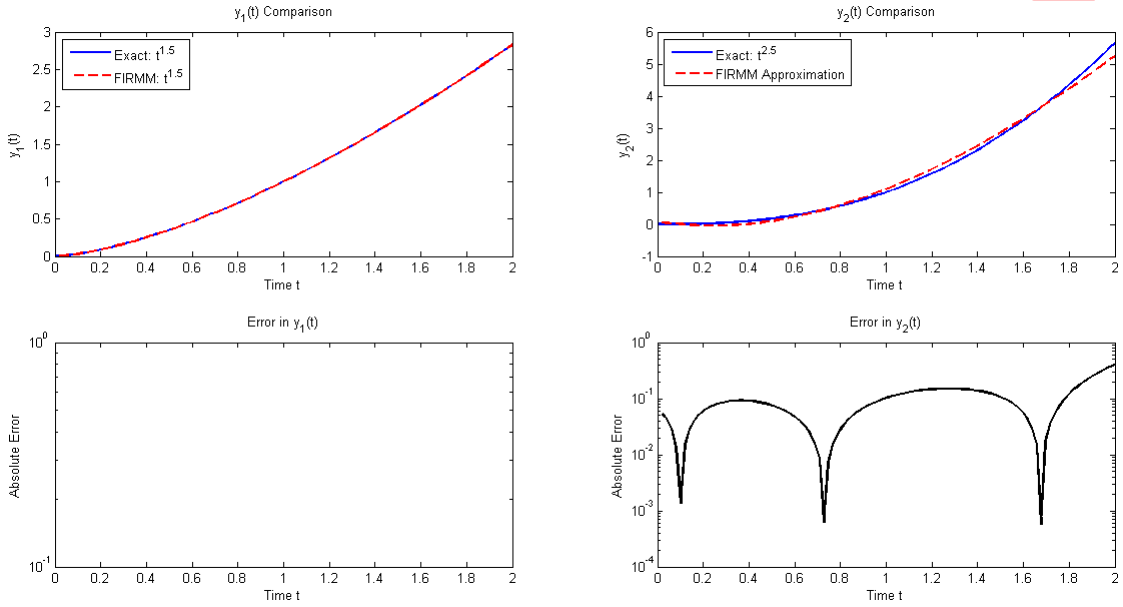

Figure 5: Comparison between FIRMM and exact solution (third order) for the Problem 4

The fractional glucose decay model [18, 24] is given by

$$^{C}D_t^{\alpha}G(t) = -kG(t), \quad G(0) = G_0 = 137.2\,\mathrm{mg/dL},$$

where t is time in minutes, $G(t)$ is the blood glucose concentration (mg/dL), k is the decay rate constant (min^{-1}) and α is the fractional order (memory effects) ($0 < \alpha < 1$). Let the parameters be $k = 0.0067 \pm 0.0003$, and $\alpha = 0.86 \pm 0.02$

Reformulating as a Volterra integral equation and applying FIRMM gives

$$\begin{aligned} G(t) &= G_0 - \frac{k}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} G(s)ds \\ &\approx G_0 \left[1 - \frac{kt^{\alpha}}{\Gamma(\alpha+1)} + \frac{k^2 t^{2\alpha}}{\Gamma(2\alpha+1)} - \frac{k^3 t^{3\alpha}}{\Gamma(3\alpha+1)} + \frac{k^4 t^{4\alpha}}{\Gamma(4\alpha+1)} \right] \end{aligned}$$

with parameters $k = 0.0067\,\mathrm{min}^{-1}$, $\alpha = 0.86$ (see Table 3), $\Gamma(\alpha+1) = \Gamma(1.87) \approx 0.9518$, $\Gamma(2\alpha+1) = \Gamma(2.74) \approx 1.5953$, $\Gamma(3\alpha+1) = \Gamma(3.61) \approx 3.7595$, $\Gamma(3\alpha+1) = \Gamma(4.48) \approx 11.3136$. Therefore

$$G(t) = 137.2 \left[1 - 0.007039t^{0.86} + 0.00002814t^{1.72} - 0.0000000800t^{2.58} + 0.000000000178t^{3.44} \right] \quad (20)$$

Table 4: Comparing FIRMM predictions and clinical data [8] with errors (second order)

Time (min)	Actual Glucose (mg/dL)	FIRMM Prediction (mg/dL)	Abs. Error (mg/dL)	Rel. Error (%)
0	137.20	137.20	0.00	0.00
30	112.70	115.09	2.39	2.12
60	105.60	99.99	5.61	5.31
90	89.40	89.15	0.25	0.28
120	75.60	81.89	6.29	8.32

Table 5: Comparing FIRMM predictions and clinical data [8] with errors (third order)

Time (min)	Actual Glucose (mg/dL)	FIRMM Prediction (mg/dL)	Abs. Error (mg/dL)	Rel. Error (%)
0	137.20	137.20	0.00	0.00
30	112.70	114.91	2.21	1.96
60	105.60	98.81	6.79	6.43
90	89.40	85.62	3.78	4.23
120	75.60	74.21	1.39	1.83

Table 6: Comparing FIRMM predictions and clinical data [8] with errors (fourth order)

Time (min)	Actual Glucose (mg/dL)	FIRMM Prediction (mg/dL)	Abs. Error (mg/dL)	Rel. Error (%)
0	137.20	137.20	0.00	0.00
30	112.70	114.92	2.22	1.97
60	105.60	98.94	6.66	6.31
90	89.40	86.16	3.24	3.63
120	75.60	75.73	0.13	0.17

Here, equation (20) provides an explicit approximation of the glucose decay for 120 minutes from the initial glucose spike $G(0) = 137.2\,\mathrm{mg/dL}$ due to insulin response. In addition, $\alpha = 0.86 \pm 0.02$ shows sub-diffusive transport due to the delayed response of insulin, and $k = 0.0067 \pm 0.0003$ shows the metabolic clearance efficiency with natural variations. Figure 6 shows the comparison of FIRMM predictions and clinical data [8] with errors (Fourth Order).

8 Conclusion

A new numerical method, FIRMM, has been proposed for solving time-fractional differential equations in the Caputo sense. The convergence analysis (Theorem 1) shows that the error in the approximation is controlled by the best approximation error available in the finite-dimensional space, multiplied by a constant that depends on the Lipschitz constant and the length of the time interval. The stability analysis

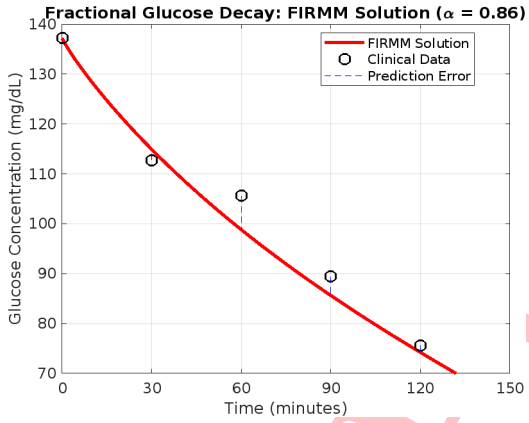

Figure 6: Comparing FIRMM predictions and clinical data [8] with errors (fourth order)

demonstrates that small perturbations in the initial data or the forcing term lead to proportionally small changes in the approximate solution, thereby ensuring that the method is robust.

We then generalised the FIRMM to solve the system of Caputo-type fractional differential equations. The convergence analysis establishes that the FIRMM approximation converges to the exact solution of the given system, with the error bounded by the best approximation under the standard Lipschitz and regularity assumptions. Furthermore, we proved the stability of the method, which guarantees the robustness and suitability of the method for practical applications with some uncertainty.

Numerical experiments on a fractional relaxation equation and a fractional logistic equation demonstrate that the FIRMM method reproduces the leading terms of the exact series solution. The numerical results of other problem shows that the FIRMM exhibits systematic convergence and stable behaviour, with accuracy comparable to VIM at similar truncation levels, thereby clarifying both the strengths and limitations of the proposed approach

On the application side, FIRMM was applied to a fractional glucose decay model, and FIRMM for the system is applied to a fractional SIR epidemic system modelling COVID-19 dynamics, demonstrating both accuracy and versatility. The proposed method provides benefits like simplicity and an inherent ability to handle non-local effects. However, its present form is confined to certain types of FDEs. Future research could explore the extension of this approach to higher-order spatial FDEs.

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Conflict of interest

There is no conflict of interest in this paper.

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