**Journal of Algebra and Related Topics**
Vol. , No. -, 2026, pp. -. Research Article

# On characterization of $J_{\delta_{ss}}$-supplemented modules

**Engin Kaynar** [‡*]

[†] *Vocational School of Technical Sciences, Amasya University, Amasya, Turkey.*
*Email: engin.kaynar@amasya.edu.tr*

**Abstract.** Let an $S$-module $J$ be the ideal of $S$. An $S$-module $X$ is called $J_{\delta_{ss}}$-supplemented provided that there is a direct summand W of X with $X = Y + W$, $Y \cap W \leq Soc_{\delta}(W)$ and $Y \cap W \subseteq WJ$ for each submodule $Y$ of a right module. In this article, the important features of this notion is presented, its comparison with $\oplus_{ss}$-supplemented and $\delta ss$-supplemented modules is given.

*Keywords*: $\oplus_{ss}$-supplemented modules, $\delta_{ss}$-supplemented modules, Semisimple module, Semiperfect ring.
*AMS Subject Classification 2010*: 65F05, 46L05, 11Y50.

## 1 Introduction

All rings mentioned in the text will be associative with an identity element, and S will always represents a ring. I denote the Jacobson radical of $S$ by $J(S)$. Whole modules will be unitial right $S$-modules. Let $X$ be an $S$-module. A submodule $Y$ of $X$ is called $\delta$-small in $X$ denoted by $Y <<_{\delta} X$, if $Y + W \neq X$ for every proper submodule $W$ of $X$ such that $\frac{X}{W}$ is singular.The sum of all $\delta$-small submodules of $X$ is denoted as $\delta(X)$ [11]. Recall from [8] that $X$ is called $\oplus - \delta ss$-supplemented if for each submodule $Y \leq M$, there exists a direct summand $W$ of $X$ such that $Y + W = X$ and $Y \cap W \leq Soc_{\delta}(W)$. Recall from [10] that $\delta_{ss}$-supplemented if each submodule $Y$ of $X$ has a $\delta_{ss}$-supplemented $W$ in $X$ such that $X = Y + W$ and $Y \cap K \subseteq Soc_{\delta}(W)$, where $Soc_{\delta}(W)$ is the sum of $\delta$-small submodules of $W$. Let $Y$ be a submodule of a module $X$. Then $Y \subseteq Soc_{\delta}(M)$ if and only if $Y \subseteq Soc(M)$ and $Y <<_{\delta} X$. A module $X$ is said to be $\delta$-local if $\delta(X) <<_{\delta} X$ and $\delta(X)$ is maximal in [1]. Let an $S$-module $J$ be any ideal of $S$. In [6] A module A is said to be $J_{\delta_{ss}}$- supplemented if for each submodule B of A, there exists a direct summand C of A such that $A = B + C$, $B \cap C \leq JC$, and $B \cap C \leq Soc_{\delta}(C)$.
In section 2 of this text, I define the concept of $J_{\delta_{ss}}$-supplemented $S$-modules, where $J$ is an ideal of $S$. I compare this concept with the notion of $\oplus - \delta ss$-supplemented modules. In this

*Corresponding author
Received: 21 March 2025/ Revised: 20 May 2025/ Accepted: 19 February 2026
DOI: 10.22124/JART.2026.30173.1784

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section I will prove that having semisimple factor modules via $J_{\delta_{ss}}$-supplemented modules for an ideal $J$ of $S$. Then I will characterize indecomposable $J_{\delta_{ss}}$-supplemented modules via an ideal $J$ of $S$.

## 2 $J_{\delta_{ss}}$-supplemented modules

In this section , I present several fundamental properties of these modules.

**Lemma 1.** *Let $X$ be an $S$-module and $J$ be an ideal of $S$. If $Y$ is a direct summand of $X$, then $Y \cap XJ = YJ$.*

*Proof.* By hypothesis there exists a submodule $T$ of $X$ such that $X = Y \oplus T$. Then I obtain $XJ = YJ \oplus TJ$. By applying modular law, I get $Y \cap XJ = Y \cap (YJ \oplus TJ) = YJ \oplus (Y \cap TJ) = YJ$. □

**Proposition 1.** *Let $X$ be an $S$-module and $J$ be an ideal of $S$ such that $Soc_\delta(X) \subseteq XJ$. Then $X$ is $J_{\delta_{ss}}$-supplemented if and only if $X$ is $\oplus - \delta ss$-supplemented.*

*Proof.* ($\Rightarrow$) Clear.
($\Leftarrow$) Let $X$ be an $\oplus{-}\delta ss$-supplemented $S$-module and $J$ be an ideal of $S$ with $Soc_{\delta(X)} \subseteq XJ$. Let $Y$ be a submodule of $X$. Then there exists a direct summnand $W$ of $X$ such that $X = Y + W$, $Y \cap W \subseteq Soc_\delta(W)$. Note that $W \cap XJ = WJ$ by Lemma 1. Since $Soc_\delta(X) \subseteq XJ$, I have $Y \cap W \subseteq Soc_\delta(W) \subseteq W \cap Soc_\delta(W) \subseteq W \cap Soc_\delta(X) \subseteq W \cap XJ = WJ$ by [10, Proposition 3.5(3)]. Thus $X$ is $J_{\delta_{ss}}$-supplemented. □

**Example 1.** Consider the $\mathbb{Z}$-module. $X = \bigoplus_{p\in\Omega} \mathbb{Z}_p$ where $\Omega$ is a subset of the set of whole prime integers. As $X$ is a semisimple $\mathbb{Z}$-module, $X$ is $J_{\delta_{ss}}$-supplemented for any ideal $J$ of $\mathbb{Z}$.

A module $M$ is said to $\oplus - \delta ss$ supplemented module if any submodule of $M$ has a $\delta_{ss}$-supplement, which is a direct summand of $M$. It is clear that every semisimple module is a $\oplus - \delta ss$ supplemented module [8].

**Remark 1.** *Using Proposition 1, I have every $\oplus - \delta ss$-supplemented $S$-module is $S - \oplus - \delta ss$-supplemented.*

**Theorem 1.** *Let $J$ be an ideal of $S$ and $X$ be an $S$-module. If there is a submodule $W$ of $X$ with $X = Y + W$ and $Y \cap W \subseteq WJ$ for each submodule $Y$ of $X$. Then $\frac{X}{XJ}$ is semisimple.*

*Proof.* Let $Y$ be a submodule of $X$ such that $XJ \subseteq Y$. Then $\frac{Y}{XJ} \leq \frac{X}{XJ}$. By hypothesis, there is a submodule $W$ of $X$ such that $X = Y + W$ and $Y \cap W \subseteq WJ$. It follows that $\frac{X}{XJ} \leq \frac{Y}{XJ} + \frac{W+XJ}{XJ}$. Then I have $\frac{Y}{XJ} \cap \frac{W+XJ}{XJ} = \frac{Y\cap(W+XJ)}{XJ} = \frac{(Y\cap W)+XJ}{XJ} \subseteq \frac{WJ+XJ}{XJ} = \frac{XJ}{XJ}$ by modular law. Therefore, $\frac{Y}{XJ}$ is a direct summand of $\frac{X}{XJ}$. So, $\frac{X}{XJ}$ is semisimple. □

The following Corollary follows from Theorem 1.

**Lemma 2.** *Let $X$ be an $\oplus - \delta ss$-supplemented $S$-module. Then there exists a decomposition $X = Y \oplus W$ with $Soc_\delta(Y) <<_\delta Y$ and $Soc_\delta(W) = W$ for some $Y$,$W$ submodule of $X$.*

*Proof.* By hypothesis, there are submodules $Y$ and $W$ of $X$ with $X = Y \oplus W$, $Soc_\delta(X)+Y = X$ and $Soc_\delta(X) \cap Y <<_\delta Y$. Since $X = Y \oplus W$, then $Soc_\delta(X) = Soc_\delta(Y) \oplus Soc_\delta(W)$ by [10, Proposition 3.1(5)]. It follows from [10, Proposition 3.1(2)] that $Soc_\delta(Y) <<_\delta Y$. $Soc_\delta(X)+Y = Soc_\delta(Y) \oplus Soc_\delta(W) + Y$ Then I have $Y \oplus Soc_\delta(W) = X$ and $(Y \cap Soc_\delta(X)) \oplus (Soc_\delta(W) \cap Soc_\delta(X)) = X \cap Soc_\delta(X)$. Then I have $(Soc_\delta(X) \cap Y) \oplus Soc_\delta(W) = Soc_\delta(X)$. Therefore $Soc_\delta(W) = W$. □

**Theorem 2.** *Let $X$ be a $J_{\delta_{ss}}$-supplemented $S$-module. Then there is a decomposition $X = Y \oplus W$ with $Soc_\delta(Y) \leq YJ$, $Soc_S(Y) <<_\delta (Y)$ and $Soc_\delta(W) = W$.*

*Proof.* Since $X$ is a $J - \oplus - \delta ss$-supplemented, $Soc_\delta(X)$ has a $\delta_{ss}$-supplement of $Y$ in $X$. So, it is obtained that $Soc_\delta(X) \cap Y \subseteq YJ$ by using Lemma 2 and Proposition 1. □

**Corollary 1.** *Let $X$ be a $J - \oplus - \delta ss$-supplemented $S$-module. Then $Soc_\delta(X) \leq XI$.*

*Proof.* Let $X$ be a $J - \oplus - \delta ss$-supplemented $S$-module. By considering the trivial decomposition $X = X \oplus 0$. It follows from Theorem 2 that $Soc_\delta(X) \leq XJ$. □

Recall from [10] that a module $X$ is said to be strongly $\delta$-local if it is $\delta$-local, and satisfies the condition $\delta(X) \subseteq Soc(X)$. It is clear that every strongly $\delta$-local module is $\delta$-local. However, the converse does not always hold. Recall from [5] that a module $X$ is called ss-supplemented if every submodule $Y$ of $X$ has a supplement $W$ in $X$ with $Y \cap W$ being semisimple. A submodule $Y$ of $X$ is said to have ample ss-supplements in $X$ if each submodule $W$ of $X$ such that $X = Y + W$ contains an $ss$-supplement of $Y$ in $X$. If every submodule of $X$ has amply ss-supplements in $X$, the module $X$ is said to be amply ss-supplemented.

**Proposition 2.** *Let $Y$ be a semisimple submodule of $X$.Then $Y << X$ if and only if $Y \subseteq Soc_\delta(X)$.*

*Proof.* Demonstrating the sufficient condition is sufficient to complete the proof. Let $X = Y + W$ for a submodule $W$ of $X$. Since $Y <<_\delta X$, there is a productive semisimple submodule $Z$ of $Y$ such that $X = Z \oplus W$. By [10, Corollary 2.3], it follows that $\delta(Z) = Z$ and $Z$ is semisimple.Suppose that $Z$ is a non-zero submodule of $X$. Then there exists a simple direct summand $T$ of $Z$. Since $Y \subseteq Soc_\delta(X)$, then $T \subseteq Rad(X)$. Therefore $T << X$. Thus $T = 0$. It contradicts with simplicity of $T$. Hence $Z = 0$. I obtain that $Y << X$. It is $Y \subseteq Soc(X)$ in the given [5, Lemma 2]. So $Y$ is semisimple. □

**Corollary 2.** *Let $X$ be a module and $Y$ is a submodule of $X$ with $Soc_s(X) = X$. Then $Y <<_\delta X$ if and only if $Y << X$.*

**Proposition 3.** *Let $X$ is a module with $Soc_s(X) = X$. Then $X$ is $\oplus - \delta_{ss}$-supplemented if and only if $X$ is $\oplus_{ss}$-supplemented.*

*Proof.* Since $Soc_s(X) = X$, $Soc_s(T) = T$ for each direct summnad $T$ of $X$. So the proof follows from Corollary 2. □

**Theorem 3.** *Let $J$ be an ideal of $S$ and $X$ be a $J_{\delta_{ss}}$-supplemented module with $XJ \leq Soc_\delta(X)$. Then $X$ is an $\oplus_{ss}$-supplemented module.*

*Proof.* Let $Y$ be a submodule of $X$. Then there exists a direct summand $W$ of $X$ such that $X = Y + W$, $Y \cap W \subseteq WJ$, and $Y \cap W \subseteq Soc_\delta(W)$. Since $XJ \leq Soc_\delta(X)$, I get $WJ = W \cap XJ \leq W \cap Soc_\delta(X) = Soc_\delta(W)$ by Lemma 1 and [10, Proposition 3.1.(1)]. It follows from Proposition 2 that $Y \cap W << W$ with $Y \cap W$ is semisimple, So $X$ is $\oplus_{ss}$-supplemented. □

**Proposition 4.** *Let $J$ be an ideal of $S$ and let $X$ be a $\oplus_{ss}$-supplemented $S$-module with $Soc_\delta(X) \subseteq XJ$. Then $X$ is $J_{\delta_{ss}}$- supplemented.*

*Proof.* Let $Y$ be a submodule of $X$. In this case, there is a direct summand $W$ of $X$ such that $X = Y + W$, $Y \cap W << W$, and $Y \cap W$ is semisimple. Thus $Y \cap W \subseteq Soc_\delta(W)$. It follows from Lemma 1 that $WJ = W \cap XJ$. Since $Soc_\delta(X) \leq XJ$, then I have $Soc_\delta(W) \leq W \cap Soc_\delta(X) \leq W \cap XJ = WJ$. So $Y \cap W \leq WJ$. Thus $X$ is $J_{\delta_{ss}}$-supplemented. □

**Corollary 3.** *Let $X$ be a $\oplus_{ss}$ -supplemented $S$-module with $XJ = X$ for each ideal $J$ of $S$. Then $X$ is $J_{\delta_{ss}}$-supplemented.*

The following Corollary follows from Proposition 4 and [2, Lemma 3]

**Corollary 4.** *Let $J'$ be a maximal ideal of a commutative ring $S$ and let $X$ be an $\oplus_{ss}$-supplemented $S$-module. If $XJ = XJ'$ for some ideal $J$ of $S$, Then $X$ is $J_{\delta_{ss}}$-supplemented.*

Recall from [4] that an $S$-module $X$ is said to be divisible over a commutative integral domain $S$ provided that $Xa = X$ for each non-zero element a of $S$. The following Corollary is obtained directly by Corollary 3.

**Corollary 5.** *Let $X$ be a divisible $S$-module over a commutative intregral domains $S$. If $X$ is $\oplus_{ss}$-supplemented, then $X$ is $J_{\delta_{ss}}$-supplemented for each non-zero ideal $J$ os $S$.*

By using Lemma1 and [8, Theorem 2.1] I can prove the following Theorem directly.

**Theorem 4.** *Let $J$ be an ideal of $S$. Then any finite direct sum of $J_{\delta_{ss}}$-supplemented $S$-modules is also $J_{\delta_{ss}}$-supplemented.*

It follows from [7] that an $S$-module $X$ is said to be duo provided if every submodule $Y$ of $X$ is fully invariant, that is, $\Psi(Y) \leq Y$ for each $S$- endomorphism $\Psi$ of $X$.

**Proposition 5.** *Let $J$ be an ideal of $S$ and $X = \bigoplus_{\alpha\in\Theta} X_\alpha$ duo $S$-module where $X_\alpha$ is $J_{\delta_{ss}}$-supplemented for each $\alpha \in \Theta$, then $X$ is $J_{\delta_{ss}}$-supplemented.*

*Proof.* Let $Y$ be a submodule of $X$ . It follows from [7, Lemma 2.1] that $Y = \bigoplus_{\alpha\in\Theta}(Y \cap X_\alpha)$. By hypothesis, there is a direct summand $W_\alpha$ of $X_\alpha$ such that $X_\alpha = (Y \cap X_\alpha) + W_\alpha$, $(Y \cap X_\alpha) \cap W_\alpha \leq W_\alpha J$, and $(Y \cap X_\alpha) \cap W_\alpha \leq Soc_\delta(W_\alpha)$. Say, $W = \bigoplus_{\alpha\in\Theta} W_\alpha$. Since $W$ is a direct summand of $X$, and $X = Y + W$. I have $Y \cap W = \bigoplus_{\alpha\in\Theta}(Y \cap W_\alpha) \leq WJ$, and $Y \cap W \leq Soc_\delta(W)$ by [9, Proposition 2.8]. □

In [6], quotient properties of $J_{\delta_{ss}}$-supplemented modules is given at Proppposition 2.28. Now, I give an example that the $J_{\delta_{ss}}$-supplemented modules does not have the same feature for each submodule of the $J_{\delta_{ss}}$-supplemented module.
The group consisting of all bijective permutations of a set with n elements is called a symmetric group. It is denoted by $S_{(n)}$.

**Example 2.** Let $K$ be a field. Suppose the local ring $S = \frac{K[x^2,x^3]}{(x^4)}$, and let $J$ denotes the maximal ideal of $S$. Consider the module $X = S_{(n)}$, where n is an integer with $n \geq 2$. Using Theorem 4 and [9, Proposition 3.7], it can be seen that $X$ is a $J_{\delta_{ss}}$ supplemented module. Since $S$ is an artinian local ring but not prime ideal ring, there exists a submodule $Y$ of $X$ such that $\frac{X}{Y}$ is not $\oplus$-supplemented by [3, Example 2.2]. Consequently, by [9, Corollary 3.16], $\frac{X}{Y}$ is not $J-\oplus$-supplemented.Therefore, $\frac{X}{Y}$ is not $J_{\delta_{ss}}$ supplemented.

**Proposition 6.** *Let $X$ be an $S$-module and $J$ an ideal of $S$. Assume that $Y$ is a fully invariant direct summand of $X$. Then the following statements are equivalent to each other.*
*(1) $X$ is $J_{\delta_{ss}}$-supplemented.*
*(2) Both $Y$ and $\frac{X}{Y}$ are $J_{\delta_{ss}}$-supplemented.*

*Proof.* (1)⇒(2) clear by , [6, proposition 2.28(2)]
(2)⇒(1) Clear by Theorem 4. □

## Acknowledgments

The authors would like to thank the referee for careful reading.

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