Journal of Algebra and Related Topics
Vol. 14, No. 2, 2026, pp. 73-78. Research Article

On characterization of $J_{\delta_{ss}}$-supplemented modules

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Abstract. Let an S-module J be the ideal of S. An S-module X is called $J_{\delta_{ss}}$-supplemented provided that there is a direct summand W of X with $X = Y + W$, $Y \cap W \leq Soc_{\delta}(W)$ and $Y \cap W \subseteq WJ$ for each submodule Y of a right module. In this article, the important features of this notion is presented, its comparison with $\oplus_{ss}$-supplemented and δss-supplemented modules is given.

Keywords: $\oplus_{ss}$-supplemented modules, δ_{ss}-supplemented modules, Semisimple module, Semiperfect ring.
AMS Subject Classification 2010: 65F05, 46L05, 11Y50.

1 Introduction

All rings mentioned in the text will be associative with an identity element, and S will always represents a ring. I denote the Jacobson radical of S by $J(S)$. Whole modules will be unitial right S-modules. Let X be an S-module. A submodule Y of X is called δ-small in X denoted by $Y <<_{\delta} X$, if $Y + W \neq X$ for every proper submodule W of X such that $\frac{X}{W}$ is singular.The sum of all δ-small submodules of X is denoted as $\delta(X)$ [11]. Recall from [9] that X is called $\oplus - \delta ss$-supplemented if for each submodule $Y \leq M$, there exists a direct summand W of X such that $Y + W = X$ and $Y \cap W \leq Soc_{\delta}(W)$. Recall from [6] that δ_{ss}-supplemented if each submodule Y of X has a δ_{ss}-supplemented W in X such that $X = Y + W$ and $Y \cap K \subseteq Soc_{\delta}(W)$, where $Soc_{\delta}(W)$ is the sum of δ-small submodules of W. Let Y be a submodule of a module X. Then $Y \subseteq Soc_{\delta}(M)$ if and only if $Y \subseteq Soc(M)$ and $Y <<_{\delta} X$. A module X is said to be δ-local if $\delta(X) <<_{\delta} X$ and $\delta(X)$ is maximal in [1]. Let an S-module J be any ideal of S. In [7] A module A is said to be $J_{\delta_{ss}}$- supplemented if for each submodule B of A, there exists a direct summand C of A such that $A = B + C$, $B \cap C \leq JC$, and $B \cap C \leq Soc_{\delta}(C)$.
In section 2 of this text, I define the concept of $J_{\delta_{ss}}$-supplemented S-modules, where J is an ideal of S. I compare this concept with the notion of $\oplus - \delta ss$-supplemented modules. In this

*Corresponding author
Received: 21 March 2025/ Revised: 20 May 2025/ Accepted: 19 February 2026
DOI: 10.22124/JART.2026.30173.1784

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section I will prove that having semisimple factor modules via $J_{\delta_{ss}}$-supplemented modules for an ideal J of S. Then I will characterize indecomposable $J_{\delta_{ss}}$-supplemented modules via an ideal J of S.

2 $J_{\delta_{ss}}$-supplemented modules

In this section , I present several fundamental properties of these modules.

Lemma 1. *Let X be an S-module and J be an ideal of S. If Y is a direct summand of X, then $Y \cap XJ = YJ$.*

Proof. By hypothesis there exists a submodule T of X such that $X = Y \oplus T$. Then I obtain $XJ = YJ \oplus TJ$. By applying modular law, I get $Y \cap XJ = Y \cap (YJ \oplus TJ) = YJ \oplus (Y \cap TJ) = YJ$. □

Proposition 1. *Let X be an S-module and J be an ideal of S such that $Soc_\delta(X) \subseteq XJ$. Then X is $J_{\delta_{ss}}$-supplemented if and only if X is $\oplus - \delta ss$-supplemented.*

Proof. ($\Rightarrow$) Clear.
($\Leftarrow$) Let X be an $\oplus-\delta ss$-supplemented S-module and J be an ideal of S with $Soc_{\delta(X)} \subseteq XJ$. Let Y be a submodule of X. Then there exists a direct summnand W of X such that $X = Y + W$, $Y \cap W \subseteq Soc_\delta(W)$. Note that $W \cap XJ = WJ$ by Lemma 1. Since $Soc_\delta(X) \subseteq XJ$, I have $Y \cap W \subseteq Soc_\delta(W) \subseteq W \cap Soc_\delta(W) \subseteq W \cap Soc_\delta(X) \subseteq W \cap XJ = WJ$ by [6, Proposition 3.5(3)]. Thus X is $J_{\delta_{ss}}$-supplemented. □

Example 1. Consider the $\mathbb{Z}$-module. $X = \bigoplus_{p\in\Omega} \mathbb{Z}_p$ where Ω is a subset of the set of whole prime integers. As X is a semisimple $\mathbb{Z}$-module, X is $J_{\delta_{ss}}$-supplemented for any ideal J of $\mathbb{Z}$.

A module M is said to $\oplus - \delta ss$ supplemented module if any submodule of M has a δ_{ss}-supplement, which is a direct summand of M. It is clear that every semisimple module is a $\oplus - \delta ss$ supplemented module [9].

Remark 1. *Using Proposition 1, I have every $\oplus - \delta ss$-supplemented S-module is $S - \oplus - \delta ss$-supplemented.*

Theorem 1. *Let J be an ideal of S and X be an S-module. If there is a submodule W of X with $X = Y + W$ and $Y \cap W \subseteq WJ$ for each submodule Y of X. Then $\frac{X}{XJ}$ is semisimple.*

Proof. Let Y be a submodule of X such that $XJ \subseteq Y$. Then $\frac{Y}{XJ} \leq \frac{X}{XJ}$. By hypothesis, there is a submodule W of X such that $X = Y + W$ and $Y \cap W \subseteq WJ$. It follows that $\frac{X}{XJ} \leq \frac{Y}{XJ} + \frac{W+XJ}{XJ}$. Then I have $\frac{Y}{XJ} \cap \frac{W+XJ}{XJ} = \frac{Y\cap(W+XJ)}{XJ} = \frac{(Y\cap W)+XJ}{XJ} \subseteq \frac{WJ+XJ}{XJ} = \frac{XJ}{XJ}$ by modular law. Therefore, $\frac{Y}{XJ}$ is a direct summand of $\frac{X}{XJ}$. So, $\frac{X}{XJ}$ is semisimple. □

Lemma 2. *Let X be an $\oplus - \delta ss$-supplemented S-module. Then there exists a decomposition $X = Y \oplus W$ with $Soc_\delta(Y) <<_\delta Y$ and $Soc_\delta(W) = W$ for some Y,W submodule of X.*

Proof. By hypothesis, there are submodules Y and W of X with $X = Y \oplus W$, $Soc_\delta(X)+Y = X$ and $Soc_\delta(X) \cap Y <<_\delta Y$. Since $X = Y \oplus W$, then $Soc_\delta(X) = Soc_\delta(Y) \oplus Soc_\delta(W)$ by [6, Proposition 3.1(5)]. It follows from [6, Proposition 3.1(2)] that $Soc_\delta(Y) <<_\delta Y$. $Soc_\delta(X)+Y = Soc_\delta(Y) \oplus Soc_\delta(W) + Y$ Then I have $Y \oplus Soc_\delta(W) = X$ and $(Y \cap Soc_\delta(X)) \oplus (Soc_\delta(W) \cap Soc_\delta(X)) = X \cap Soc_\delta(X)$. Then I have $(Soc_\delta(X) \cap Y) \oplus Soc_\delta(W) = Soc_\delta(X)$. Therefore $Soc_\delta(W) = W$. □

Theorem 2. *Let X be a $J_{\delta_{ss}}$-supplemented S-module. Then there is a decomposition $X = Y \oplus W$ with $Soc_\delta(Y) \leq YJ$, $Soc_S(Y) <<_\delta (Y)$ and $Soc_\delta(W) = W$.*

Proof. Since X is a $J-\oplus-\delta ss$-supplemented, $Soc_\delta(X)$ has a δ_{ss}-supplement of Y in X. So, it is obtained that $Soc_\delta(X) \cap Y \subseteq YJ$ by using Lemma 2 and Proposition 1. □

Corollary 1. *Let X be a $J-\oplus-\delta ss$-supplemented S-module. Then $Soc_\delta(X) \leq XI$.*

Proof. Let X be a $J-\oplus-\delta ss$-supplemented S-module. By considering the trivial decomposition $X = X \oplus 0$. It follows from Theorem 2 that $Soc_\delta(X) \leq XJ$. □

Recall from [6] that a module X is said to be strongly δ-local if it is δ-local, and satisfies the condition $\delta(X) \subseteq Soc(X)$. It is clear that every strongly δ-local module is δ-local. However, the converse does not always hold. Recall from [5] that a module X is called ss-supplemented if every submodule Y of X has a supplement W in X with $Y \cap W$ being semisimple. A submodule Y of X is said to have ample ss-supplements in X if each submodule W of X such that $X = Y + W$ contains an ss-supplement of Y in X. If every submodule of X has amply ss-supplements in X, the module X is said to be amply ss-supplemented.

Proposition 2. *Let Y be a semisimple submodule of X.Then $Y << X$ if and only if $Y \subseteqq Soc_\delta(X)$.*

Proof. Demonstrating the sufficient condition is sufficient to complete the proof. Let $X = Y + W$ for a submodule W of X. Since $Y <<_\delta X$, there is a productive semisimple submodule Z of Y such that $X = Z \oplus W$. By [6, Corollary 2.3], it follows that $\delta(Z) = Z$ and Z is semisimple.Suppose that Z is a non-zero submodule of X. Then there exists a simple direct summand T of Z. Since $Y \subseteq Soc_\delta(X)$, then $T \subseteq Rad(X)$. Therefore $T << X$. Thus $T = 0$. It contradicts with simplicity of T. Hence $Z = 0$. I obtain that $Y << X$. It is $Y \subseteqq Soc(X)$ in the given [5, Lemma 2]. So Y is semisimple. □

Corollary 2. *Let X be a module and Y is a submodule of X with $Soc_s(X) = X$. Then $Y <<_\delta X$ if and only if $Y << X$.*

Proposition 3. *Let X is a module with $Soc_s(X) = X$. Then X is $\oplus-\delta_{ss}$-supplemented if and only if X is $\oplus_{ss}$-supplemented.*

Proof. Since $Soc_s(X) = X$, $Soc_s(T) = T$ for each direct summnad T of X. So the proof follows from Corollary 2. □

Theorem 3. *Let J be an ideal of S and X be a $J_{\delta_{ss}}$-supplemented module with $XJ \leq Soc_\delta(X)$. Then X is an $\oplus_{ss}$-supplemented module.*

Proof. Let Y be a submodule of X. Then there exists a direct summand W of X such that $X = Y + W$, $Y \cap W \subseteq WJ$, and $Y \cap W \subseteq Soc_\delta(W)$. Since $XJ \leq Soc_\delta(X)$, I get $WJ = W \cap XJ \leq W \cap Soc_\delta(X) = Soc_\delta(W)$ by Lemma 1 and [6, Proposition 3.1.(1)]. It follows from Proposition 2 that $Y \cap W << W$ with $Y \cap W$ is semisimple, So X is $\oplus_{ss}$-supplemented. □

Proposition 4. *Let J be an ideal of S and let X be a $\oplus_{ss}$-supplemented S-module with $Soc_\delta(X) \subseteq XJ$. Then X is $J_{\delta_{ss}}$- supplemented.*

Proof. Let Y be a submodule of X. In this case, there is a direct summand W of X such that $X = Y + W$, $Y \cap W << W$, and $Y \cap W$ is semisimple. Thus $Y \cap W \subseteq Soc_\delta(W)$. It follows from Lemma 1 that $WJ = W \cap XJ$. Since $Soc_\delta(X) \leq XJ$, then I have $Soc_\delta(W) \leq W \cap Soc_\delta(X) \leq W \cap XJ = WJ$. So $Y \cap W \leq WJ$. Thus X is $J_{\delta_{ss}}$-supplemented. □

Corollary 3. *Let X be a $\oplus_{ss}$ -supplemented S-module with $XJ = X$ for each ideal J of S. Then X is $J_{\delta_{ss}}$-supplemented.*

The following Corollary follows from Proposition 4 and [2, Lemma 3]

Corollary 4. *Let J' be a maximal ideal of a commutative ring S and let X be an $\oplus_{ss}$-supplemented S-module. If $XJ = XJ'$ for some ideal J of S, Then X is $J_{\delta_{ss}}$-supplemented.*

Recall from [4] that an S-module X is said to be divisible over a commutative integral domain S provided that $Xa = X$ for each non-zero element a of S. The following Corollary is obtained directly by Corollary 3.

Corollary 5. *Let X be a divisible S-module over a commutative intregral domains S. If X is $\oplus_{ss}$-supplemented, then X is $J_{\delta_{ss}}$-supplemented for each non-zero ideal J os S.*

By using Lemma1 and [9, Theorem 2.1] I can prove the following corollary directly.

Corollary 6. *Let J be an ideal of S. Then any finite direct sum of $J_{\delta_{ss}}$-supplemented S-modules is also $J_{\delta_{ss}}$-supplemented.*

It follows from [8] that an S-module X is said to be duo provided if every submodule Y of X is fully invariant, that is, $\Psi(Y) \leq Y$ for each S- endomorphism Ψ of X.

Proposition 5. *Let J be an ideal of S and $X = \bigoplus_{\alpha\in\Theta} X_\alpha$ duo S-module where X_α is $J_{\delta_{ss}}$-supplemented for each $\alpha \in \Theta$, then X is $J_{\delta_{ss}}$-supplemented.*

Proof. Let Y be a submodule of X . It follows from [8, Lemma 2.1] that $Y = \bigoplus_{\alpha\in\Theta}(Y \cap X_\alpha)$. By hypothesis, there is a direct summand W_α of X_α such that $X_\alpha = (Y \cap X_\alpha) + W_\alpha$, $(Y \cap X_\alpha) \cap W_\alpha \leq W_\alpha J$, and $(Y \cap X_\alpha) \cap W_\alpha \leq Soc_\delta(W_\alpha)$. Say, $W = \bigoplus_{\alpha\in\Theta} W_\alpha$. Since W is a direct summand of X, and $X = Y + W$. I have $Y \cap W = \bigoplus_{\alpha\in\Theta}(Y \cap W_\alpha) \leq WJ$, and $Y \cap W \leq Soc_\delta(W)$ by [10, Proposition 2.8]. □

In [7], quotient properties of $J_{\delta_{ss}}$-supplemented modules is given at Propposition 2.28. Now, I give an example that the $J_{\delta_{ss}}$-supplemented modules does not have the same feature for each submodule of the $J_{\delta_{ss}}$-supplemented module.
The group consisting of all bijective permutations of a set with n elements is called a symmetric group. It is denoted by $S_{(n)}$.

Example 2. Let K be a field. Suppose the local ring $S = \frac{K[x^2,x^3]}{(x^4)}$, and let J denotes the maximal ideal of S. Consider the module $X = S_{(n)}$, where n is an integer with $n \geq 2$. Using Theorem 6 and [10, Proposition 3.7], it can be seen that X is a $J_{\delta_{ss}}$ supplemented module. Since S is an artinian local ring but not prime ideal ring, there exists a submodule Y of X such that $\frac{X}{Y}$ is not $\oplus$-supplemented by [3, Example 2.2]. Consequently, by [10, Corollary 3.16], $\frac{X}{Y}$ is not $J-\oplus$-supplemented.Therefore, $\frac{X}{Y}$ is not $J_{\delta_{ss}}$ supplemented.

Proposition 6. *Let X be an S-module and J an ideal of S. Assume that Y is a fully invariant direct summand of X. Then the following statements are equivalent to each other.*
(1) X is $J_{\delta_{ss}}$-supplemented.
(2) Both Y and $\frac{X}{Y}$ are $J_{\delta_{ss}}$-supplemented.

Proof. (1)⇒(2) clear by , [7, proposition 2.28(2)]
(2)⇒(1) Clear by Theorem 6. □

Acknowledgments

The authors would like to thank the referee for careful reading.

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