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Saturated and absolutely closed posemigroups

Shabir A. Ahanger[†*], **Shabir A. Mir**[‡], **Ashfaq Hussian Bhat**[§]

[†]*Department of Mathematics, Central University of Kashmir, Ganderbal, India*
[‡]*Desh Bhagat University, Punjab, India*
[§]*Goverment Degree College Shopian, Jammu and Kashmir, India*
Emails: shabir@cukashmir.ac.in, theshabir2@gmail.com, ashfaqbhat2016@gmail.com@gmail.com

Abstract. We show that commutative inverse posemigroups and finite monogenic posemigroups are saturated in the category of all posemigroups. Further, we show that the variety of pobands satisfying the identity $axya = ayxa$ is closed as well as saturated. Finally, we show that the convex finite monogenic posemigroups and inverse posemigroups are absolutely closed in the category of all commutative posemigroups.

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1 Introduction and summary

Epimorphism is a generalization of a surjective morphism in any concrete category. Since the inclusion map $i : (\mathbb{N}, +) \to (\mathbb{Z}, +)$ is an epimorphism in the category of semigroups, but obviously is not surjective; therefore, the opposite is not true in general. Also, a morphism is epimorphism in the category of posemigroups if it is so in the category of semigroups. Therefore, the above inclusion map i is also a non-surjective epimorphism in posemigroups, and thus reverse is also not true in posemigroups. The main tool used to study epimorphisms in semigroups and posemigroups is zigzag equations and inequalities, respectively. These are a useful characterizations of dominions in these categories. This approach was initiated in semigroups by Howie and Isbell [6] and later by Khan and Higgins (see [7] and references contained therein). S. Nasir and L. Tart [9] initiated investigating epimorphisms and dominions in the category of posemigroups, and they provided zigzag inequalities. Recently, S. A. Ahanger, A. H. Shah, and N. M. Khan (see [2], [3] and [1]) extensively applied the techniques of zigzag inequalities to the

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study of epimorphisms, dominions and varieties of posemigroups, and extended many classical results in the category of posemigroups. The present paper extends this work.

In the category of all semigroups, Howie and Isbell [6], demonstrated that inverse semigroups and finite monogenic semigroups are absolutely closed. As a result, inverse semigroups and finite monogenic semigroups are particularly saturated in the category of all semigroups. Thus, whether the same is true for inverse posemigroups and finite monogenic posemigroups is of relevance, and this subject is addressed in section 3.

Scheiblich [8] and Higgins [5] have demonstrated that the variety of all bands meeting the identity $axya = ayxa$ is closed and saturated, respectively. In section 4, we extend these findings to a wider variety of pobands.

Finally, in section 5, we show that the convex finite monogenic posemigroups and inverse posemigroups are absolutely closed in the category of all commutative posemigroups. This partially generalizes the results of Howie and Isbell [6], from semigroups to posemigroups.

2 Preliminaries

If B is a subsemigroup of a semigroup A, then we say that A is an oversemigroup of a semgroup B. By a partially ordered semigroup, briefly a posemigroup we mean a pair $(A, \leq)$ where A is a semigroup and $\leq$ is a compatible partial order on A, i.e., for all $a, b, a', b' \in A$ with $a \leq b$ and $a' \leq b'$ implies $aa' \leq bb'$. A posemigroup A is said to be an overposemigroup of B if A is an oversemigroup of B and $\leq_B \; = \; \leq_A \cap (B \times B)$. The most natural example of a posemigroup is $(\mathbb{N}, \; \leq)$, where $\mathbb{N}$ is a multiplicative semigroup of positive integers and $\leq$ is the usual natural order on $\mathbb{N}$. In the remainder of this paper, an arbitrary posemigroup shall be denoted by A without explicit mention. A posemigroup morphism h between the posemigroups A and C is a semigroup morphism which also respects the order, which means $a \leq b$ in A implies $h(a) \leq h(b)$ in C. For example, $h : (\mathbb{N}, \; \leq) \to (\mathbb{N}, \; \leq)$ defined by

$$h(m) = m^2$$

is a posemigroup morphism.

A morphism $h : A \to B$ in a category $\mathcal{C}$ is said be an *epimorphism* if for all objects C in $\mathcal{C}$ and any pair of morphisms $h_1, \; h_2 : B \to C$ in $\mathcal{C}$, whenever $h_1 \circ h = h_2 \circ h$,we have $h_1 = h_2$. Without mentioning further, we will refer an epimorphism as epi.

Example 1. Let $A = (\mathbb{N}, \; \leq)$ and $B = (\mathbb{Q}^+, \; \leq)$ be the multiplicative posemigroup of positive integers and positive rational numbers with the usual natural order $\leq$. Then B is an overposemigroup of A, and clearly the inclusion map $i : A \to B$ is a posemigroup epi.

Let A be an overposemigroup of B, and $x \in A$. We say that x is dominated by B if for any pair of morphisms α_1, α_2 in posemigroups from A into a posemigroup C, $\alpha_1|_B = \alpha_2|_B$ implies $\alpha_1(x) = \alpha_2(x)$. $\widehat{Dom}(B, A)$ represents the set of all elements of A that are dominated by B, and is known as the *posemigroup dominion* of B in A. It is an overposemigroup of B and is contained in A. Also, it is obviously verified that a posemigroup morphism $\alpha : A \to C$ is epi if and only if $\widehat{Dom}(\alpha A, \; C) = C$, and the embedding $i : B \to A$ is epi if and only if $\widehat{Dom}(B \; , A) = A$. If A is an oversemigroup of a semigroup B then the semigroup dominion of B in A is defined in a similar way as posemigroup dominion and is denoted by $Dom(B, A)$. Consider the posemigroups A and B with A as an overposemigroup of B then, one can always treat A and B as semigroups by disregarding the order and consider $Dom(B, A)$ in the category of semigroups and we call it as the algebraic dominion of B in A. It is simple to verify that $Dom(B, A) \subseteq \widehat{Dom}(B, A)$. Conversely, if A and B as semigroups with A as an oversemigroup of B then treating A and B as posemigroups with trivial order $(=)$, it is evident that $Dom(B, A) \subseteq \widehat{Dom}(B, A)$. Hence, if B and A are semigroups (posemigroups) with A as an oversemigroup (overposemigroup) of B then

$$B \subseteq Dom(B, A) \subseteq \widehat{Dom}(B, A) \subseteq A. \tag{1}$$

Let A be an overposemigroup of B. We say that B is *closed* in A if $\widehat{Dom}(B, A) = B$. If $\widehat{Dom}(B, A) \neq A$ for every proper overposemigroup A of B, then we say that B is *saturated*. By inclusion (1), it is simple to verify that if a posemigroup B is saturated as a posemigroup, then it is so as a semigroup. If each member of a family of posemigroups $\mathcal{S}$ is saturated, then we say that $\mathcal{S}$ is saturated. If $\mathcal{S}$ is saturated, and closed under morphic images, then one can easily verify that every epi is surjective from any member of $\mathcal{S}$. Therefore, if a variety of posemigroups is saturated then epis in such a variety are surjective.

Isbell [7, Theorem 8.3.5] gave the classification of dominion elements in semigroups, known as Isbell's Zigzag Theorem. The analogue of this result in posemigroups is established by Sohail and Tart [9], known as the Zigzag Theorem for posemigroups. We shall mainly use this theorem as a tool to prove our results.

Theorem 1. *[[9], Theorem 5] Let A be an overposemigroup of a posemigroup B and $z \in A$. Then $z \in \widehat{Dom}(B, A)$ if and only if $z \in B$ or*

$$\begin{aligned} z &\le w_1 u_0, & u_0 &\le u_1 z_1 \\ w_i u_{2i-1} &\le w_{i+1} u_{2i}, & u_{2i} z_i &\le u_{2i+1} w_{i+1} \quad (i = 1, 2, \ldots, m-1) \\ w_m u_{2m-1} &\le u_{2m}, & u_{2m} z_m &\le z; \ \text{and} \end{aligned} \tag{2}$$

$$\begin{aligned} a_0 &\le b_1 v_1, & z &\le a_0 c_1 \\ b_j a_{2j} &\le b_{j+1} a_{2j+1}, & a_{2j-1} c_j &\le b_{2j} c_{j+1} \quad (j = 1, 2, \ldots, m'-1) \\ b_{m'} a_{2m'} &\le z, & a_{2m'-1} c_{m'} &\le a_{2m'}; \end{aligned} \tag{3}$$

where, $u_0, a_0, \ldots, u_{2m}, a_{2m'} \in B, w_1, z_1, \ldots, w_m, z_m, b_1, c_1, \ldots, b_{m'}, c_{m'} \in A$.

The set of inequalities (2) and (3) are known as zigzag inequalities in A over B of length (m, m') and value z. The length (m, m') of the zigzag inequalities (2) and (3) is said to be minimal if m and m' are the smallest positive integers. From the zigzag inequalities (2), we have

$$z = w_1 u_0 = w_1 u_1 z_1 = w_2 u_2 z_1 = \cdots = w_m u_{2m-1} z_m = u_{2m} z_m = z. \tag{4}$$

Similarly, from the zigzag inequalities (3), we have

$$z = a_0 c_1 = b_1 a_1 c_1 = b_1 a_2 c_2 = \cdots = b_{m'} a_{2m'-1} c_{m'} = b_{m'} a_{2m'} = z. \tag{5}$$

In [1], Ahanger and Shah established that the Zigzag Theorem is also valid in the category of all commutative posemigroups.

The next results are from [2] and [3] and are very important for our investigations.

Lemma 1 ([2], Lemma 3.2)**.** *Let A be an overposemigroup of a posemigroup B and $z \in \widehat{Dom}(B,A) \setminus B$. If (2) and (3) are zigzag inequalities with value z having minimal length (m,m'). Then for each $p = 1,2,\dots,m,\ q = 1,2,\dots,m', w_p, z_p, b_q, c_q \in S \setminus U$.*

Lemma 2 ([2], Lemma 3.3)**.** *Let A be an overposemigroup of a posemigroup B with $\widehat{Dom}(B,A) = A$ and $z \in A \backslash B$. Then for each positive integers l and l' there exist $b_1, b_2, \dots, b_l, a_1, a_2, \dots, a_{l'} \in B$ and $z', z'' \in B \setminus A$ such that $z = b_1 b_2 \cdots b_l z' = z'' a_{l'} a_{l'-1} \cdots a_2 a_1$.*

The next result is from [1].

Theorem 2 ([1], Theorem 2.1)**.** *If B is a posemigroup satisfying a permutation identity $y_1 y_2 \cdots y_n = y_{i_1} y_{i_2} \cdots y_{i_n}$ and A is an overposemigroup of B such that $\widehat{Dom}(B,A) = A$. Then A satisfies the same permutation identity.*

Wherever bracketed assertions are used, they must be the dual of the other claims. Background material and data on semigroups assumed in the following can be found in Clifford and Preston [4], Howie [7], and will be used without explicit reference throughout.

3 Saturated posemigroups

We say that an element s in a semigroup A is a regular element if one can find an element $t \in A$ such that $sts = s$. A semigroup A is regular if each element of A is regular. In a semigroup A an element $s' \in A$ is called an inverse of $s \in A$ if $ss's = s$ and $s'ss' = s'$. One can easily verify that if s is a regular element of A then s has an inverse. An inverse semigroup A is a regular semigroup in which every element possess unique inverse. A posemigroup S is said to be a regular (inverse) posemigroup if the underlying semigroup is regular (inverse). A semigroup A is an inverse semigroup and vice versa if it is regular and its idempotents commute. Howie and Isbell [6], established that inverse semigroups are absolutely closed. Therefore, in particular inverse semigroups are saturated. In the next theorem, we extend the result to commutative inverse posemigroups. First, we prove the following lemma.

Lemma 3. *Let A be an overposemigroup of a commutative posemigroup B such that $\widehat{Dom}(B,A) = A$. Let $z \in A \setminus B$ and let (2) and (3) be zigzag inequalities for z of length (m,m'). Then $z = w_1 u_1 z_1 (\prod_{i=1}^{l} u_{2i-1})(\prod_{i=1}^{l} u_{2i-1}^{-1})$ for all $l = 1,2,\dots,m$.*

Proof. By Theorem 2, A is also a commutative posemigroup. We prove the lemma by induction on l. For the base step, we have

$$\begin{aligned} z &= w_1 u_1 z_1 \quad \text{(by zigzag equations (4))} \\ &= w_1 u_1 u_1^{-1} u_1 z_1 \quad \text{(as } B \text{ is an inverse posemigroup)} \\ &= w_1 u_1 z_1 u_1 u_1^{-1} \quad \text{(as } A \text{ is commutative).} \end{aligned}$$

This proves the base step. Assume inductively that for $k < m$

$$z = w_1 u_1 z_1 (\prod_{i=1}^{k} u_{2i-1}^{-1})(\prod_{i=1}^{k} u_{2i-1}). \tag{6}$$

Now,

$$\begin{aligned}
z &= w_{k+1} u_{2k+1} z_{k+1} (\prod_{i=1}^{k} u_{2i-1}^{-1})(\prod_{i=1}^{k} u_{2i-1}) \quad \text{(by equation (6) and} \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{zigzag equations (4))} \\
&= w_{k+1} u_{2k+1} u_{2k+1}^{-1} u_{2k+1} z_{k+1} (\prod_{i=1}^{k} u_{2i-1})(\prod_{i=1}^{k} u_{2i-1}^{-1}) \quad \text{(as } B \text{ is an inverse posemigroup)} \\
&= w_{k+1} u_{2l+1} z_{k+1} (\prod_{i=1}^{k+1} u_{2i-1})(\prod_{i=1}^{k+1} u_{2i-1}^{-1}) \quad \text{(as } A \text{ is commutative)} \\
&= w_1 u_1 z_1 (\prod_{i=1}^{k+1} u_{2i-1})(\prod_{i=1}^{k+1} u_{2i-1}^{-1}) \quad \text{(by zigzag equations (4)).}
\end{aligned}$$

This proves the lemma completely. □

Theorem 3. *Commutative inverse posemigroups are saturated.*

Proof. Assume on contrary that there exist a commutative inverse posemigroup B and an overposemigroup A of B with $\widehat{Dom}(B, A) = A$. Now, take any $z \in A \setminus B$ and let (2) and (3) be the zigzag inequalities for z of length (m, m'). Now,

$$\begin{aligned}
z &= w_1 u_1 z_1 (\prod_{i=1}^{m} u_{2i-1})(\prod_{i=1}^{m} u_{2i-1}^{-1}) \quad \text{(by Lemma 3)} \\
&= w_1 u_0 (\prod_{i=1}^{m} u_{2i-1})(\prod_{i=1}^{m} u_{2i-1}^{-1}) \quad \text{(by zigzag equations (4))} \\
&= w_1 u_1 u_0 (\prod_{i=2}^{m} u_{2i-1})(\prod_{i=1}^{m} u_{2i-1}^{-1}) \quad \text{(as } A \text{ is commutative)} \\
&\leq w_2 u_2 u_0 (\prod_{i=2}^{m} u_{2i-1})(\prod_{i=1}^{m} u_{2i-1}^{-1}) \quad \text{(by zigzag inequalities (2))} \\
&= w_2 u_3 u_0 u_2 (\prod_{i=3}^{m} u_{2i-1})(\prod_{i=1}^{m} u_{2i-1}^{-1}) \quad \text{(as } A \text{ is commutative).}
\end{aligned}$$

By carrying on in this manner, we arrive at the following

$$z \leq w_m u_{2m-1} (\prod_{i=0}^{m-1} u_{2i})(\prod_{i=1}^{m} u_{2i-1}^{-1})$$

$$\leq (\prod_{i=0}^{m} u_{2i})(\prod_{i=1}^{m} u_{2i-1}^{-1}) \ \text{(by zigzag inequalities (2))}.$$

By a similar token, we obtain that $z \geq (\prod_{i=0}^{m} u_{2i})(\prod_{i=1}^{m} u_{2i-1}^{-1})$ and thus, $z = (\prod_{i=0}^{m} u_{2i})(\prod_{i=1}^{m} u_{2i-1}^{-1}) \in U$, a contradiction as required. □

A posemigroup B is said to be finite monogenic posemigroup if it is such as a semigroup. Let $B = \langle u \rangle$ be a finite monogenic posemigroup of index n and period r. Then

$$u^n = u^{n+qr} \text{ for every positive integer } q. \tag{7}$$

Let A be an overposemigroup of B, and $z \in \widehat{Dom}(B, A) \setminus A$. By Theorem 1, the following types of zigzag inequalities of minimal length (m, m') exists for z

$$\begin{aligned} z &\leq w_1 u^{p_0}, & u^{p_0} &\leq u^{p_1} z_1 \\ w_i u^{p_{2i-1}} &\leq w_{i+1} u^{p_{2i}}, & u^{p_{2i}} z_i &\leq u^{p_{2i+1}} z_{i+1} \quad (i = 1, 2, \ldots, m-1) \\ w_m u^{p_{2m-1}} &\leq u^{p_{2m}}, & u^{p_{2m}} z_m &\leq z, \end{aligned} \tag{8}$$

$$\begin{aligned} a^{q_0} &\leq b_1 a^{q_1}, & z &\leq a^{q_0} c_1 \\ b_j a^{q_{2j}} &\leq b_{j+1} a^{q_{2j+1}}, & a^{q_{2j-1}} c_j &\leq a^{q_{2j}} c_{j+1} \quad (j = 1, 2, \ldots, m'-1) \\ b_{m'} a^{q_{2m'}} &\leq z, & a^{q_{2m'-1}} c_{m'} &\leq a^{q_{2m'}}, \end{aligned} \tag{9}$$

where, $p_0, q_0, p_1, q_1, \ldots, p_{2m}, q_{2m'}$ are positive integers, and by Lemma 1, $w_i, z_i, b_j, c_j \in A \backslash B$, $1 \leq i \leq m$ and $1 \leq j \leq m'$. The zigzag inequalities (8) reduces to the following equations,

$$d = w_1 u^{p_0} = w_1 u^{p_1} w_1 = \cdots = w_m u^{p_{2m-1}} z_m = u^{p_{2m}} z_m. \tag{10}$$

In Lemma 4 and Lemma 5, $B = \langle u \rangle$ be a finite monogenic posemigroup of index n, and period r and A be a proper overposemigroup of B such that $\widehat{Dom}(B, A) = A$.

Lemma 4. *Let $z \in A \setminus B$ and let (8) and (9) be zigzag inequalities for z in A over B of minimal length (m, m'). Then for all $i = 1, 2, \ldots, m$ there exist w_i', $z_i' \in A$ such that $w_i = w_i' u^n$ and $z_i = u^n w_i'$.*

Proof. We prove $w_i = w_i' u^n$ the other part follows on similar lines. By Lemma 2, for all $i = 1, 2, \ldots, m$ there exist $u^{l_1}, u^{l_2}, \ldots, u^{l_n} \in B$, $a_i' \in A \setminus B$ such that

$$w_i = a_i' u^{l_1} u^{l_2} \cdots u^{l_n}.$$

This gives,

$$w_i = w_i' u^n, \quad \text{where } w_i' = a_i' u^{l_1 + l_2 + \cdots + l_n - n} \in S.$$

□

Lemma 5. *If $l = 1, 2, \ldots, m$ then* $z \leq w_k u^{p_{2k-1}} (\prod_{i=1}^{l} u^{p_{2i-1} r - p_{2i-1}})(\prod_{i=1}^{l} u^{p_{2(i-1)}}) u^{nr}$.

Proof. For the base step we have

$$\begin{aligned} z &\leq w_1 u^{p_0} \quad \text{(by zigzag inequalities (8))} \\ &= w_1' u^n u^{p_0} \quad \text{(by Lemma 4)} \\ &= w_1' u^{n+nr} u^{p_0} \quad \text{(by equation (6))} \\ &= w_1' u^n u^n u^{nr-n} u^{p_0} \\ &= w_1 u^n u^{p_0} u^{nr-n} \quad \text{(by Lemma 4)} \\ &= w_1 u^{n+p_1 r} u^{p_0} u^{nr-n} \quad \text{(by equation (6))} \\ &= w_1 u^{p_1} u^n u^{p_1 r - p_1} u^{p_0} u^{nr-n} \\ &= w_1 u^{p_1} u^{p_1 r - p_1} u^{p_0} u^{nr}. \end{aligned}$$

This proves the base step. Assume inductively that the result holds for $k < m$. Then we have

$$z \leq w_k u^{p_{2k-1}} (\prod_{i=1}^{k} u^{p_{2i-1} r - p_{2i-1}}) (\prod_{i=1}^{k} u^{p_{2(i-1)}}) u^{nr}. \tag{11}$$

Now,

$$\begin{aligned} z &\leq w_k u^{p_{2k-1}} (\prod_{i=1}^{k} u^{p_{2i-1} r - p_{2i-1}}) (\prod_{i=1}^{k} u^{p_{2(i-1)}}) u^{nr} \quad \text{(by equation (11))} \\ &= w_k u^{p_{2k-1}} u^n (\prod_{i=1}^{k} u^{p_{2i-1} r - p_{2i-1}}) (\prod_{i=1}^{k} u^{p_{2(i-1)}}) u^{nr-n} \\ &\leq w_{k+1} u^{p_{2k}} u^n (\prod_{i=1}^{k} u^{p_{2i-1} r - p_{2i-1}}) (\prod_{i=1}^{k} u^{p_{2(i-1)}}) u^{nr-n} \quad \text{(by zigzag inequalities (8))} \\ &= w_{k+1} u^{n+p_{2k+1} r} (\prod_{i=1}^{k} u^{p_{2i-1} r - p_{2i-1}}) (\prod_{i=1}^{k+1} u^{p_{2(i-1)}}) u^{nr-n} \quad \text{(by equation (6))} \\ &= w_{k+1} u^{p_{2k+1}} u^n (\prod_{i=1}^{k+1} u^{p_{2i-1} r - p_{2i-1}}) (\prod_{i=1}^{k+1} u^{p_{2(i-1)}}) u^{nr-n} \\ &= w_{k+1} u^{p_{2k+1}} (\prod_{i=1}^{k+1} u^{p_{2i-1} r - p_{2i-1}}) (\prod_{i=1}^{k+1} u^{p_{2(i-1)}}) u^{nr}, \end{aligned}$$

as required. □

Theorem 4. *Finite monogenic posemigroups are saturated.*

Proof. Let $B = \langle u \rangle$ be a finite monogenic posemigroup of index n, and period r, and A be a proper overposemigroup of B. We show that $\widehat{Dom}(B, A) \neq A$. Assume on contrary that $\widehat{Dom}(B, A) = A$. Let $z \in A \setminus B$ and let (8) and (9) be zigzag inequalities for z in A over B of minimal length (m, m'). Now,

$$z \leq w_m u^{p_{2m-1}} (\prod_{i=1}^{m} u^{p_{2i-1} r - p_{2i-1}}) (\prod_{i=1}^{m} u^{p_{2(i-1)}}) u^{nr} \quad \text{(by Lemma 5)}$$

$$\leq u^{p_{2m}}(\prod_{i=1}^{m} u^{p_{2i-1}r-p_{2i-1}})(\prod_{i=1}^{m} u^{p_{2(i-1)}})u^{nr} \quad \text{(by zigzag inequalities (8))}$$
$$= (\prod_{i=1}^{m} u^{p_{2i-1}r-p_{2i-1}})(\prod_{i=1}^{m+1} u^{p_{2(i-1)}})u^{nr}.$$

By a similar token, we obtain $z \geq (\prod_{i=1}^{m} u^{p_{2i-1}r-p_{2i-1}})(\prod_{i=1}^{m+1} u^{p_{2(i-1)}})u^{nr}$. Thus, $z = (\prod_{i=1}^{m} u^{p_{2i-1}r-p_{2i-1}})(\prod_{i=1}^{m+1} u^{p_{2(i-1)}})u^{nr} \in U$, a contradiction, as required. □

4 Closed and saturated varieties of posemigroups

A variety $\mathcal{V}$ of posemigroups is said to be closed if for any posemigroups $A,\ B \in \mathcal{V}$ with A as an overposemigroup of B, implies $\widehat{Dom}(B, A) = B$, i.e., B is closed in A. It is said to be saturated if for any $B \in \mathcal{V}$ and for any proper overposemigroup A of B, $\widehat{Dom}(B, A) \neq A$. Scheiblich [8] and Higgins [5] have shown that the variety of bands satisfying the identity $axya = ayxa$ is closed and saturated respectively. We extend both the results to the variety of pobands.

Lemma 6. *Let B and A be pobands satisfying the identity*

$$axyb = ayxb, \tag{12}$$

such that A is an overpoband of B. Let $z \in \widehat{Dom}(B, A) \setminus B$ and let (2) and (3) be zigzag inequalities for z in A over B of length (m, m'). Then for all $k = 1, 2, \ldots, m,\ z \geq w_k u_{2k-1} z_k (\prod_{i=1}^{k} u_{2i-1}) u_0$.

Proof. We use induction on k. For the base step, we have

$$\begin{aligned} z &= w_1u_1z_1 \quad \text{(by zigzag equations (4))} \\ &= w_1u_1z_1u_1z_1 \quad \text{(as } A \text{ is a poband)} \\ &\geq w_1u_1z_1u_0 \quad \text{(by zigzag inequalities (2))} \\ &= w_1u_1u_1z_1u_0 \quad \text{(as } B \text{ is a poband)} \\ &= w_1u_1z_1u_1u_0 \quad \text{(as } A \text{ satisfies (12))}. \end{aligned}$$

This proves the base step. Assume inductively that for $k = l < m$, we have

$$z \geq w_l u_{2l-1} z_l (\prod_{i=1}^{l} a_{2i-1}) u_0 \tag{13}$$

Now,

$$z \geq w_{l+1}u_{2l+1}z_{l+1}(\prod_{i=1}^{l} u_{2i-1})u_0 \quad \text{(by equation (13) and zigzag equations (4))}$$
$$= w_{l+1}u_{2l+1}u_{2l+1}z_{l+1}(\prod_{i=1}^{l} u_{2i-1})u_0 \quad \text{(as } B \text{ is a poband)}$$

$$= w_{l+1}u_{2l+1}z_{l+1}(\prod_{i=1}^{l+1} u_{2i-1})u_0 \quad \text{(as } A \text{ satisfies (12))}.$$

Thus, the result holds for $k = l + 1$, as required. □

Lemma 7. *Let B and A be pobands satisfying the identity (12) such that A is an overpoband of B. Let $z \in \widehat{Dom}(B, A) \setminus B$ and let (2) and (3) be zigzag inequalities for z in A over B of length (m, m'). Then for all $k = 1, 2, \ldots, m,\ z \leq u_{2m}(\prod_{i=1}^{k} u_{2i-1})w_k u_{2k-1} z_k$.*

Proof. The proof follows on the similar lines as the proof of the Lemma 6. □

Theorem 5. *The variety of pobands satisfying the identity (12) is closed.*

Proof. Let B and A be any two pobands satisfying the identity (12) with A as an overpoband of B. We show that $\widehat{Dom}(B, A) = B$. Assume, on the contrary, and take any $z \in \widehat{Dom}(B, A) \setminus B$. Let (2) and (3) be the zigzag inequalities for z in A over B, and length (m, m'). Now,

$$\begin{aligned}
z &\geq w_m u_{2m-1} z_m (\prod_{i=1}^{m} u_{2i-1})u_0 \quad \text{(by Lemma 6)} \\
&= u_{2m} z_m (\prod_{i=1}^{m} u_{2i-1})u_0 \quad \text{(by zigzag equations (4))} \\
&= u_{2m} u_{2m-1} z_m (\prod_{i=1}^{m-1} u_{2i-1})u_0 \quad \text{(as } A \text{ satisfies (12))} \\
&\geq (\prod_{i=0}^{1} u_{2m-2i}) z_{m-1} (\prod_{i=1}^{m-1} u_{2i-1})u_0 \quad \text{(by zigzag inequalities (2))}.
\end{aligned}$$

Proceeding in this way, we get

$$\begin{aligned}
z &\geq (\prod_{i=0}^{m-1} u_{2m-2i}) z_1 (\prod_{i=1}^{1} u_{2i-1})u_0 \\
&= (\prod_{i=0}^{m-1} u_{2m-2i}) u_1 y_1 u_0 \quad \text{(as } A \text{ satisfies (12))} \\
&\geq (\prod_{i=0}^{m} u_{2m-2i}) u_0 \quad \text{(by zigzag inequalities (2))} \\
&= \prod_{i=0}^{m} u_{2m-2i} \quad \text{(as } B \text{ is a poband)}.
\end{aligned}$$

Now on similar lines, by Lemma 7 and zigzag inequalities (2), we obtain $z \leq \prod_{i=0}^{m} u_{2m-2i}$. Thus $z = \prod_{i=0}^{m} u_{2m-2i} \in B$, a contradiction, as required. □

Theorem 6. *The variety $\mathcal{V}$ of pobands satisfying the identity (12) is saturated.*

Proof. Assume, on the contrary that the variety $\mathcal{V}$ is not saturated and let $B \in \mathcal{V}$ and A be a proper overposemigroup B such that $\widehat{Dom}(B, A) = A$. By Theorem 2, $A \in \mathcal{V}$. By Theorem 5, $\widehat{Dom}(B, A) = B$ and therefore, $B = A$, a contradiction, as required. □

5 Absolutely closed posemigroups

Let A be an overposemigroup of B. Then we say B is convex in A if for any $b,\ b' \in B$ and $a \in A$ with $b \le a \le b'$, implies $a \in B$, and we say a posemigroup B is convex if it is convex in every overposemigroup. In [6], Howie and Isbell have shown that the finite monogenic semigroups and the inverse semigroups are absolutely closed. Here, we extend these results by showing that the convex finite monogenic posemigroups and inverse posemigroups are absolutely closed in the category of all commutative posemigroups.

From Lemma 8 to Lemma 10, $B = \langle u \rangle$ is a convex finite monogenic posemigroup of index n and period r, and A is an overposemigroup of B. Further, $z \in \widehat{Dom}(B, A) \setminus A$, and (8) and (9) are zigzag inequalities for z in A over B of length (m, m').

Lemma 8. *If $p_0 < p_1$, then for every positive integer k, $z = w_1 u^{p_0} u^{k(p_1 - p_0)} z_1^k$.*

Proof. For the base step, we have

$$\begin{aligned} z &= w_1 u^{p_1} z_1 \quad \text{(by zigzag equations (10))} \\ &= w_1 u^{p_0} u^{p_1 - p_0} z_1 \quad (\text{as } p_0 < p_1). \end{aligned}$$

This proves the base step. Assume inductively that the following holds

$$z = w_1 u^{p_0} u^{l(p_1 - p_0)} z_1^l. \tag{14}$$

Now,

$$\begin{aligned} z &= w_1 u^{p_1} z_1 u^{l(p_1 - p_0)} z_1^l \quad \text{(by equation (14) and zigzag equations (10))} \\ &= w_1 u^{p_0} u^{p_1 - p_0} z_1 u^{l(p_1 - p_0)} z_1^l \\ &= w_1 u^{p_0} u^{(l+1)(p_1 - p_0)} z_1^{l+1} \quad (\text{as } A \text{ is commutative}). \end{aligned}$$

This proves the lemma completely. □

Lemma 9. *If $p_0 < p_1$, then $z = u^{np_1 - n} u^{(n+1)(p_1 r - p_1)} u^{p_0} w_1 u^{p_1} u^n$.*

Proof. We have

$$\begin{aligned} z &= w_1 u^{p_0} u^{n(p_1 - p_0)} z_1^n \quad \text{(by Lemma 8)} \\ &= u^{p_0} u^{n(p_1 - p_0) - n} w_1 u^n z_1^n \quad (\text{as } A \text{ is commutative}) \\ &= u^{p_0} u^{n(p_1 - p_0) - n} w_1 u^{n + p_1 r} z_1^n \quad \text{(by equation (6))} \\ &= u^{p_0} u^{n(p_1 - p_0) - n} u^n u^{p_1 r - p_1} w_1 u^{p_1} z_1 z_1^{n-1} \quad (\text{as } A \text{ is commutative}) \end{aligned}$$

$$
\begin{aligned}
&= u^{p_0}u^{n(p_1-p_0)-n}u^nu^{p_1r-p_1}w_1u^{p_0}z_1^{n-1} \quad \text{(by zigzag equations (10))}\\
&= u^{2p_0}u^{n(p_1-p_0)-n}u^{p_1r-p_1}w_1u^nz_1^{n-1} \quad \text{(as } A \text{ is commutative).}
\end{aligned}
$$

By continuing in this manner, we arrive to the following

$$
\begin{aligned}
z &= u^{np_0}u^{n(p_1-p_0)-n}u^{(n-1)(p_1r-p_1)}w_1u^nz_1\\
&= u^{np_0}u^{n(p_1-p_0)-n}u^{(n-1)(p_1r-p_1)}w_1u^{n+p_1r}z_1 \quad \text{(by equation (6))}\\
&= u^{np_0}u^{n(p_1-p_0)-n}u^{n(p_1r-p_1)}u^nw_1u^{p_1}z_1 \quad \text{(as } A \text{ is commutative)}\\
&= u^{np_1-n}u^{n(p_1r-p_1)}u^nw_1u^{p_0} \quad \text{(by zigzag equations (10))}\\
&= u^{np_1-n}u^{n(p_1r-p_1)}u^{p_0}w_1u^n \quad \text{(as } A \text{ is commutative)}\\
&= u^{np_1-n}u^{n(p_1r-p_1)}u^{p_0}w_1u^{n+p_1r} \quad \text{(by equation (6))}\\
&= u^{np_1-n}u^{(n+1)(p_1r-p_1)}u^{p_0}w_1u^{p_1}u^n \quad \text{(as } A \text{ is commutative).}
\end{aligned}
$$

This completes the proof of the lemma. □

Lemma 10. *If $p_0 < p_1$ and $p_{2m} < p_{2m-1}$, then $z \in B$.*

Proof. We have

$$
\begin{aligned}
z &= u^{np_1-n}u^{(n+1)(p_1r-p_1)}u^{p_0}w_1u^{p_1}u^n \quad \text{(by Lemma 9, as } p_0 < p_1)\\
&\leq u^{np_1-n}u^{(n+1)(p_1r-p_1)}u^{p_0}w_2u^{p_2}u^n \quad \text{(by zigzag inequalities (8))}\\
&= u^{np_1-n}u^{(n+1)(p_1r-p_1)}u^{p_0}w_2u^{p_2}u^{n+p_3r} \quad \text{(by equation (6))}\\
&= u^{np_1-n}u^{n(p_1r-p_1)}\Big(\prod_{i=0}^{1}u^{p_{2i}}\Big)\Big(\prod_{i=1}^{2}u^{p_{2i-1}r-p_{2i-1}}\Big)w_2u^{p_3} \quad \text{(as } A \text{ is commutative).}
\end{aligned}
$$

By doing so, we arrive to the following

$$
\begin{aligned}
z &\leq u^{np_1-n}u^{n(p_1r-p_1)}\Big(\prod_{i=0}^{m-1}u^{p_{2i}}\Big)\Big(\prod_{i=1}^{m}u^{p_{2i-1}r-p_{2i-1}}\Big)w_mu^{p_{2m-1}}\\
&\leq u^{np_1-n}u^{n(p_1r-p_1)}\Big(\prod_{i=0}^{m-1}u^{p_{2i}}\Big)\Big(\prod_{i=1}^{m}u^{p_{2i-1}r-p_{2i-1}}\Big)u^{p_{2m}} \quad \text{(by zigzag inequalities (8))}\\
&= u^{np_1-n}u^{n(p_1r-p_1)}\Big(\prod_{i=0}^{m}u^{p_{2i}}\Big)\Big(\prod_{i=1}^{m}u^{p_{2i-1}r-p_{2i-1}}\Big) = a_1 \in U.
\end{aligned}
$$

On similar lines by using $p_{2m} < p_{2m-1}$, zigzag equations (10) and zigzag inequalities (8) in reverse order there exists $a_2 \in U$ such that $z \geq a_2$. Hence $z \in B$, as B is convex in A. This proves the lemma completely. □

Lemma 11. *If $p_0 > p_1 > \cdots > p_{2m}$, then $z \in B$.*

Proof. We have

$$\begin{aligned} z &= w_1 u^{p_0} \quad \text{(by zigzag equation (10))} \\ &= w_1 u^{p_1} u^{p_0 - p_1} \quad (\text{as } p_0 > p_1) \\ &\leq w_2 u^{p_2} u^{p_0 - p_1} \quad \text{(by zigzag inequalities (8))} \\ &= w_2 u^{p_3} u^{p_2 - p_3} u^{p_0 - p_1} \quad (\text{as } p_2 > p_3). \end{aligned}$$

Proceeding in this manner, we arrive to the following

$$\begin{aligned} z &\leq w_m u^{p_{2m-1}} \prod_{k=0}^{m-1} u^{p_{2k} - p_{2k+1}} \\ &\leq u^{p_{2m}} \prod_{k=0}^{m-1} u^{p_{2k} - p_{2k+1}} = a_1 \in B \quad \text{(by zigzag inequalities (8))}. \end{aligned}$$

Since $p_{2m} < p_{2m-1}$, by Lemma 10, there exists $a_2 \in B$ such that $a_2 \leq z$. Thus $z \in U$, as B is convex in A. □

Lemma 12. *Let i be the least positive integer such that $p_{2i-1} \leq p_{2i}$ $[p_{2i} \leq p_{2i+1}]$, then the length m of the zigzag inequalities (8) can be reduced to $m-1$.*

Proof. We prove the lemma for the case, $p_{2i-1} \leq p_{2i}$, the other case follows dually. If $i = 1$, then $p_0 > p_1$ and $p_1 \leq p_2$. Now,

$$\begin{aligned} z &\leq w_1 u^{p_0} \quad \text{(by zigzag inequalities (8))} \\ &= w_1 u^{p_1} u^{p_0 - p_1} \quad (\text{as } p_0 > p_1) \\ &\leq w_2 u^{p_2} u^{p_0 - p_1} \quad \text{(by zigzag inequalities (8))}. \end{aligned}$$

And

$$\begin{aligned} u^{p_2} u^{p_0 - p_1} &= u^{p_2 - p_1} u^{p_0} \quad \text{as } p_2 \geq p_1 \\ &\leq u^{p_2 - p_1} u^{p_1} z_1 \quad \text{(by zigzag inequalities (8))} \\ &= u^{p_2} z_1 \\ &\leq u^{p_3} z_2 \quad \text{(by zigzag inequalities (8))}. \end{aligned}$$

Therefore, the length m of the zigzag inequalities (8) can be reduced to $m-1$ if $i = 1$. In case $i \geq 2$. Since i is the least positive integer such that $p_{2i-1} \leq p_{2i}$, therefore $p_{2i-2} > p_{2i-1}$. Now,

$$\begin{aligned} w_{i-1} u^{p_{2i-3}} &\leq w_i u^{p_{2i-2}} \quad \text{(by zigzag inequalities (8))} \\ &= w_i u^{p_{2i-1}} u^{p_{2i-2} - p_{2i-1}} \quad (\text{as } p_{2i-2} > p_{2i-1}) \\ &\leq w_{i+1} u^{p_{2i}} u^{p_{2i-2} - p_{2i-1}} \quad \text{(by zigzag inequalities (8))}, \end{aligned}$$

and

$$u^{p_{2i}} u^{p_{2i-2} - p_{2i-1}} w_{i-1} = u^{p_{2i} - p_{2i-1}} u^{p_{2i-2}} w_{i-1} \quad (\text{as } p_{2i} \geq p_{2i-1})$$

$$
\begin{aligned}
&\le u^{p_{2i}-p_{2i-1}}u^{p_{2i-1}}w_i \quad \text{(by zigzag inequalities (8))}\\
&= u^{p_{2i}}w_i\\
&\le u^{p_{2i+1}}w_{i+1} \quad \text{(by zigzag inequalities (8)).}
\end{aligned}
$$

Therefore, the length m of the zigzag inequalities (8) can be reduced to $m-1$ if $i \geq 2$. □

Theorem 7. *In the category of all commutative posemigroups every finite monogenic posemigroup is absolutely closed.*

Proof. Let $B = \langle u \rangle$ be a finite monogenic posemigroup of index n and period r, and let A be an commutative posemigroup such that A is an overposemigroup of B. We show that $\widehat{Dom}(B,A) = A$. Take any $z \in \widehat{Dom}(U,S)$ and let (8) and (9) be zigzag inequalities for z in A over B of length (m, m'). If $p_0 > p_1 > \cdots > p_{2m}$, then $z \in B$, by Lemma 11, as requird. Now, let i be the least positive integer such that $p_{2i-1} \leq p_{2i}$ $[p_{2i} \leq p_{2i+1}]$, then by Lemma 12, the length m of the zigzag inequalities (8) can be reduced to $m-1$. Thus we can assume that $m = 1$ in the zigzag inequalities (8). If $p_0 < p_1$ and $p_2 < p_1$, then proof follows from the Lemma 10. If $p_0 > p_1 > p_2$ $[p_0 < p_1 < p_2]$, then proof follows from Lemma 11. Finally, if $p_0 \geq p_1$ and $p_1 \leq p_2$, then,

$$
\begin{aligned}
z &\le w_1u^{p_0} \quad \text{(by zigzag inequalities (8))}\\
&= w_1u^{p_1}u^{p_0-p_1} \quad (\text{as } p_0 \ge p_1)\\
&= u^{p_2}u^{p_0-p_1} = u^{p_0+p_2-p_1} \quad \text{(by zigzag inequalities(8))}
\end{aligned}
$$

and

$$
\begin{aligned}
z &\ge u^{p_2}z_1 \quad \text{(by zigzag inequalities (8))}\\
&= u^{p_2-p_1}u^{p_1}y_1 \quad (\text{as } p_1 \le p_2)\\
&\ge u^{p_2-p_1}u^{p_0} = u^{p_0+p_2-p_1} \quad \text{(by zigzag inequalities (8)).}
\end{aligned}
$$

Thus, $z = u^{p_0+p_2-p_1} \in U$, as required. □

In the next theorem, we show that inverse posemigroups are absolutely closed in the category of commutative posemigroups.

Lemma 13. *Let B be an inverse posemigroup and A be an overposemigroup of B such that the idempotents of B are central in $A \setminus B$. Then $\widehat{Dom}(B,A) = A$.*

Proof. Assume on contrary and take any $z \in \widehat{Dom}(B,A) \setminus A$ has zigzag inequalities (2) and (3) of minimal length (m, m'). Then $w_i, z_i \in A \setminus B$ for all $i = 1, 2, \ldots, m$. For all $k = 1, 2, \ldots, m$, we show that

$$z = w_k u_{2k-1} z_k (\prod_{i=1}^{k} u_{2i-1}^{-1} a_{2i-1}). \tag{15}$$

To prove (15), we use induction on k. For the base step, we have

$$z = w_1 u_1 z_1 \quad \text{(by zigzag equations (4))}$$

$$= w_1u_1u_1^{-1}u_1z_1 \text{ (as } B \text{ is an inverse posemigroup)}$$
$$= w_1u_1z_1u_1^{-1}u_1 \text{ (as } u_1^{-1}u_1 \text{ is an idempotent of } B).$$

This proves the base step. Assume inductively that for $k = l < m$, we have

$$z = w_l u_{2l-1} z_l (\prod_{i=1}^{l} u_{2i-1}^{-1} u_{2i-1}). \tag{16}$$

Now,

$$\begin{aligned} z &= w_{l+1}u_{2l+1}z_{l+1}(\prod_{i=1}^{l} u_{2i-1}^{-1}u_{2i-1}) \text{ (by equation (15) and zigzag equations (4))} \\ &= w_{l+1}u_{2l+1}u_{2l+1}^{-1}u_{2l+1}z_{l+1}(\prod_{i=1}^{l} u_{2i-1}^{-1}u_{2i-1}) \text{ (as } B \text{ is an inverse posemigroup)} \\ &= w_{l+1}u_{2l+1}z_{l+1}u_{2l+1}^{-1}u_{2l+1}(\prod_{i=1}^{l} u_{2i-1}^{-1}u_{2i-1}) \text{ (as } u_{2l+1}^{-1}u_{2l+1} \text{ is an idempotent)} \\ &= w_{l+1}u_{2l+1}z_{l+1}(\prod_{i=1}^{l+1} u_{2i-1}^{-1}u_{2i-1}) \text{ (as idempotent commute).} \end{aligned}$$

This completes the induction. Now to complete the proof of the lemma, we have

$$\begin{aligned} z &= w_m u_{2m-1} z_m (\prod_{i=1}^{m} u_{2i-1}^{-1}u_{2i-1}) \text{ (by equations (15))} \\ &= u_{2m}z_m(\prod_{i=1}^{m} u_{2i-1}^{-1}u_{2i-1}) \text{ (by zigzag equations (4))} \\ &= u_{2m}u_{2m-1}^{-1}u_{2m-1}z_m(\prod_{i=1}^{m-1} u_{2i-1}^{-1}u_{2i-1}) \text{ (as } u_{2m-1}^{-1}u_{2m-1} \text{ is an idempotent)} \\ &\geq u_{2m}u_{2m-1}^{-1}u_{2m-2}z_{m-1}(\prod_{i=1}^{m-1} u_{2i-1}^{-1}u_{2i-1}) \text{ (by zigzag inequalities (3))} \\ &= u_{2m}u_{2m-1}^{-1}u_{2m-2}u_{2m-3}^{-1}u_{2m-3}z_{m-1}(\prod_{i=1}^{m-2} u_{2i-1}^{-1}u_{2i-1}) \text{ (as } u_{2m-3}^{-1}u_{2m-3} \text{ is an idempotent)} \\ &= (\prod_{i=0}^{1} u_{2m-2i}u_{2m-2i-1}^{-1})u_{2m-3}z_{m-1}(\prod_{i=1}^{m-2} u_{2i-1}^{-1}u_{2i-1}). \end{aligned}$$

Proceeding like wise, we arive to the following

$$z \geq (\prod_{i=0}^{m-2} u_{2m-2i}u_{2m-2i-1}^{-1})u_3z_2(\prod_{i=1}^{1} u_{2i-1}^{-1}u_{2i-1})$$

$$\geq (\prod_{i=0}^{m-2} u_{2m-2i}u_{2m-2i-1}^{-1})u_2y_1u_1^{-1}u_1 \quad \text{(by zigzag inequalities (3))}$$
$$= (\prod_{i=0}^{m-2} u_{2m-2i}u_{2m-2i-1}^{-1})u_2u_1^{-1}u_1z_1 \quad \text{(as } u_1^{-1}u_1 \text{ is an idempotent)}$$
$$\geq (\prod_{i=0}^{m-1} u_{2m-2i}u_{2m-2i-1}^{-1})u_0 \quad \text{(by zigzag inequalities (2)).}$$

By a similar token, we obtain $z \leq (\prod_{i=0}^{m-1} u_{2(m-i)}u_{2(m-i)-1}^{-1})u_0$ and hence $z \in B$, as required. □

Theorem 8. *In the category of all commutative posemigroups, inverse posemigroups are absolutely closed.*

Proof. It follows from the Lemma 13. □

Now, we pose the following problems.

Problem 1. Is the variety of po-left[right] regular bands closed?

Problem 2. Is the variety of po-left[right] regular bands closed in the variety of po-bands?

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